Unitarily localizable entanglement of Gaussian states

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We consider generic $(m\times n)$-mode bipartitions of continuous-variable systems, and study the associated bisymmetric multimode Gaussian states. They are defined as $(m+n)$-mode Gaussian states invariant under local mode permutations on the m-mode and n-mode subsystems. We prove that such states are equivalent, under local unitary transformations, to the tensor product of a two-mode state and of $m+n-2$ uncorrelated single-mode states. The entanglement between the m-mode and the n-mode blocks can then be completely concentrated on a single pair of modes by means of local unitary operations alone. This result allows us to prove that the PPT (positivity of the partial transpose) condition is necessary and sufficient for the separability of $(m+n)$-mode bisymmetric Gaussian states. We determine exactly their negativity and identify a subset of bisymmetric states whose multimode entanglement of formation can be computed analytically. We consider explicit examples of pure and mixed bisymmetric states and study their entanglement scaling with the number of modes.

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I. INTRODUCTION

In quantum information and computation science, it is of particular relevance to provide theoretical methods to determine the entanglement of systems susceptible to encompass many parties. Such an interest does not stem only from pure intellectual curiosity, but also from practical needs in the implementations of realistic information protocols. This is especially true as soon as one needs to encode two-party information in a multipartite structure in order to minimize possible errors and decoherence effects [1,2]. The study of the structure of multipartite entanglement poses many formidable challenges, concerning both its qualification and quantification, and so far little progress has been achieved for multiqubit systems and in general for multiparty systems in finite-dimensional Hilbert spaces. However, the situation looks somehow more promising in the arena of continuous variable systems, where some aspects of genuine multipartite entanglement can be at least qualitatively understood in the study of the entanglement of multimode bipartitions.

In the present work, we study in detail the entanglement properties of multimode Gaussian states of continuous-variable (CV) systems (for an introduction to CV quantum information, see Ref. [3]). After the seminal analysis on the separability of two-mode Gaussian states [4,5], much progress has been accomplished on the separability conditions of multimode Gaussian states under various bipartitions [6–9]. On the other hand, much less is known on the quantification of the entanglement of multimode, multipartite Gaussian states [10]. In a previous work [11], we have presented a theoretical scheme to exactly determine the entanglement of pure or mixed $(n+1)$-mode Gaussian states, under $(1\times n)$-mode bipartitions, endowed with full or partial symmetries under mode exchange. More recently, a measure of genuine multipartite CV entanglement has been proposed [12] that extends the approach introduced by Coffman, Kundu, and Wootters for multiqubit systems [13], and possesses a precise operational meaning related to the optimal fidelity of teleportation in a continuous-variable teleportation network [14].

In this paper, we generalize the analysis introduced in Ref. [11] to bisymmetric $(m+n)$-mode Gaussian states of $(m\times n)$-mode bipartitions. The main result of the present paper is that the bipartite entanglement of bisymmetric $(m+n)$-mode Gaussian states is *unitarily localizable*, i.e., that, through local unitary operations, it may be fully concentrated in a single pair of modes, each of them owned by one of the two parties (blocks). Here the notion of localizable entanglement is different from that introduced by Verstraete, Popp, and Cirac for spin systems [15]. There, it was defined as the maximal entanglement concentrable on two chosen spins through local *measurements* on all the other spins. Here, the local operations that concentrate all the multimode entanglement on two modes are *unitary* and involve the two chosen modes as well, as parts of the respective blocks.

The consequences of the unitary localizability are manifold. In particular, the PPT (positivity of the partial transpose) criterion is proved to be a necessary and sufficient condition for the separability of $(m+n)$-mode bisymmetric Gaussian states. Moreover, the block entanglement (i.e., the entanglement between blocks of modes) of bisymmetric (generally mixed) Gaussian states can be determined. The entanglement can be quantified by the logarithmic negativity in the general instance because the PPT criterion holds, but we will also show some explicit cases in which the entanglement of formation between m-mode and n-mode parties can be exactly computed.

The plan of the paper is as follows. In Sec. II, we introduce the notation and review some basic facts about Gaussian states and their entanglement properties. In Sec. III, we show that a bisymmetric Gaussian state reduces to the tensor product of a correlated two-mode state and of uncorrelated single-mode states. In Sec. IV, we exploit such a result to explicitly determine the entanglement of bisymmetric Gaussian states. In Sec. V, the scaling of the block entanglement and the evaluation of the unitarily localizable entanglement involving different partitions of (generally mixed) symmetric states are studied in detail. