

**Viscous relaxation and collective oscillations in a trapped Fermi gas near the unitarity limit**

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The viscous relaxation time of a trapped two-component gas of fermions in its normal phase is calculated as a function of temperature and scattering length, with the collision probability being determined by an energy-dependent  $s$ -wave cross section. The result is used for calculating the temperature dependence of the frequency and damping of collective modes studied in recent experiments, starting from the kinetic equation for the fermion distribution function with mean-field effects included in the streaming terms.

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**I. INTRODUCTION**

The trapping and cooling of fermions has become one of the central areas of research within the field of ultracold atomic gases. Such gases offer the exciting prospect of examining the properties of interacting Fermi gases with unprecedented flexibility. Due to the existence of Feshbach resonances, the interactions between atoms can be varied almost at will by changing an external magnetic field, allowing one to study the intriguing problem of a two-component Fermi gas with a unitarity-limited interaction. It has been predicted that the gas at low temperatures consists of a Bose-Einstein condensate (BEC) of tightly bound molecules on the molecular side of the resonance where  $0 < k_F a \ll 1$ , with  $a$  being the scattering length and  $k_F$  the magnitude of the Fermi wave vector, and a Bardeen-Cooper-Schrieffer (BCS) superfluid state on the other side of the resonance where  $0 < -k_F a \ll 1$ , with an interesting crossover regime in between [1]. Several experimental groups have now reported clear experimental evidence for a BEC of diatomic molecules on the molecular side of the resonance from measurements of the momentum distribution of the gas [2]. Experiments have also probed the  $a < 0$  side of the resonance using a magnetic-field sweep to the BEC side together with radio frequency (rf) spectroscopy [3].

The study of collective modes is an important spectroscopic tool for probing the many-body dynamics of atomic gases [4]. Three recent papers report measurements of the collective modes of a two-component Fermi gas ( ${}^6\text{Li}$ ) close to the unitarity limit [5–7]. Although the collective mode spectrum of a normal gas in the hydrodynamic regime and that of a bulk superfluid gas are identical [8], it was argued in Ref. [6] that the damping of the modes as a function of temperature should allow one to distinguish between a normal gas in the hydrodynamic regime and a superfluid gas.

In the following we shall examine the collective mode spectrum of a two-component Fermi gas in its normal phase. Our starting point is the Boltzmann-Vlasov equation for the fermion distribution function  $f(\mathbf{r}, \mathbf{p}, t)$ . The effects of atom-

atom interactions enter both the collision integral and the streaming terms of the equation. Several recent theoretical papers have considered the effect of collisions on the collective oscillations of a trapped gas of fermions [9–12]. Our approach differs from that of previous authors in that we take into account both the energy dependence of the scattering cross section in the collision integral and the effects of the mean field in the streaming terms. Both these effects are expected to be important in the region near the resonance. We determine the appropriate relaxation rate from an approximate solution to the kinetic equation which is known to give very accurate results for the viscous relaxation time in the limits of high and low temperatures [13]. We then use it to extract the frequency and damping of the collective oscillations in the trapped gas.

Since our approach takes full account of the collision processes and includes the effects of the mean field, it allows for a direct comparison between the measured and calculated values of the frequency and damping of collective modes. Our results indicate that the transverse oscillations measured experimentally [5–7] are never truly hydrodynamic in the normal phase, and it identifies the regions at high and low temperatures in which collisionless behavior is to be expected. Our calculations suggest that further insight into the properties of the gas near the unitarity limit may be obtained by more extensive measurements of the oscillation frequencies at magnetic fields near resonance as a function of temperature in a regime where the temperature is comparable to or higher than the Fermi temperature.

The plan of the paper is as follows. In Sec. II we introduce the viscous relaxation time and derive its temperature dependence for both a uniform and a trapped gas from approximate solutions to the kinetic equation for the fermion distribution function. The kinetic equation also forms the starting point for the calculation of collective-mode frequencies and their damping. In Sec. III we obtain these by taking moments of the equation and present our results for a trapped gas of  ${}^6\text{Li}$  atoms in an axially symmetric trap. The effects of interaction in the streaming terms are taken into account to

leading order by adding a mean field to the trap potential. For realistic scattering lengths and trap parameters the frequencies of the collective modes are close to their values in the collisionless limit at high and low temperatures. In the intermediate-temperature regime the gas approaches hydrodynamic behavior in the sense that the viscous relaxation rate becomes comparable to the oscillation frequency. In Sec. IV we compare the calculated frequencies and attenuation of the collective modes to experiment, and our main findings are summarized in the concluding section V. A detailed account of the moment method used for solving the Boltzmann-Vlasov equation is given in the Appendix.

## II. VISCOUS RELAXATION TIME

We shall consider a two-component Fermi gas of atoms with mass  $m$  in its normal phase. The gas may be uniform or trapped in a potential  $V(\mathbf{r})$ . We assume that the dynamics is described by a semiclassical distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  which satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = -I[f], \quad (1)$$

where  $I$  is the collision integral. The time development of  $\mathbf{r}$  and  $\mathbf{p}$  is given by the equations of motion

$$\dot{\mathbf{r}} = \mathbf{v} = \frac{\mathbf{p}}{m}, \quad \dot{\mathbf{p}} = -\frac{\partial V}{\partial \mathbf{r}}. \quad (2)$$

Since this paper concerns dynamics for which the two components of the gas move together, we only need to introduce one distribution function referring to a definite set of internal quantum numbers, for brevity denoted by ‘‘spin’’ with the two values  $\sigma = \uparrow, \downarrow$ . The distribution function  $f$  always refers to a definite spin value, and we have  $f = f_{\uparrow} = f_{\downarrow}$ . Since we consider  $s$ -wave scattering, the interaction only involves particles with opposite spin. In the present section the streaming terms on the left-hand side of Eq. (1) do not contain any effects of the interaction, but later, in Sec. III, we shall add these as a mean field in the equations of motion (2).

The viscous relaxation time, which plays an important part in the following, is defined in terms of the viscosity. Let us briefly recall how one determines the viscosity by linearizing the kinetic equation in the spatial derivatives of the flow velocity  $\mathbf{u}(\mathbf{r})$ . For simplicity we take the direction of the flow velocity as our  $x$  axis and assume that it varies in the  $y$  direction,  $\mathbf{u} = (u_x(y), 0, 0)$ . To calculate the viscosity we insert a local equilibrium distribution  $f_{\text{loc}}$  given by

$$f_{\text{loc}}(\mathbf{p}) = f^0(\epsilon - \mathbf{u} \cdot \mathbf{p}), \quad (3)$$

where  $\epsilon = p^2/2m$  and  $f^0$  denotes the equilibrium Fermi function, and linearize in the gradient  $\partial u_x / \partial y$ . Under stationary conditions Eq. (1) then becomes

$$-\frac{\partial u_x}{\partial y} v_y p_x \frac{\partial f^0}{\partial \epsilon} = -I[f]. \quad (4)$$

The shear viscosity  $\eta$  relates the momentum current density  $\Pi_{xy}$ , given by

$$\Pi_{xy} = 2 \int \frac{d^3 p}{(2\pi\hbar)^3} v_y p_x f, \quad (5)$$

to the gradient of the flow velocity according to  $\Pi_{xy} = -\eta \partial u_x / \partial y$ . The factor of 2 appearing on the right-hand side of Eq. (5) arises from summing over the contributions of the two components ( $\uparrow$  and  $\downarrow$ ). In order to establish the concept of a viscous relaxation time let us make a relaxation time approximation to the collision integral with a relaxation time  $\tau_\eta$  which so far is an unknown quantity:

$$I[f] \approx \frac{f - f^0}{\tau_\eta}. \quad (6)$$

The collision integral (6) yields together with Eqs. (4) and (5) the viscosity

$$\eta = 2\tau_\eta \int \frac{d^3 p}{(2\pi\hbar)^3} v_y^2 p_x^2 \left( -\frac{\partial f^0}{\partial \epsilon} \right). \quad (7)$$

The total particle density  $n_{\text{tot}}$  of the two components is

$$n_{\text{tot}} = 2 \int \frac{d^3 p}{(2\pi\hbar)^3} f^0. \quad (8)$$

The angular integration in momentum space yields a factor of  $1/15$ , and the ratio  $\eta/n_{\text{tot}}$  may then after partial integration be written in terms of energy integrals as

$$\frac{\eta}{n_{\text{tot}}} = \frac{2}{5} \tau_\eta \frac{\int_0^\infty \epsilon^{5/2} f^0 (1 - f^0) d\epsilon}{\int_0^\infty \epsilon^{3/2} f^0 (1 - f^0) d\epsilon}. \quad (9)$$

Analytic expressions for the integrals occurring in Eq. (9) may be obtained when  $kT$  is either small or large compared to the Fermi energy  $\epsilon_F = \hbar^2 k_F^2 / 2m$ , where the magnitude of the Fermi wave vector,  $k_F$ , is given by  $k_F^3 = 3\pi^2 n_{\text{tot}}$ . When the temperature is much larger than the Fermi temperature,  $T_F = \epsilon_F / k$ , the equilibrium distribution function is  $f^0 \approx \exp[(\mu - \epsilon) / kT]$ , with  $\mu$  being the chemical potential, and Eq. (9) becomes

$$\frac{\eta}{n_{\text{tot}}} = kT \tau_\eta. \quad (10)$$

At low temperatures ( $T \ll T_F$ ) we obtain from Eq. (9) that

$$\frac{\eta}{n_{\text{tot}}} = \frac{1}{5} m v_F^2 \tau_\eta, \quad (11)$$

where  $v_F = \hbar k_F / m$  is the Fermi velocity. We shall use Eq. (9) [and the limiting forms (10) and (11)] to define the viscous relaxation time  $\tau_\eta$  in terms of the viscosity calculated by taking the full collision integral into account. The viscous relaxation time at low temperatures ( $T \ll T_F$ ) would thus be defined in terms of the calculated low-temperature viscosity  $\eta_{\text{low}}$  according to  $\tau_\eta = 5 \eta_{\text{low}} / n_{\text{tot}} m v_F^2$ .

In the presence of the full collision integral  $I[f]$ , which is a functional of the distribution function, we linearize the Boltzmann equation in terms of a small deviation  $\delta f$  from the equilibrium distribution by writing  $f(\mathbf{r}, \mathbf{p}, t) = f^0(\mathbf{r}, \mathbf{p}) + \delta f(\mathbf{r}, \mathbf{p}, t)$  with

$$\delta f(\mathbf{r}, \mathbf{p}, t) = f^0(\mathbf{r}, \mathbf{p})[1 - f^0(\mathbf{r}, \mathbf{p})]\Phi(\mathbf{r}, \mathbf{p}, t). \quad (12)$$

The linearized collision integral becomes a functional of  $\Phi$  given by

$$I[\Phi] = \int \frac{d^3 p_1}{(2\pi\hbar)^3} \int d\Omega \frac{d\sigma}{d\Omega} |\mathbf{v} - \mathbf{v}_1| \times [\Phi + \Phi_1 - \Phi' - \Phi'_1] f_1^0 (1 - f^{0'}) (1 - f_1^{0'}), \quad (13)$$

where  $d\sigma/d\Omega$  is the differential cross section and  $\Omega$  is the solid angle for the direction of the relative outgoing momentum  $\mathbf{p}'_r = (\mathbf{p}' - \mathbf{p}'_1)/2$  with respect to the relative incoming momentum  $\mathbf{p}_r = (\mathbf{p} - \mathbf{p}_1)/2$  [4]. In the present paper, we take the cross section to be given by the resonant form

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{1 + (p_r/\hbar)^2 a^2}. \quad (14)$$

In the unitarity limit, where  $|a|$  tends toward infinity, the calculated viscous relaxation rate approaches a finite value that depends on temperature, since the cross section in this case is determined by the typical value of the wave number  $p_r/\hbar$  for the relative motion.

It should be noted that we neglect in the cross section (14) any effects of the medium, which can be significant at very low temperatures [14].

#### A. Viscosity of a uniform gas

We shall now calculate the viscosity of a homogeneous gas starting from Eqs. (1) and (13), using a variational principle commonly employed in transport theory. The Boltzmann equation has the form of a linear, inhomogeneous integral equation,  $X = H\Phi$ , where the inhomogeneous term is  $X$  and  $H$  is an integral operator with eigenvalues greater than or equal to zero. The use of the Schwarz inequality  $(U, HU)(\Phi, H\Phi) \geq (U, H\Phi)^2$ , where  $(\dots, \dots)$  denotes a suitably defined scalar product and  $U$  is a trial function, allows one to put a lower bound on the viscosity, which is proportional to  $(\Phi, X) = (\Phi, H\Phi)$ . In the present case we choose  $X$  to be equal to  $v_y p_x$  and define the scalar product according to

$$(A, B) = \int \frac{d^3 p}{(2\pi\hbar)^3} A(\mathbf{p}) B(\mathbf{p}) f^0 (1 - f^0), \quad (15)$$

where  $A$  and  $B$  are functions of the momentum. Since the momentum-space dependence of the left-hand side of the Boltzmann equation involves  $v_y p_x$ , it is natural to use a trial function proportional to  $v_y p_x$  as a first approximation. The viscosity can therefore be approximated by the lower bound

$$\eta = \frac{2}{kT} \frac{(X, X)^2}{(X, HX)}, \quad (16)$$

where the integral operator  $H$  is defined as

$$H\Phi = \frac{1}{f^0(1-f^0)} I[\Phi]. \quad (17)$$

The explicit factor of 2 in the numerator of Eq. (16) arises from summing the contributions of the two components. The corresponding viscous relaxation time is

$$\tau_\eta = \frac{(X, X)}{(X, HX)}, \quad (18)$$

which is seen to be independent of the normalization of  $X$ . The final expression (18) is thus an approximate expression, obtained by a trial function proportional to  $v_y p_x$ , but it is known [13] to differ at high and low temperatures by only a few percent from the viscosity obtained from the exact solution to the Boltzmann equation.

In the classical limit,  $T \gg T_F$ , and for an energy-dependent scattering cross section given by Eq. (14) the viscosity, when expressed in terms of the viscous relaxation time  $\tau_\eta = \eta/n_{\text{tot}} kT$ , is (see, e.g., Ref. [13])

$$\frac{1}{\tau_\eta} = \frac{8}{5\sqrt{\pi}} n_{\text{tot}} \left( \frac{kT}{m} \right)^{1/2} \bar{\sigma}. \quad (19)$$

Here  $\bar{\sigma}$  is an effective cross section, which depends on the ratio  $T/T_a$ , where the temperature  $T_a$  is defined by

$$kT_a = \frac{\hbar^2}{ma^2}. \quad (20)$$

In general, we have

$$\bar{\sigma} = \frac{4\pi a^2}{3} \int_0^\infty dx x^7 e^{-x^2} (1 + x^2 T/T_a)^{-1}. \quad (21)$$

For  $T \ll T_a$  we obtain from Eq. (21) the classical result  $\bar{\sigma} = 4\pi a^2$ , while in the opposite limit  $T \gg T_a$ , Eq. (21) yields

$$\bar{\sigma} = \frac{4\pi a^2 T_a}{3T} = \frac{4\pi}{3} \frac{\hbar^2}{mkT}, \quad (22)$$

which is seen to be independent of the scattering length  $a$  and, apart from a numerical constant, equal to the square of the thermal de Broglie wavelength.

At low temperatures  $T \ll T_F$ , one expects on general grounds that  $1/\tau_\eta \propto T^2$  due to the restrictions on the available phase space caused by the occupied states, the so-called Pauli blocking. The magnitude of  $1/\tau_\eta$  depends on the dimensionless quantity  $\gamma = (k_F a)^2 = 2T_F/T_a$ . The corresponding variational solution to the Landau-Boltzmann equation of a Fermi liquid (see Ref. [13], Sec. 6.2.1) yields

$$\frac{1}{\tau_\eta} = 2\pi \frac{kT^2}{\hbar T_a} F(\gamma), \quad (23)$$

where the function  $F(\gamma)$  is given by the integral

$$F(\gamma) = 2 \int_0^1 dx \frac{x^5}{\sqrt{1-x^2}} \frac{1}{1+\gamma x^2}, \quad (24)$$

the variable  $x$  being equal to the sine of half the angle between the two incoming particle momenta in a collision. The function  $F(\gamma)$  decreases monotonically from its  $\gamma=0$  value

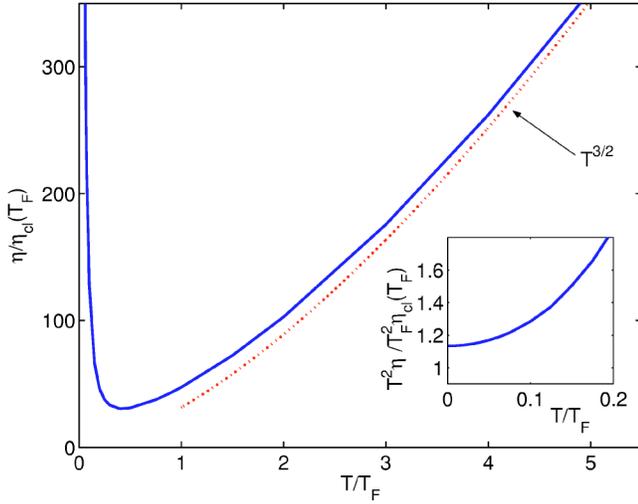


FIG. 1. (Color online) The viscosity  $\eta$  as a function of temperature for  $k_F|a|=4.5$ , in units of  $\eta_{cl}(T_F)=5(mkT_F/\pi)^{1/2}/32a^2$ , the classical value of the viscosity for an energy-independent scattering cross section, evaluated at the Fermi temperature  $T=T_F$ . The inset illustrates the low-temperature  $T^{-2}$  dependence of the viscosity.

$F(0)=16/15$  to its asymptotic expression  $F(\gamma)\approx 4/3\gamma$  for  $\gamma\gg 1$ .

In the unitarity limit ( $|a|\rightarrow\infty$ ) the viscous relaxation rate (23) becomes independent of the magnitude of the scattering length, since  $F(\gamma)$  in this limit is proportional to  $1/a^2$ . In general, the calculated relaxation rate tends toward a well-defined value which depends on temperature, when the scattering length tends toward infinity. The value of  $1/\tau_\eta$  at unitarity vanishes as  $T^2$  at low temperatures and as  $T^{-1/2}$  at high temperatures.

In Fig. 1 we plot the calculated viscosity as a function of temperature for the value  $k_F|a|=4.5$ . The inset shows the viscosity multiplied by  $T^2$  in order to illustrate its characteristic low-temperature behavior given by Eq. (23). Since  $T_a=0.1T_F$  for this value of  $k_F|a|$ , the viscosity at high temperatures is proportional to  $T^{3/2}$ . This may be seen by combining the high-temperature relation  $\eta=n_{tot}kT\tau_\eta$  with Eqs. (19) and (22), which show that  $\tau_\eta$  is proportional to  $T^{1/2}$  at high temperatures, resulting in  $\eta\propto T^{3/2}$ . In the case of energy-independent scattering ( $T_a\gg T_F$ ) the high-temperature viscosity is proportional to  $T^{1/2}$ .

### B. Viscous relaxation rate of a trapped gas

In order to apply these results to a trapped atomic cloud we now include the trap potential in the equilibrium Fermi function. We consider the harmonic-oscillator potential

$$V(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \quad (25)$$

The average viscous relaxation rate  $1/\tau$  is defined by

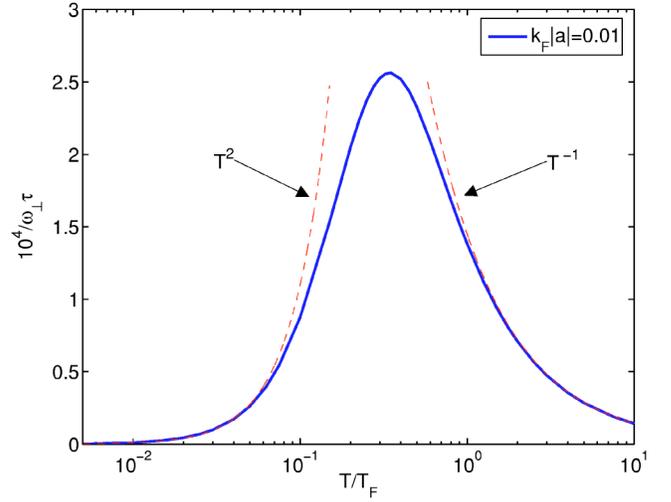


FIG. 2. (Color online) The average viscous relaxation rate  $1/\tau$  divided by the transverse trap frequency  $\omega_\perp$  as a function of temperature, for  $k_F|a|=0.01$ . The asymptotic temperature dependences are indicated by the dashed lines. Note that the system is highly collisionless, since the maximum value of  $1/\omega_\perp\tau$  is about 0.000 25.

$$\frac{1}{\tau} = \frac{\int d^3r(X, HX)}{\int d^3r(X, X)}. \quad (26)$$

Note that the spatial average of Eq. (18) is carried out here for the denominator and numerator separately. As demonstrated in the Appendix, this is the quantity that enters as an effective relaxation rate when we take moments of the kinetic equation in order to determine the frequency and attenuation of the collective modes. The calculation of the average viscous relaxation rate proceeds as in Ref. [15] [see, e.g., Eq. (38)], where the corresponding rate was obtained for bosons above the Bose-Einstein condensation temperature, the only modification being the change of sign in the equilibrium distribution function. The resulting five-dimensional integrations were carried out numerically, with varying step sizes until convergence was achieved.

The results shown in Fig. 2 and all following figures were obtained for a total number  $N$  of particles given by  $N=2.8\times 10^5$ , which represents a typical value for the experiments on  $^6\text{Li}$  reported in Refs. [5–7]. We use the trap frequencies for the cigar-shaped cloud of Ref. [6]—i.e., an axial frequency  $\omega_z=2\pi\times 70$  Hz and a transverse frequency  $\omega_\perp=2\pi\times 1550$  Hz, giving an anisotropy ratio equal to  $\lambda=\omega_z/\omega_\perp=0.045$ .

The resulting average viscous relaxation rate is shown in Figs. 2 and 3 for two different values of the parameter  $k_F|a|$ , one characterizing the regime of weak coupling and the other the regime near the unitarity limit, where  $k_F$  is the magnitude of the Fermi wave vector in the center of the trap. At low temperatures the relaxation rates are proportional to  $T^2$ , and they exhibit in both cases a pronounced maximum at a temperature somewhat below  $T_F$ . The asymptotic behavior at high temperatures differs in the two cases. When  $k_F|a|$  is

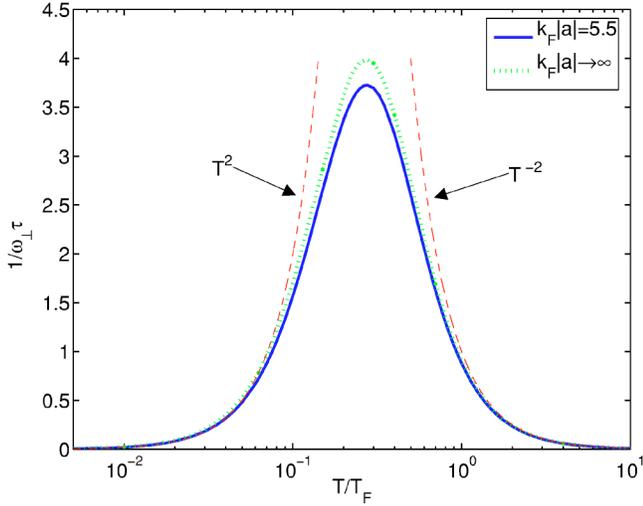


FIG. 3. (Color online) The average viscous relaxation rate  $1/\tau$  divided by the transverse trap frequency  $\omega_{\perp}$  as a function of temperature, for  $k_F|a|=5.5$ , corresponding to the experiment of Ref. [6] at a magnetic field of 870 G. The asymptotic temperature dependences are indicated by the dashed lines. The dotted line is the result obtained in the unitarity limit  $|a| \rightarrow \infty$ .

much less than unity, the average viscous relaxation rate decreases as  $1/T$  at high temperatures. This may seem at odds with the fact that for a uniform gas  $1/\tau_{\eta}$  according to Eq. (19) is proportional to  $T^{1/2}$  in this limit, since  $\bar{\sigma}$  is independent of temperature. However, the average density in a trapped gas is not a constant, but decreases at high temperatures in proportion to  $T^{-3/2}$ , resulting in an average relaxation rate proportional to  $T^{-1}$ . When  $k_F|a|$  is much greater than unity, the temperature-dependent cross section (22) causes the relaxation rate to decrease even more strongly, in proportion to  $T^{-2}$ . In the unitarity limit, when  $|a|$  approaches infinity, the average viscous relaxation rate approaches a limiting value indicated by the dotted curve in Fig. 3. This is further illustrated in Fig. 4 where we plot the average viscous relaxation rate as a function of  $1/k_F|a|$  for various temperatures.

We have normalized in Figs. 2–4 the viscous relaxation rate to the transverse trap frequency  $\omega_{\perp}$  used in the experiments [6,7]. The limiting value of the average viscous relaxation rate for  $|a| \rightarrow \infty$  is seen never to be large compared to  $\omega_{\perp}$ , which demonstrates that hydrodynamics cannot be applied to the transverse motion of the trapped atomic clouds in the normal phase. In the next section we determine the frequency and attenuation of the collective modes and obtain results in support of this general conclusion.

### III. FREQUENCY AND ATTENUATION OF COLLECTIVE MODES

We proceed now to extract the dispersion relation of the low-lying collective modes by solving the linearized Boltzmann-Vlasov equation (1) with an appropriate trial function  $\Phi$ . The equations of motion (2) are modified to take into account the mean-field potential  $U$  given by

$$U(\mathbf{r}) = gn(\mathbf{r}) = g[n^0(\mathbf{r}) + \delta n(\mathbf{r})], \quad (27)$$

where  $g = 4\pi\hbar^2 a/m$  is the interaction constant. The density  $n^0$  denotes the equilibrium density for a single spin—that is

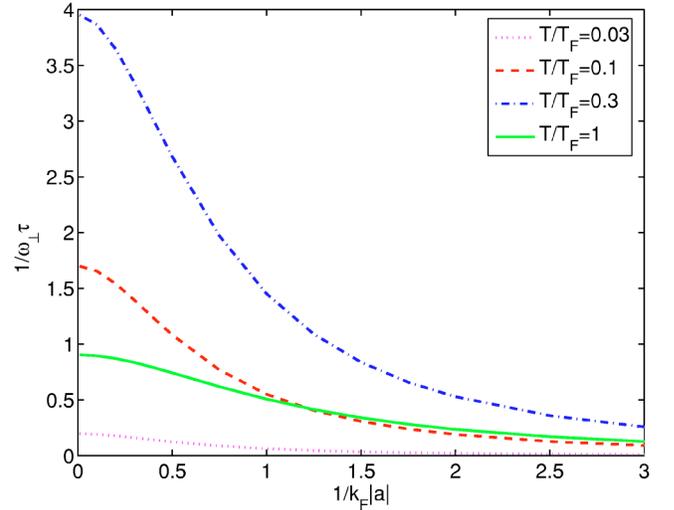


FIG. 4. (Color online) The average viscous relaxation rate  $1/\tau$  divided by the transverse trap frequency  $\omega_{\perp}$  as a function of  $1/k_F|a|$  for four different temperatures. The parameters  $T/T_F=0.03$  and  $T/T_F=0.1$  correspond to the experimental conditions of Refs. [5,7], respectively.

$n^0 = n_{\uparrow}^0 = n_{\downarrow}^0 = n_{\text{tot}}^0/2$ —and  $\delta n(\mathbf{r})$  similarly denotes the nonequilibrium change in density for a single spin. The effective potential is thus the sum of  $U$  and the harmonic oscillator potential  $V(\mathbf{r})$  given by Eq. (25), yielding the equations of motion

$$\dot{\mathbf{r}} = \mathbf{v} = \frac{\mathbf{p}}{m}, \quad \dot{\mathbf{p}} = -\frac{\partial(V+U)}{\partial\mathbf{r}}. \quad (28)$$

Let us first establish some useful relations between equilibrium quantities. We consider the kinetic equation in equilibrium, where the right-hand side of Eq. (1) vanishes, and insert the equations of motion (28), which results in

$$\sum_{i=x,y,z} \omega_i \left[ p_i \frac{\partial f^0}{\partial \tilde{r}_i} - \left( \tilde{r}_i + mg \frac{\partial n^0}{\partial \tilde{r}_i} \right) \frac{\partial f^0}{\partial p_i} \right] = 0 \quad (29)$$

or, equivalently,

$$\sum_{i=x,y,z} \omega_i \left[ p_i \frac{\partial [f^0(1-f^0)]}{\partial \tilde{r}_i} \right] = \sum_{i=x,y,z} \omega_i \left[ \left( \tilde{r}_i + mg \frac{\partial n^0}{\partial \tilde{r}_i} \right) \times \frac{\partial [f^0(1-f^0)]}{\partial p_i} \right]. \quad (30)$$

For convenience we have here introduced the variable  $\tilde{r}_i = m\omega_i r_i$  in terms of which the potential is  $V(\mathbf{r}) = \tilde{r}^2/2m$ . If we now multiply Eq. (30) by  $\tilde{x}p_x p_y^2$  and integrate over both position and momentum variables, we obtain

$$\langle p_x^2 p_y^2 \rangle - \langle \tilde{x}^2 p_y^2 \rangle + \frac{m^2}{2} kTg \int d^3r (n^0)^2 = 0, \quad (31)$$

where  $\langle \dots \rangle$  denotes multiplication by  $f^0(1-f^0)$  and integration over the whole phase space:

$$\langle \dots \rangle = \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} \dots f^0(1-f^0). \quad (32)$$

Using  $f^0(1-f^0) = -(mkT/p)\partial f^0/\partial p$ , we can calculate analytically the integrals appearing in Eq. (31) and obtain the virial theorem [16]

$$\frac{1}{3}E_{\text{kin}} - \frac{1}{3}E_{\text{pot}} + \frac{1}{2}E_{\text{int}} = 0. \quad (33)$$

Here

$$E_{\text{kin}} = 2 \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{2m} f^0 \quad (34)$$

is the kinetic energy and

$$E_{\text{pot}} = 2 \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} V(\mathbf{r}) f^0 \quad (35)$$

is the potential energy, while

$$E_{\text{int}} = g \int d^3r (n^0)^2 \quad (36)$$

is the interaction energy.

If instead we multiply the equilibrium equation (30) by the combination  $\tilde{x}^2 y p_y$  and integrate as before, we obtain the equality

$$\langle \tilde{x}^2 p_y^2 \rangle - \langle \tilde{x}^2 \tilde{y}^2 \rangle - mg \left\langle \tilde{x}^2 \tilde{y} \frac{\partial n^0}{\partial \tilde{y}} \right\rangle = 0, \quad (37)$$

which will be used in the Appendix to simplify the matrix that we diagonalize to obtain the frequencies of the collective modes.

To lowest order in the coupling constant  $g$ , the linearized version of Eq. (1) reads

$$\begin{aligned} \frac{\partial \Phi}{\partial t} + \sum_{i=x,y,z} \omega_i \left[ p_i \frac{\partial \Phi}{\partial \tilde{r}_i} - \left( \tilde{r}_i + mg \frac{\partial n^0}{\partial \tilde{r}_i} \right) \frac{\partial \Phi}{\partial p_i} \right. \\ \left. - \frac{mg}{f^0(1-f^0)} \frac{\partial \delta n}{\partial \tilde{r}_i} \frac{\partial f^0}{\partial p_i} \right] = - \frac{I[\Phi]}{f^0(1-f^0)}, \end{aligned} \quad (38)$$

where  $n^0$  as before denotes the equilibrium density for a single spin while the corresponding nonequilibrium change in the density is

$$\delta n = \int \frac{d^3p}{(2\pi\hbar)^3} f^0(1-f^0) \Phi. \quad (39)$$

We shall in the following consider modes for which the drift velocity  $\mathbf{u}$  has a spatial dependence given by  $u_i \propto r_i$ . The deviation function  $\Phi$  of a fluid moving with velocity  $\mathbf{u}$  is proportional to  $\mathbf{u} \cdot \mathbf{p}$ . Since acting on  $\mathbf{u} \cdot \mathbf{p}$  with the left-hand side of Eq. (1) generates terms like  $x^2$ ,  $p_x^2$ , etc., we follow Ref. [17] in choosing the trial function as

$$\Phi = e^{-i\omega t} \sum_{i=x,y,z} (a_i \tilde{r}_i^2 + b_i \tilde{r}_i p_i + c_i p_i^2). \quad (40)$$

We insert this ansatz into the kinetic equation (38) and calculate moments by multiplying with the product of

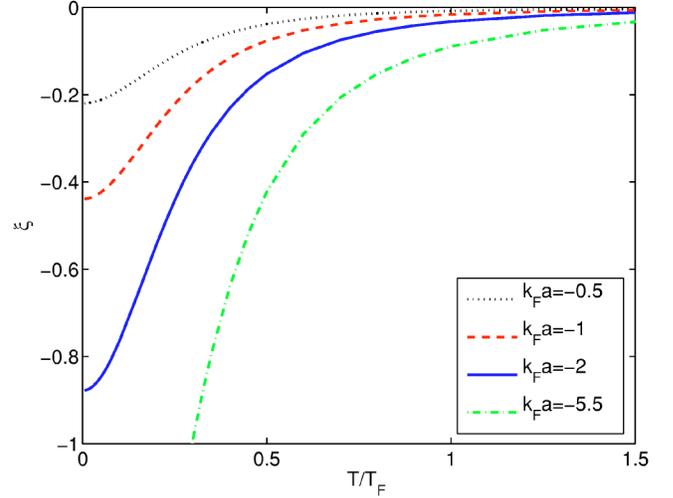


FIG. 5. (Color online) The temperature dependence of the parameter  $\xi$  given by Eq. (41) for different values of  $k_F a$ . At high temperatures  $|\xi|$  decreases as  $T^{-5/2}$ .

$f^0(1-f^0)$  and any of the terms  $\tilde{x}^2, \tilde{y}^2, \dots, p_z^2$  appearing in  $\Phi$ , and subsequently integrating over both  $\mathbf{r}$  and  $\mathbf{p}$ . The result is a homogeneous set of nine coupled equations for the nine coefficients  $a_x, a_y, \dots, c_z$  and the frequencies of the collective modes emerge as the roots of the determinant. The details of the calculation are given in the Appendix for the general case when all three trap frequencies are different.

#### IV. RESULTS AND COMPARISON WITH EXPERIMENT

In order to make contact with recent experiments [5–7] we consider an axially symmetric trap with  $\omega_x = \omega_y = \omega_\perp$  and  $\omega_z = \lambda \omega_\perp$ . We introduce the parameter

$$\xi = \frac{3E_{\text{int}}}{2E_{\text{pot}}}, \quad (41)$$

which, as we shall see, determines the sign and relative magnitude of the frequency shifts. We shall expand our results to first order in  $\xi$ , since our mean-field treatment of the interaction in the streaming terms of the kinetic equation is only valid when  $|\xi|$  is small compared to unity. The temperature dependence of  $\xi$  is shown in Fig. 5. In accordance with our first-order treatment of the mean field we calculate  $\xi$  by approximating the equilibrium Fermi function, which enters  $E_{\text{int}}$  as well as  $E_{\text{pot}}$ , by its value in the absence of interaction. At high temperatures one finds from Eq. (41) that  $|\xi| \propto T^{-5/2}$ , since the interaction energy  $E_{\text{int}}$  in the classical regime is inversely proportional to the volume of the cloud ( $E_{\text{int}} \propto T^{-3/2}$ ), while the potential energy is proportional to the temperature.

The determinant of the matrix, which is derived in the Appendix, has the form of a polynomial in the frequency  $\omega$ . The vanishing of the determinant yields the equation

$$0 = \omega[(\omega^2 - \omega_{\text{hd}}^2) - i\omega\tau(\omega^2 - \omega_{\text{cl}}^2)] \\ \times [(\omega^2 - \omega_{\text{hd}+}^2)(\omega^2 - \omega_{\text{hd}-}^2) - i\omega\tau(\omega^2 - \omega_{\text{cl}+}^2)(\omega^2 - \omega_{\text{cl}-}^2)], \quad (42)$$

where  $\tau$  is defined by Eq. (26). Note that the average viscous relaxation rate  $1/\tau$  depends on temperature as illustrated in Figs. 2–4.

In general, the solutions to Eq. (42) have a real and an imaginary part,  $\omega = \text{Re}(\omega) + i \text{Im}(\omega)$ , which determine the frequency and the damping of the collective modes, respectively. The (purely real) frequencies appropriate to the hydrodynamic limit,  $\omega\tau \rightarrow 0$ , are denoted by subscript (hd), while those for the collisionless limit,  $\omega\tau \rightarrow \infty$ , carry the subscript (cl). To first order in  $\xi$ , we find

$$\omega_{\text{hd}}^2 = 2\omega_{\perp}^2, \quad (43)$$

$$\omega_{\text{cl}}^2 = 4\omega_{\perp}^2 \left(1 - \frac{\xi}{2}\right), \quad (44)$$

$$\frac{\omega_{\text{hd}\pm}^2}{\omega_{\perp}^2} = \frac{5 + 4\lambda^2 \pm \gamma}{3} \pm \xi \frac{4\lambda^4 - \lambda^2(5 \mp \gamma) + 2(5 \pm \gamma)}{6\gamma}, \quad (45)$$

with  $\gamma = (25 - 32\lambda^2 + 16\lambda^4)^{1/2}$ . For  $\lambda \ll 1$  the latter are approximately given by

$$\omega_{\text{hd}+}^2 = \frac{10}{3}\omega_{\perp}^2 \left(1 + \frac{\xi}{5}\right) \quad (46)$$

and

$$\omega_{\text{hd}-}^2 = \frac{12}{5}\omega_{\perp}^2 \left(1 + \frac{\xi}{20}\right). \quad (47)$$

The modes labeled + and – are the transverse and axial modes, respectively, which are studied in the experiments [5–7].

In the collisionless limit we obtain, for elongated traps ( $\lambda \ll 1$ ),

$$\omega_{\text{cl}+}^2 = 4\omega_{\perp}^2, \quad (48)$$

$$\omega_{\text{cl}-}^2 = 4\omega_{\perp}^2 \left(1 - \frac{\xi}{4}\right), \quad (49)$$

while, for spherical traps ( $\lambda = 1$ ),

$$\omega_{\text{cl}+}^2 = 4\omega_{\perp}^2 \left(1 + \frac{\xi}{4}\right), \quad (50)$$

$$\omega_{\text{cl}-}^2 = 4\omega_{\perp}^2 \left(1 - \frac{\xi}{2}\right). \quad (51)$$

These results, valid to first order in  $\xi$ , are in agreement with those of *Pedri et al.* [12] when expanded to first order in  $\xi$ , but our results differ to second and higher order. This is understandable since the form of Eq. (40) is more general than the scaling ansatz used in Ref. [12], which involves six rather than nine parameters. However, since our calculation

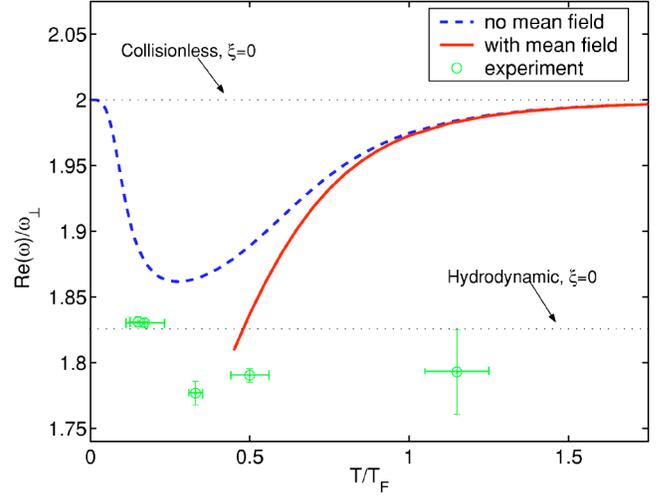


FIG. 6. (Color online) The calculated frequency of the transverse (+) mode as a function of temperature, with and without the mean-field correction for values of  $|\xi|$  less than or equal to 0.5. The experimental values from Ref. [6] are indicated for comparison with the mean-field-corrected curve, with the estimated error bars included.

of the frequency shifts caused by the interaction cannot be trusted beyond first order in  $\xi$ , our results are in essential agreement with those of Ref. [12]. Our work thus extends that of Ref. [12] in the sense that we determine  $\xi$  and  $\tau$  as functions of temperature, thereby allowing a direct comparison with experiment.

