Lamb shift in muonic hydrogen

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The Lamb shift in muonic hydrogen continues to be a subject of experimental and theoretical investigation. Here my older work on the subject is updated to provide a complementary calculation of the energies of the 2*p*-2*s* transitions in muonic hydrogen.

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I. INTRODUCTION

The energy levels of muonic atoms are very sensitive to effects of quantum electrodynamics (QED), nuclear structure, and recoil, since the muon is about 206 times heavier than the electron [1]. In view of a proposed measurement of the Lamb shift in muonic hydrogen [2], an improved theoretical analysis seems to be desirable. Since the first theoretical analysis [3], the subject of the Lamb shift (the 2*p*-2*s* transition) in light muonic atoms has been investigated with increasing precision by a number of authors [4–10]. The present paper provides an independent recalculation of some of the most important effects, including hyperfine structure, and a new calculation of some terms that were omitted in the most recent literature, such as the virtual Delbrück effect [11]. An alternative calculation of the relativistic recoil correction is presented.

In the numerical calculations the fundamental constants from the CODATA 1998 [12] are used: i.e., α^{-1}, $\hbar c$, m_μ, m_e, and m_u=137.035 999 8, 197.326 96 MeV fm, 105.658 357 MeV, 0.510 998 9 MeV, and 931.4940 MeV, respectively. The changes in these constants in the CODATA 2002 compared with CODATA 1998 are too small to make any relevant difference in the results.

II. VACUUM POLARIZATION

The most important QED effect for muonic atoms is the virtual production and annihilation of a single e^+e^- pair It has as a consequence an effective interaction of order $\alpha Z\alpha$ which is usually called the Uehling potential [13,14]. This interaction describes the most important modification of Coulomb's law. Numerically it is so important that it should not be treated using perturbation theory; instead the Uehling potential should be added to the nuclear electrostatic potential before solving the Dirac equation. However, a perturbative treatment is also useful in the case of very light atoms, such as hydrogen.

Unlike some other authors, we prefer to use relativistic (Dirac) wave functions to describe the muonic orbit. This is more exact, and as will be seen below, it makes a difference for at least the most important contributions. The wave functions are given in the book of Akhiezer and Berestetskii [15] and will not be given here. In perturbation theory, the energy shift due to an effective potential ΔV is given by

$$\Delta E_{n\kappa} = \frac{1}{2\pi^2}\int_0^\infty q^2 dq \Delta V(q) \int_0^\infty dr j_0(qr)[F_{n\kappa}^2 + G_{n\kappa}^2], \quad (1)$$

where $F_{n\kappa}$ and $G_{n\kappa}$ are the small and large components of the wave function, n is the principal quantum number, and κ is equal to $-(\ell+1)$ if $j=\ell+\frac{1}{2}$ and $+\ell$ if $j=\ell-\frac{1}{2}$. $\Delta V(q)$ is the Fourier transform of the physical potential:

$$\Delta V(q) = 4\pi\int_0^\infty r^2 j_0(qr)\Delta V(r) dr, \quad (2)$$

$$\Delta V(r) = \frac{1}{2\pi^2}\int_0^\infty q^2 j_0(qr)\Delta V(q) dq. \quad (3)$$

As is well known [1], the Uehling potential in momentum space is given by

$$V_{Uehl}(q) = -\frac{4\alpha(\alpha Z)}{3} G_E(q) F(\phi) = -4\pi(\alpha Z) G_E(q) U_2(q),$$

where G_E is the proton charge form factor, $\sinh(\phi)=q/(2m_e)$, and

$$F(\phi) = \frac{1}{3} + [\coth^2(\phi) - 3][1 + \phi\coth(\phi)], \quad (4)$$

$U_2(q)$ is defined in [1]. The vacuum polarization corrections were calculated in momentum space; formulas (124), (125), and (127) of [1] are completely equivalent to (200) in [10]. If the correction to the transition $2p_{1/2}$-$2s_{1/2}$ is calculated in lowest-order perturbation theory using nonrelativistic point Coulomb wave functions, the result is 205.0074 meV, in agreement with other authors [10].

The same procedure was used to calculate the two-loop corrections; the corresponding diagrams were first calculated by Källen and Sabry [16]. The Fourier transform of the corresponding potential is given in [1,4]. The result for a point nucleus is 1.5080 meV.

In momentum space including the effect of nuclear size on the Uehling potential is trivial, since the corresponding expression for $\Delta V(q)$ is simply multiplied by the form factor. The numbers obtained were the same for a dipole form factor and for a Gaussian form factor, provided the parameters were adjusted to reproduce the experimental rms radius of the proton. The correction can be regarded as taking into account the effect of finite nuclear size on the virtual electron-positron pair in the loop. The contribution of the Uehling potential to the $2p$-$2s$ transition is reduced by 0.0081 meV with a proton radius of 0.862 fm [17] and by 0.0085 meV with a proton radius of 0.880 fm [18]. This result is consistent with the number given in [10] [Eq. (266)]. More recent values for the proton radius have been given by Sick [19] (0.895 ± 0.018 fm) and in the newest CODATA compilation [20] (0.875 ± 0.007 fm).

The numerical values given below were calculated as the expectation value of the Uehling potential using point Coulomb-Dirac wave functions with reduced mass.

	Point nucleus		$R_p = 0.875$ fm	
	$2p_{1/2}$-$2s_{1/2}$	$2p_{3/2}$-$2s_{1/2}$	$2p_{1/2}$-$2s_{1/2}$	$2p_{3/2}$-$2s_{1/2}$
Uehling	205.0282	205.0332	205.0199	205.0250
Kaellen-Sabry	1.50814	1.50818	1.50807	1.50811

The effect of finite proton size calculated here can be parametrized as $-0.0109\langle r^2\rangle$. However, higher iterations can change these results. For a very crude estimate, one can scale previous results for helium [5] and assume that the ratio of nonperturbative to perturbative contributions was the same, giving a contribution of 0.175 meV.

The contributions due to two and three iterations have been calculated by [8,23], respectively, giving a total of 0.151 meV. An additional higher iteration including finite size and vacuum polarization is given in Ref. [8] [Eqs. (66) and (67)] and Ref. [10] [Eqs. (264) and (268)]. These amount to $-0.0164\langle r^2\rangle$. The best way to calculate this would be an accurate numerical solution of the Dirac equation in the combined Coulomb plus Uehling potential.

The mixed muon-electron vacuum polarization correction was recalculated and gave the same result as obtained previously: namely, 0.000 07 meV [10,21].

The Wichmann-Kroll [22] contribution was calculated using the parametrization for the potential given in [1]. The result obtained (−0.001 03 meV) is consistent with that given in [10], but not with that given in [8].

The equivalent potential for the virtual Delbrück effect was recomputed from the Fourier transform given in [11,1]. The resulting potential was checked by reproducing previously calculated results for the $2s$-$2p$ transition in muonic helium and the $3d$-$2p$ transitions in muonic Mg and Si. The result for hydrogen is $+(0.001\,35 \pm 0.000\,15)$ meV. As in the case of muonic helium, this contribution very nearly cancels the Wichmann-Kroll contribution. The contribution corresponding to three photons to the muon and one to the proton should be analogous to the light-by-light contribution to the muon anomalous moment; to my knowledge, the corresponding contribution to the muon form factor has never been calculated. It will be comparable to the other light-by-light contributions. For an estimate, the correction to the Lamb shift due to the contribution to the anomalous magnetic moment was calculated; it amounts to (−)0.000 02 meV; the contribution to the muon form factor is one of the most significant unknown corrections.

The sixth-order vacuum polarization corrections to the Lamb shift in muonic hydrogen have been calculated by Kinoshita and Nio [23]. Their result for the $2p$-$2s$ transition is

$$\Delta E^{(6)} = 0.120045(\alpha Z)^2 m_r \left(\frac{\alpha}{\pi}\right)^3 \approx 0.00761 \text{ meV}.$$

It is entirely possible that the as-yet uncalculated light-by-light contribution will give a comparable contribution.

The hadronic vacuum polarization contribution has been estimated by a number of authors [10,24,25]. It amounts to about 0.012 meV. One point that should not be forgotten about the hadronic VP correction is the fact that the sum rule or dispersion relation that everyone (including myself) used does not take into account the fact that the proton (nucleus) can in principle interact strongly with the hadrons in the virtual hadron loop. This is irrelevant for the anomalous magnetic moment but probably not for muonic atoms. An estimation of this effect appears to be extremely difficult and could easily change the correction by up to 50%. Eides *et al.* [10] point out that the graph related to hadronic vacuum polarization can also contriibute to the measured value of the nuclear charge distribution (and polarizability). It is not easy to determine where the contribution should be assigned.

III. FINITE NUCLEAR SIZE AND NUCLEAR POLARIZATION

The main contribution due to finite nuclear size has been given analytically to order $(\alpha Z)^6$ by Friar [26]. The main result is

$$\Delta E_{ns} = -\frac{2\alpha Z}{3}\left(\frac{\alpha Z m_r}{n}\right)^3 \bigg[\langle r^2\rangle - \frac{\alpha Z m_r}{2}\langle r^3\rangle_{(2)} + (\alpha Z)^2 (F_{REL} + m_r^2 F_{NR})\bigg], \quad (5)$$

where $\langle r^2\rangle$ is the mean-square radius of the proton. For muonic hydrogen, the coefficient of $\langle r^2\rangle$ is 5.1975 (meV fm^{-2}), giving an energy shift (for the leading term) of 3.862 ± 0.108 meV if the proton rms radius is 0.862 ± 0.012 fm. The shift is 4.163 ± 0.188 meV if the proton rms radius is 0.895 ± 0.018 fm, and 3.979 ± 0.076 meV if the proton rms radius of 0.875 ± 0.007 fm.

The second term in Eq. (5) contributes −0.0232 meV for a dipole form factor and −0.0212 meV for a Gaussian form factor. The parameters were fitted to the proton rms radius. This can be written as $-0.0347\langle r^2\rangle^{3/2}$ or $0.0317\langle r^2\rangle^{3/2}$, respectively. This differs slightly from the value given by Pachucki [9]. The model dependence introduces an uncertainty of about ±0.002 meV. The remaining terms contribute 0.00046 meV. This estimate includes all of the terms given in [26], while other authors [9] give only some of them. Clearly the neglected terms are not negligible. There is also a contribution of -3×10^{-6} meV to the binding energy of the $2p_{1/2}$ level and a recoil correction of 0.012 meV to the binding energy of the $2s$ level.

As mentioned previously, the finite-size contributions to vacuum polarization can be parametrized as

$-0.0109\langle r^2\rangle - 0.0164\langle r^2\rangle$, giving a total of $-0.0273\langle r^2\rangle$ or $-0.0209(6)$ meV if the proton radius is 0.875 fm.

The contribution due to nuclear polarization has been calculated by Rosenfelder [27] to be 0.017 ± 0.004 meV and by Pachuki [9] to be 0.012 ± 0.002 meV. Other calculations [28,29] give intermediate values (0.013 meV and 0.016 meV, respectively). The value appearing in Table I is an average of the three most recent values, with the largest quoted uncertainty, which is probably underestimated.

IV. RELATIVISTIC RECOIL

As is well known, the center-of-mass motion can be separated exactly from the relative motion only in the nonrelativistic limit. Relativistic corrections have been studied by many authors and will not be reviewed here. The relativistic recoil corrections summarized in [1] include the effect of finite nuclear size to leading order in m_μ/m_N properly.

Up to now this method has been used to treat recoil corrections to vacuum polarization only in the context of extensive numerical calculations that include the Uehling potential in the complete potential, as described in [1]. They can be included explicitly, as a perturbation correction to point Coulomb values. Recall that (to leading order in $1/m_N$), the energy levels are given by

$$E = E_r - \frac{B_0^2}{2m_N} + \frac{1}{2m_N}\langle h(r) + 2B_0 P_1(r)\rangle, \tag{6}$$

where E_r is the energy level calculated using the reduced mass and B_0 is the unperturbed binding energy. Also

$$h(r) = -P_1(r)\left(P_1(r) + \frac{1}{r}Q_2(r)\right) - \frac{1}{3r}Q_2(r)[P_1(r) + Q_4(r)/r^3]. \tag{7}$$

Here

$$P_1(r) = 4\pi\alpha Z\int_r^\infty r' \rho(r')dr' = -V(r) - rV'(r),$$

$$Q_2(r) = 4\pi\alpha Z\int_0^r r'^2\rho(r')dr' = r^2V'(r),$$

$$Q_4(r) = 4\pi\alpha Z\int_0^r r'^4\rho(r')dr'. \tag{8}$$

An effective charge density ρ_{VP} for vacuum polarization can be derived from the Fourier transform of the Uehling potential. Recall that (for a point nucleus)

$$V_{Uehl}(r) = -\frac{\alpha Z}{r}\frac{2\alpha}{3\pi}\chi_1(2m_e r) = -(\alpha Z)\frac{2\alpha}{3\pi}\int_1^\infty dz\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\left(\frac{2}{\pi}\int_0^\infty \frac{q^2 j_0(qr)}{q^2+4m_e^2z^2}dq\right),$$

where $\chi_n(x)$ is defined in [1]. In momentum space, the Fourier transform of $\nabla^2 V$ is obtained by multiplying the Fourier transform of V by $-q^2$. Note that using the normalizations of [1,6], one has $\nabla^2 V = -4\pi\alpha Z\rho$ where ρ is the charge density. One then obtains

$$4\pi\rho_{VP}(r) = \frac{2\alpha}{3\pi}\int_1^\infty dz\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\times\left(\frac{2}{\pi}\int_0^\infty\frac{q^4 j_0(qr)}{q^2+4m_e^2z^2}dq\right) = \frac{2}{\pi}\int_0^\infty q^2 U_2(q) j_0(qr)dq, \tag{9}$$

where $U_2(q)$ is defined in [1]. It is also easy to show that

$$\frac{dV_{Uehl}}{dr} = +\frac{\alpha Z}{r}\frac{2\alpha}{3\pi}\left[\frac{1}{r}\chi_1(2m_e r) + 2m_e\chi_0(2m_e r)\right] = -\frac{1}{r}V_{Uehl}(r) + (\alpha Z)\frac{2\alpha}{3\pi}\frac{2m_e}{r}\chi_0(2m_e r).$$

Keeping only the Coulomb and Uehling potentials, one finds

$$P_1(r) = -\alpha Z\frac{2\alpha}{3\pi}(2m_e)\chi_0(2m_e r),$$

$$Q_2(r) = \alpha Z\left(1 + \frac{2\alpha}{3\pi}[\chi_1(2m_e r) + (2m_e r)\chi_0(2m_e r)]\right),$$

$$Q_4(r) = \alpha Z\frac{2\alpha}{3\pi}\int_1^\infty dz\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\left(\frac{2}{\pi}\right)\int_0^\infty\frac{1}{q^2+4m_e^2z^2}\times\frac{[6qr-(qr)^3]\cos(qr)+[3(qr)^2-6]\sin(qr)}{q}dq,$$

where $\chi_n(x)$ is defined in [1]. Corrections due to finite nuclear size can be included when a model for the charge distribution is given. This done by Friar [26] (and confirmed independently for two different model charge distributions); the contribution due to finite nuclear size to the recoil correction for the binding energy of the $2s$ level is -0.013 meV. The factor $1/m_n$ is replaced by $1/(m_\mu + m_N)$, also consistent with the calculations presented in [26].

Since vacuum polarization is assumed to be a relatively small correction to the Coulomb potential, it will be sufficient to approximate $Q_2(r)$ by $\alpha Z/r$. After some algebra, one can reduce the expectation values to single integrals:

$$\langle P_1(r)\rangle = 2m_e\alpha Z\frac{2\alpha}{3\pi}\int_1^\infty\frac{(z^2-1)^{1/2}}{z}\left(1+\frac{1}{2z^2}\right)\times\left(\frac{(az)^2-az+1}{(1+az)^5}\delta_{\ell 0} + \frac{1}{(1+az)^5}\delta_{\ell 1}\right)dz, \tag{10}$$

$$\left\langle \frac{\alpha Z}{r} P_1(r) \right\rangle = -(\alpha Z)^3 m_r m_e \frac{2\alpha}{3\pi} \int_1^\infty \frac{(z^2-1)^{1/2}}{z} \left(1 + \frac{1}{2z^2}\right) \times \left(\frac{2(az)^2+1}{2(1+az)^4} \delta_{\ell 0} + \frac{1}{2(1+az)^4} \delta_{\ell 1} \right) dz, \quad (11)$$

with $a=2m_e/(\alpha Z m_r)$. When Eq. (10) is multiplied by $-2B_0/(m_\mu+m_N)$ this results in a shift of $-0.000\,15$ meV for the $2s$ state and of $-0.000\,01$ meV for the $2p$ state, and when Eq. (11) is multiplied by $1/(m_\mu+m_N)$ this results in a shift of 0.004 89 meV for the $2s$ state and of 0.000 17 meV for the $2p$ state. These expectation values also appear when vacuum polarization is included in the Breit equation [31].

Finally,

$$\left\langle \frac{\alpha Z}{3r^4} Q_4(r) \right\rangle = -\frac{(\alpha Z)^4 m_r^2}{6} \frac{2\alpha}{3\pi} \int_1^\infty \frac{(z^2-1)^{1/2}}{z^2} \left(1 + \frac{1}{2z^2}\right) \times \left\{ \left[-\frac{6}{az}\left(\frac{2+az}{1+az} - \frac{2}{az}\ln(1+az) \right) + \frac{3(az)^2+2az-1}{(1+az)^3} + \frac{3+az}{4(1+az)^4} \right] \delta_{\ell 0} + \frac{1-3az-2(az)^2}{4(1+az)^4} \delta_{\ell 1} \right\} dz. \quad (12)$$

When multiplied by $1/(m_\mu+m_N)$ this results in a shift of 0.002 475 meV for the $2s$ state and of 0.000 238 meV for the $2p$-state.

Combining these expectation values according to Eqs. (6) and (7), one finds a contribution to the $2p$-$2s$ transition of $-0.004\,19$ meV. To obtain the full relativistic and recoil corrections, one must add the difference between the expectation values of the Uehling potential calculated with relativistic and nonrelativistic wave functions, giving a total correction of 0.0166 meV. This is in fairly good agreement with the correction of 0.0169 meV calculated by Veitia and Pachucki [31], using a generalization of the Breit equation [32] which is similar to that given in [6]. The treatment presented here has the advantage of avoiding second order perturbation theory.

The review by Eides *et al.* [10] gives a better version of the two-photon recoil [Eq. (136)] than was available for the review by Borie and Rinker [1]. Evaluating this expression for muonic hydrogen gives a contribution of $-0.044\,97$ meV to the $2p$-$2s$ transition. Higher-order radiative recoil corrections give an additional contribution of $-0.00\,96$ meV [10]. However, some of the contributions to the expressions given in [10] involve logarithms of the mass ratio m_μ/m_N. Logarithms can only arise in integrations in the region from m_μ to m_N; in this region, the effect of the nuclear form factor should not be neglected. Pachucki [8] has estimated a finite-size correction to this of about 0.02 meV, which seems to be similar to the term proportional to $\langle r^3\rangle_{(2)}$ given in Eq. (5) as calculated in the external field approximation by Friar [26]. This two-photon correction requires further investigation. In particular, the parametrization of the form factors used in any calculation should reproduce the correct proton radius.

An additional recoil correction for states with $\ell \neq 0$ has been given by [32] (see also [10]). It is

$$\Delta E_{n,\ell,j} = \frac{(\alpha Z)^4 m_r^3}{2n^3 m_N^2}(1-\delta_{\ell 0})\left(\frac{1}{\kappa(2\ell+1)} \right). \quad (13)$$

When evaluated for the $2p$ states of muonic hydrogen, one finds a contribution to the $2p$-$2s$ transition energy of 0.0575 meV for the $2p_{1/2}$ state and -0.0287 meV for the $2p_{3/2}$ state.

V. MUON LAMB SHIFT

For the calculation of muon self-energy and vacuum polarization, the lowest-order (one-loop approximation) contribution is well known, at least in perturbation theory. Including also muon vacuum polarization (0.0168 meV) and an extra term of order $(Z\alpha)^5$ as given in [10],

$$\Delta E_{2s} = \frac{\alpha(\alpha Z)^5 m_\mu}{4}\left(\frac{m_r}{m_\mu}\right)^3\left(\frac{139}{64} + \frac{5}{96} - \ln(2) \right),$$

which contributes $-0.004\,43$ meV, one finds a contribution of $-0.667\,88$ meV for the $2s_{1/2}$-$2p_{1/2}$ transition and $-0.650\,31$ meV for the $2s_{1/2}$-$2p_{3/2}$ transition.

A misprint in the evaluation of the contribution of the higher-order muon form factors (contributing to the fourth-order terms) has been corrected. The extra electron loop contribution to $F_2(0)$ is should be $1.094\,26(\alpha/\pi)^2$. This reproduces the correct coefficient of $(\alpha/\pi)^2$ from the muon (g-2) analyses. This is 0.7658, which is equal to 1.094 26–0.328 48.

The fourth-order electron loops [30] dominate the fourth-order contribution ($-0.001\,69$ meV and $-0.001\,64$ meV, respectively). The rest is the same as for the electron [1]. The contribution of the electron loops alone is $-0.001\,68$ meV for the $2s_{1/2}$-$2p_{1/2}$ transition and $-0.001\,59$ meV for the $2s_{1/2}$-$2p_{3/2}$ transition.

Pachuki [8] has estimated an additional contribution of -0.005 meV for a contribution corresponding to a vacuum polarization insert in the external photon.

VI. SUMMARY OF CONTRIBUTIONS

Using the fundamental constants from the CODATA 1998 [12] one finds the transition energies in meV in Table I. Here the main vacuum polarization contributions are given for a point nucleus, using the Dirac equation with reduced mass. Some uncertainties have been increased from the values given by the authors, as discussed in the text.

In the case of the muon Lamb shift, the numbers in Table II are for the $2s_{1/2}$-$2p_{1/2}$ transition. The corresponding numbers for the $2s_{1/2}$-$2p_{3/2}$ transition are $-0.650\,31$ meV and $-0.001\,64$ meV, respectively.

A. Fine structure of the *2p* state

There are two possible ways to calculate the fine structure. One is to start with the point Dirac value, include the contribution due to vacuum polarization, as calculated above, as well as the spin-orbit splitting (computed perturbatively)

TABLE I. Contributions to the muonic hydrogen Lamb shift. The proton radius is taken from [20]. The various contributions are discussed in the text.

Contribution	Value (meV)	Uncertainty (meV)
Uehling	205.0282	
Källen-Sabry	1.5081	
Wichmann-Kroll	−0.00103	
Virt. Delbrueck	0.00135	0.00015
Mixed mu-*e* VP	0.00007	
Hadronic VP	0.011	0.002
Sixth order [23]	0.00761	
Recoil [10] [Eq. (136)]	−0.04497	
Recoil, higher order [10]	−0.0096	
Recoil, finite size [26]	0.013	0.001
Recoil correction to VP [1]	−0.0041	
Additional recoil [32]	0.0575	
Muon Lamb shift		
Second order	−0.66788	
Fourth order	−0.00169	
Nuclear size (R_p=0.875 fm)		0.007 fm
Main correction [26]	−3.979	0.076
Order $(\alpha Z)^5$	0.0232	0.002
Order $(\alpha Z)^6$	−0.0005	
Correction to VP	−0.0083	
Polarization [9]	0.015	0.004
Other (not checked)		
VP iterations [8]	0.151	
VP insertion in self energy [8]	−0.005	
Additional size for VP [10]	−0.0128	

due to the muon's anomalous magnetic moment, and recoil as given by Eq. (13). The results are summarized in Table II.