In Fig. 6 we plot the calculated frequency as a function of temperature for  $k_F|a| = 5.5$ , which corresponds to the parameters used in Ref. [6], along with their experimental values. Since we assume  $|\xi|$  to be small compared to unity, we show the mean-field curve only in the temperature region where  $|\xi|$  is less than 0.5. There is a clear discrepancy between our calculated frequency and the experimental result at  $T = 1.15T_F$  and further work is needed to understand the origin of this. From the point of view of theory it appears implausible that allowance for medium effects, leading to modification of the cross section (14), would resolve the discrepancy at this high temperature. The corresponding results for the damping, given by the imaginary part of the frequency, are shown in Fig. 7. The dashed curves in Figs. 6 and 7 illustrate the changeover between the collisionless regimes at high and low temperature, but should not be compared to experiment since they do not include the effects of the mean field. In order to compare with the experimental data below  $0.5T_F$  in Figs. 6 and 7, it is necessary to improve our treatment of the interaction effects in the streaming terms of the kinetic equation.

In Figs. 8 and 9 we show results for the real and imaginary parts of the frequency of the axial mode. Since  $\omega_z \ll \omega_{\perp}$ , there is a broad temperature region where the system behaves hydrodynamically. The damping shows a double-peak structure that reflects, as temperature is lowered, the transition between the different regimes, from collisionless to hydrodynamic and back to collisionless behavior [10]. The mean-field corrections in Fig. 8 are seen to be much

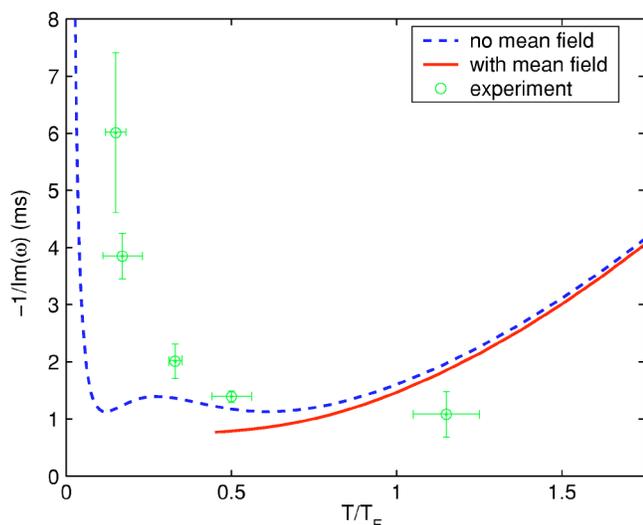


FIG. 7. (Color online) The inverse damping rate of the transverse (+) mode as a function of temperature, with and without the mean-field correction. The mean-field-corrected curve is plotted for values of  $|\xi|$  less than or equal to 0.5. The experimental values from Ref. [6] are indicated for comparison with the mean-field-corrected curve.

smaller than those of Fig. 6, in agreement with Eqs. (46) and (47).

## V. SUMMARY AND CONCLUSIONS

Starting from a kinetic equation for the semiclassical distribution function we have calculated the viscous relaxation rate which determines the frequency and attenuation of collective modes. By including interaction effects as a mean field in the streaming terms, we have solved the kinetic equation using a moment method which takes the conservation laws into account and provides an accurate account of the damping. Deep in the collisionless regime the rate of attenu-

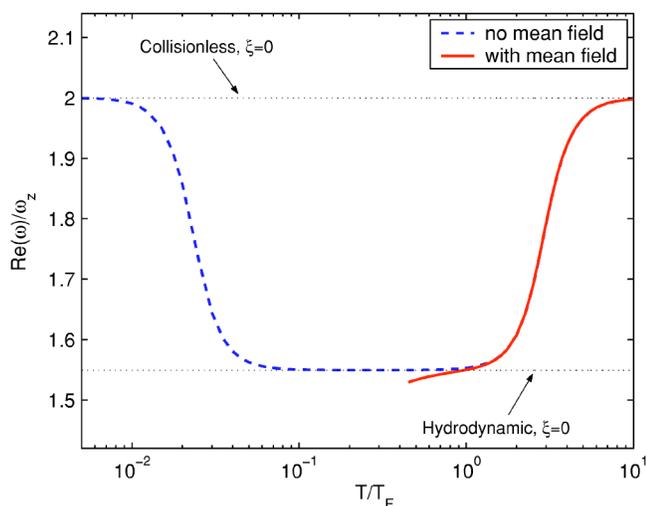


FIG. 8. (Color online) The frequency of the axial (-) mode as a function of temperature.

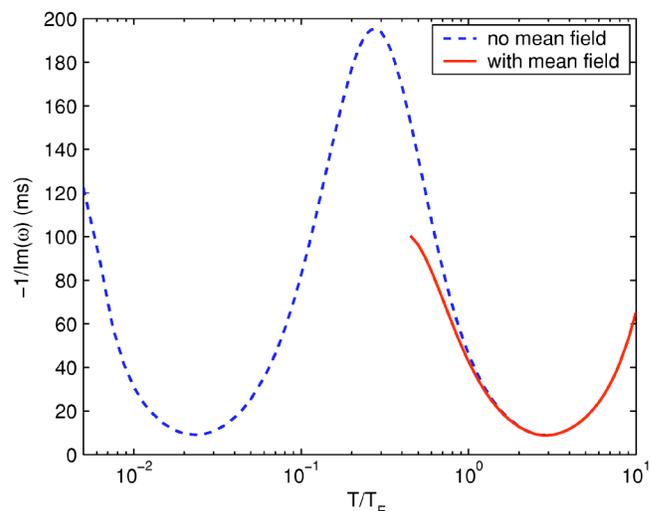


FIG. 9. (Color online) The inverse damping rate of the axial (-) mode as a function of temperature.

ation is proportional to the viscous relaxation rate, which is small compared to the oscillation frequency. If hydrodynamics is applicable, the viscous relaxation rate must be much larger than the oscillation frequency, and the rate of attenuation is then proportional to the oscillation frequency squared times the viscous relaxation time. Our treatment in the present work applies to both limits as well as to the intermediate regime and yields results that allow for a direct comparison between experiment and theory.