An alternative method is to use the formalism given in [6] (and elsewhere see, e.g. [10,32]) which gives the energy shift as the expectation value of

$$-\frac{1}{r}\frac{dV}{dr}\cdot\frac{1+a_\mu+(a_\mu+1/2)m_N/m_\mu}{m_N m_\mu}\vec{L}\cdot\vec{\sigma}_\mu. \tag{14}$$

Note that

$$\frac{1}{m_N m_\mu}+\frac{1}{2m_\mu^2}=\frac{1}{2m_r^2}-\frac{1}{2m_N^2},$$

so that the terms not involving a_μ in the spin-orbit contribution are really the Dirac fine structure plus the Barker-Glover correction [Eq. (13)].

The Uehling potential has to be included in the potential $V(r)$. For states with $\ell > 0$ in light atoms and neglecting the effect of finite nuclear size, we may take

TABLE II. Contributions to the fine structure of the $2p$ state in muonic hydrogen.

	$E(2p_{3/2})$-$E(2p_{1/2})$ (meV)
Dirac	8.41564
Uehling (VP)	0.0050
Källen-Sabry	0.00004
Anomalous moment a_μ	
Second order	0.01757
Higher orders	0.00007
Recoil [Eq. (13)]	−0.0862
Total fine structure	8.352

$$\frac{1}{r}\frac{dV}{dr}=\frac{\alpha Z}{r^3}\left[1+\frac{2\alpha}{3\pi}\int_1^\infty\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\times(1+2m_e rz)e^{-2m_e rz}dz\right], \tag{15}$$

which is obtained from the Uehling potential [13,14] by differentiation. Then, assuming that it is sufficient to use nonrelativistic point Coulomb wave functions for the $2p$ state, one finds

$$\left\langle\frac{1}{r^3}\right\rangle_{2p}\to\left\langle\frac{1}{r^3}\right\rangle_{2p}(1+\varepsilon_{2p}),$$

where

$$\varepsilon_{2p}=\frac{2\alpha}{3\pi}\int_1^\infty\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\left(\frac{1}{(1+az)^2}+\frac{2az}{(1+az)^3}\right)dz, \tag{16}$$

with $a=2m_e/(\alpha Z m_r)$. The result for the fine structure is

$$\frac{-(\alpha Z)^4 m_r^3}{n^3(2\ell+1)\kappa}\left(\frac{1}{m_N m_\mu}+\frac{1}{2m_\mu^2}+\frac{a_\mu}{m_\mu m_r}\right)(1+\varepsilon_{2p}), \tag{17}$$

where ε_{2p} is given by Eq. (16). In this case, the terms involving a_μ in the expression for the muon Lamb shift are included and should not be double counted. With a numerical value of $\varepsilon_{2p}=0.000\,365$, one finds a contribution of 0.003 05 meV (compared with 0.005 meV using Dirac wave functions).

Numerically, the terms not involving a_μ give a contribution of 8.3291 meV and the contribution from a_μ gives a contribution of 0.0176 meV, for a total of 8.3467 meV, in good agreement with Eq. (80) of [8]. When the vacuum polarization correction is added, the result is only very slightly different from the Dirac value of 8.352 meV. The contribution due to the anomalous magnetic moment of the muon is the same in both cases.

In both cases one should include the $B^2/2M_N$-type correction to the fine structure [see [10], Eq. (38)]. This is tiny (5.7×10^{-6} meV) and is not included in the table. Friar [26] has given expressions for the energy shifts of the $2p$ states due to finite nuclear size. These were calculated and found to give a negligible contribution (3.1×10^{-6}meV) to the fine structure of the $2p$ state.

VII. HYPERFINE STRUCTURE

The hyperfine structure (HFS) is calculated in the same way as was done in earlier work [6,7], but with improved accuracy. Most of the formalism and results are similar to those given by [8].

A. 2*p* state

The hyperfine structure of the $2p$ state is given by [6] (F is the total angular momentum of the state)

$$\frac{1}{4m_\mu m_N}\left\langle \frac{1}{r}\frac{dV}{dr}\right\rangle_{2p}(1+\kappa_p)\left[2(1+x)\delta_{jj'}(F(F+1)-11/4) + 6\hat{j}\hat{j}'(C_{F1}(1+a_\mu)-2(1+x))\begin{Bmatrix}\ell & F & 1\\ \frac{1}{2} & \frac{1}{2} & j\end{Bmatrix}\begin{Bmatrix}\ell & F & 1\\ \frac{1}{2} & \frac{1}{2} & j'\end{Bmatrix}\right], \tag{18}$$

where $\hat{j}=\sqrt{2j+1}$, the $6j$ symbols are defined in [33], $C_{F1}=\delta_{F1}-2\delta_{F0}-(1/5)\delta_{F2}$, and

$$x=\frac{m_\mu(1+2\kappa_p)}{2m_N(1+\kappa_p)}$$

represents a recoil correction due to Thomas precession [6,32]. The same correction due to vacuum polarization [Eq. (16)] should be applied to the HFS shifts of the $2p$ states, as well as to the spin-orbit term.

As has been known for a long time [6–8], the states with total angular momentum $F=1$ are a superposition of the states with $j=1/2$ and $j=3/2$. Let the fine structure splitting be denoted by $\delta=E_{2p3/2}-E_{2p1/2}$, and let

$$\beta=\frac{(\alpha Z)^4 m_r^3}{3m_\mu m_N}(1+\kappa_p)$$

and $\beta'=\beta(1+\varepsilon_{2p})$.

The energy shifts of the $2p$ states with total angular momentum F (notation ${}^{2F+1}L_j$) are then given in Table III where

$$\Delta=\delta-\beta'(x-a_\mu)/16,$$

$$R^2=[\delta-\beta'(1+7x/8+a_\mu/8)/6]^2+(\beta')^2(1+2x-a_\mu)^2/288$$

(here $\delta=8.352$ meV). Some minor errors in [6] have been corrected. These numbers differ slightly from those given in Ref. [10].

B. 2*s* state

The basic hyperfine splitting of the $2s$ state is given by

TABLE III. Hyperfine structure of the $2p$ state in muonic hydrogen.

State	Energy	Energy in meV
${}^1p_{1/2}$	$-\beta'(2+x+a_\mu)/8$	−5.971
${}^3p_{1/2}$	$(\Delta-R)/2$	1.846
${}^3p_{3/2}$	$(\Delta+R)/2$	6.376
${}^5p_{3/2}$	$\delta+\beta'(1+5x/4-a_\mu/4)/20$	9.624

$$\Delta\nu_F=\frac{(\alpha Z)^4 m_r^3}{3m_\mu m_N}(1+\kappa_p)(1+a_\mu)=\beta(1+a_\mu)=22.8332\text{ meV}$$

[see, for example [10], Eqs. (271) and (277)]. As was shown in [6,10], the energy shift of the $2s$ state is given by

$$\Delta E_{2s}=\Delta\nu_F(1+\varepsilon_{VP}+\varepsilon_{vertex}+\varepsilon_{Breit}+\varepsilon_{FS,rec})[\delta_{F1}-3\delta_{F0}]/4. \tag{19}$$

Here [34]

$$\varepsilon_{vertex}=\frac{2\alpha(\alpha Z)}{3}\left(\ln(2)-\frac{13}{4}\right)=-1.36\times 10^{-4}$$

and [[10], Eq. (277)]

$$\varepsilon_{Breit}=\frac{17(\alpha Z)^2}{8}=1.13\times 10^{-4}.$$

The vacuum polarization correction has two contributions. One of these is a result of a modification of the magnetic interaction between the muon and the nucleus and is given by (see [7])

$$\varepsilon_{VP1}=\frac{4\alpha}{3\pi^2}\int_0^\infty r^2dr\left(\frac{R_{ns}(r)}{R_{ns}(0)}\right)^2\int_0^\infty q^4 j_0(qr)G_M(q)dq \times\int_1^\infty\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\frac{dz}{4m_e^2[z^2+(q/2m_e)^2]}. \tag{20}$$

One can do two of the integrals analytically and obtains for $2s$ state [with $a=2m_e/(\alpha Z m_r)$ and $\sinh(\phi)=q/(2m_e)=K/a$]

$$\varepsilon_{VP1}=\frac{4\alpha}{3\pi^2}\int_0^\infty\frac{K^2}{(1+K^2)^2}F(\phi)G_M(\alpha Z m_r K)dK \times\left[2-\frac{7}{(1+K^2)}+\frac{6}{(1+K^2)^2}\right], \tag{21}$$

where $F(\phi)$ is known from the Fourier transform of the Uehling potential and is given by Eq. (4).

The other contribution, as discussed by [34,35], arises from the fact that the lower-energy hyperfine state, being more tightly bound, has a higher probability of being in a region where the vacuum polarization is large. This results in an additional energy shift of

$$2\int V_{Uehl}(r)\psi_{2s}(r)\delta_M\psi_{2s}(r)d^3r.$$

Following Ref. [34] with $y=(\alpha Zm_r/2)r$, one has

$$\delta_M\psi_{2s}(r) = 2m_\mu\Delta\nu_F\psi_{2s}(0)\left(\frac{2}{\alpha Zm_r}\right)^2\exp(-y)\times\left[(1-y)[\ln(2y)+\gamma]+\frac{13y-3-2y^2}{4}-\frac{1}{4y}\right]$$

(γ is Euler's constant), and

$$\psi_{2s}(r) = \psi_{2s}(0)(1-y)\exp(-y).$$

One finds after a lengthy integration

$$\varepsilon_{VP2} = \frac{16\alpha}{3\pi^2}\int_0^\infty\frac{dK}{1+K^2}G_E(\alpha Zm_rK)F(\phi)\times\left\{\frac{1}{2}-\frac{17}{(1+K^2)^2}+\frac{41}{(1+K^2)^3}-\frac{24}{(1+K^2)^4}+\frac{\ln(1+K^2)}{1+K^2}\left[2-\frac{7}{(1+K^2)}+\frac{6}{(1+K^2)^2}\right]+\frac{\tan^{-1}(K)}{K}\left[1-\frac{19}{2(1+K^2)}+\frac{20}{(1+K^2)^2}-\frac{12}{(1+K^2)^3}\right]\right\}. \quad (22)$$