We have demonstrated that for a normal Fermi gas over most of the temperature range studied experimentally [5–7], hydrodynamic theory does not apply even at the unitarity limit for the transverse oscillations. It would be interesting to study further the behavior of the gas very near the unitarity limit, for both negative and positive  $a$ , by more extensive measurements of the oscillation frequencies at a fixed magnetic field as a function of temperature, thereby testing the predicted shifts in frequency and attenuation over a broad range of temperatures.

## ACKNOWLEDGMENTS

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## APPENDIX: MOMENTS OF THE BOLTZMANN-VLASOV EQUATION

In this appendix we provide details of the calculation of the  $9 \times 9$  matrix, from which the frequencies and damping rates can be extracted.

First we state some useful identities involving the momentum variables  $p_i$  and the rescaled position variables  $\tilde{r}_i = m\omega_i r_i$  ( $i = x, y, z$ ),

$$\langle p_x^2 p_x^2 \rangle = 3 \langle p_x^2 p_y^2 \rangle, \quad (\text{A1})$$

together with

$$\langle p_x^2 p_y^2 \rangle = \frac{1}{3} m^2 k T E_{\text{kin}} \quad (\text{A2})$$

and

$$\langle \tilde{x}^2 p_x^2 \rangle = \frac{1}{3} m^2 k T E_{\text{pot}}. \quad (\text{A3})$$

From these it follows that

$$\frac{\langle \tilde{x}^2 p_x^2 \rangle}{\langle p_x^2 p_y^2 \rangle} = \frac{E_{\text{pot}}}{E_{\text{kin}}} = \frac{E_{\text{pot}}}{E_{\text{pot}} - 3E_{\text{int}}/2} = \frac{1}{1 - \xi}. \quad (\text{A4})$$

Similar identities hold for the other components of  $\mathbf{p}$  and  $\tilde{\mathbf{r}}$ . They are valid both in the presence and absence of the mean field in the equilibrium Fermi function  $f^0$ .

The following identity holds when the mean field is neglected in the equilibrium Fermi function,

$$\begin{aligned} mg \langle \tilde{x}^2 n^0 \rangle &= mg \int \frac{d^3 p}{(2\pi\hbar)^3} \int d^3 r \tilde{x} n^0 \left( -m k T \frac{\partial f^0}{\partial \tilde{x}} \right) \\ &= m^2 k T g \int d^3 r \left[ (n^0)^2 + \tilde{x} \frac{\partial n^0}{\partial \tilde{x}} n^0 \right] = \frac{m^2}{2} k T E_{\text{int}}, \end{aligned} \quad (\text{A5})$$

and implies that

$$\frac{mg \langle \tilde{x}^2 n^0 \rangle}{\langle \tilde{x}^2 p_x^2 \rangle} \approx \xi. \quad (\text{A6})$$

In order to obtain the matrix determining the collective modes we insert Eq. (40) into Eq. (38) and start taking moments of Eq. (38) with  $\tilde{x}^2$ , resulting in

$$\begin{aligned} -i\omega[(3a_1 + a_2 + a_3)\langle \tilde{x}^2 \tilde{y}^2 \rangle + (c_1 + c_2 + c_3)\langle \tilde{x}^2 p_x^2 \rangle] \\ + \omega_x b_1 A_1 + \omega_y b_2 A_2 + \omega_z b_3 A_3 = 0. \end{aligned} \quad (\text{A7})$$

Using the identities given above and Eq. (37), we find

$$A_1 = \langle \tilde{x}^2 p_x^2 \rangle - \langle \tilde{x}^2 \tilde{x}^2 \rangle - mg \left\langle \tilde{x}^3 \frac{\partial n^0}{\partial \tilde{x}} \right\rangle = -2\langle \tilde{x}^2 p_x^2 \rangle,$$

$$A_2 = \langle \tilde{x}^2 p_x^2 \rangle - \langle \tilde{x}^2 \tilde{y}^2 \rangle - mg \left\langle \tilde{x}^2 \tilde{y} \frac{\partial n^0}{\partial \tilde{y}} \right\rangle = 0.$$

The constant  $A_3$  also vanishes for the same reason as  $A_2$ . Collisions do not appear in Eq. (A7) since the combinations  $\tilde{r}_i^2$  (as well as  $\tilde{r}_i p_i$ ) are collision invariants—i.e.,  $I[\tilde{r}_i^2] = 0$ . We now divide Eq. (A7) by  $\langle \tilde{x}^2 p_x^2 \rangle$ . Since the integral  $mg \langle \tilde{y}^2 \tilde{x} \partial n^0 / \partial \tilde{x} \rangle$  appearing in Eq. (37) only introduces frequency shifts of second order in  $\xi$ , we use here the approximation  $\langle \tilde{x}^2 \tilde{y}^2 \rangle / \langle \tilde{x}^2 p_x^2 \rangle \approx 1$ . This yields

$$-i\omega[3a_1 + a_2 + a_3 + c_1 + c_2 + c_3] - 2\omega_x b_1 = 0. \quad (\text{A8})$$

Next we proceed to take moments with  $\tilde{x} p_x$  and obtain

$$-i\omega b_1 \langle \tilde{x}^2 p_x^2 \rangle + \omega_x [2\langle \tilde{x}^2 p_x^2 \rangle (a_1 - c_1) + c_1 m^2 k T E_{\text{int}} + B] = 0, \quad (\text{A9})$$

where  $B$  is given by

$$\begin{aligned} B &= -mg \left\langle \frac{\delta n}{f^0(1-f^0)} \left( f^0 + \tilde{x} \frac{\partial f^0}{\partial \tilde{x}} \right) \right\rangle \\ &= -mg \langle \tilde{x}^2 n^0 \rangle \sum_{i=x,y,z} a_i - \frac{m^2}{2} k T E_{\text{int}} \sum_{i=x,y,z} c_i. \end{aligned}$$

Dividing Eq. (A9) by  $\langle \tilde{x}^2 p_x^2 \rangle$ , we obtain

$$-i\omega b_1 + \omega_x [(2 - \xi)(a_1 - c_1) - \xi(a_2 + c_2 + a_3 + c_3)] = 0. \quad (\text{A10})$$

Finally we take moments with  $p_x^2$ . Unlike  $\tilde{r}_i^2$  and  $\tilde{r}_i p_i$  the quantities  $p_i^2$  are not separately collision invariants. By exploiting the fact that the sum  $p_x^2 + p_y^2 + p_z^2 = p^2$  is indeed a collision invariant we arrive at the equation

$$\begin{aligned} -i\omega[(1 - \xi)^{-1}(a_1 + a_2 + a_3) + (3 + 4i/3\omega\tau)c_1 \\ + (1 - 2i/3\omega\tau)(c_2 + c_3)] + 2\omega_x b_1 = 0, \end{aligned} \quad (\text{A11})$$

where  $\tau$  is given by Eq. (26). Equations (A8), (A10), and (A11) form, respectively, the first, second, and third lines of our system of coupled equations. The other six equations may be obtained from the three given above by simple permutation of the indices.

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