Sternheim [35] denotes the two contributions by δ_M and δ_E, respectively. An alternative expression, obtained by assuming a point nucleus, using Eq. (131) from [1] for the Uehling potential, and doing the integrations in a different order, is

$$\varepsilon_{VP2} = \frac{16\alpha}{3\pi}\int_1^\infty\frac{(z^2-1)^{1/2}}{z^2}\left(1+\frac{1}{2z^2}\right)\frac{1}{(1+az)^2}\times\left[\frac{az}{2}-\frac{1}{1+az}+\frac{23}{8(1+az)^2}-\frac{3}{2(1+az)^3}+\ln(1+az)\left(1-\frac{2}{1+az}+\frac{3}{2(1+az)^2}\right)\right]dz, \quad (23)$$

with $a=2m_e/(\alpha Zm_r)$. Both methods give the same result.

In the case of ordinary hydrogen, each of these contributes $3\alpha^2/8=1.997\times10^{-5}$. The accuracy of the numerical integration was checked by reproducing these results. One can thus expect that the muonic vacuum polarization will contribute $3\alpha^2/4\simeq4\times10^{-5}$, as in the case of normal hydrogen. This amounts to an energy shift of 0.0009 meV. Contributions due to the weak interaction or hadronic vacuum polarization should be even smaller. For muonic hydrogen, one obtains $\varepsilon_{VP1}=0.002\,11$ and $\varepsilon_{VP2}=0.003\,25$ for a point nucleus. Including the effect of the proton size [with $G_E(q)=G_M(q)$ as a dipole form factor] reduces these numbers to 0.002 06 and 0.003 21, respectively. For the case of muonic ^{3}He [7], the corresponding numbers are $\varepsilon_{VP1}=0.002\,86$ and $\varepsilon_{VP2}=0.004\,76$. The contribution to the hyperfine splitting of the 2s state is then 0.0470 meV +0.0733 meV=0.1203 meV (0.1212 meV if muonic vacuum polarization is included). The combined Breit and vertex corrections reduce this value to 0.1207 meV (0.1226 meV if the proton form factors are not taken into account).

The contribution to the hyperfine structure from the two-loop diagrams [16] can be calculated by replacing $U_2(\alpha Zm_rK)=(\alpha/3\pi)F(\phi)$ by $U_4(\alpha Zm_rK)$ (as given in [1,4]) in Eqs. (21) and (22). The resulting contributions are 1.64×10^{-5} and 2.46×10^{-5}, respectively, giving a total shift of 0.0009 meV.

The correction due to finite size and recoil have been given in [8] as −0.145 meV, while a value of −0.152 meV is given in [38]. Reference [8] also gives a correction as calculated by Zemach [36] equal to −0.183 meV. This correction is equal to

$$\varepsilon_{Zem} = -2\alpha Zm_r\langle r\rangle_{(2)},$$

where $\langle r\rangle_{(2)}$ is given in [6,26,37]. Using the value $\langle r\rangle_{(2)}=1.086\pm0.012$ fm from [37] gives $\varepsilon_{Zem}=-0.00702$ and a contribution of of −0.1742 meV to the hyperfine splitting of the 2s state. Including this, but not other recoil corrections, to the hyperfine structure of the 2s state gives a total splitting of 22.7806 meV. Additional higher-order corrections calculated in Ref. [38] amount to a total of −0.0003 meV and are not included here.

VIII. SUMMARY OF CONTRIBUTIONS AND CONCLUSIONS

The most important contributions to the Lamb shift in muonic hydrogen, including hyperfine structure, have been independently recalculated. A calculation of some terms that were omitted in the most recent literature, such as the virtual Delbrück effect [11] and an alternative calculation of the relativistic recoil correction, have been presented.

Numerically the results given in Table I add up to a total correction of $[206.032(6)-5.225\langle r^2\rangle+0.0347\langle r^2\rangle^{3/2}]$ meV =202.055±0.12 meV (for the value of the proton radius from [20]). As is well known, most of the uncertainty arises from the uncertainty in the proton radius.

However, the contribution of the light-by-light graph to the muon form factor has not yet been calculated. Also, since $m_\mu/m_p=0.1126$ is much larger than αZ, it is possible that recoil corrections of higher order in the mass ratio, which have never been calculated, could be significant at the level of the expected experimental accuracy of about 0.01 meV. In particular, the two-photon recoil corrections, including finite nuclear size, should be recalculated to resolve (small) inconsistencies among various theoretical results.

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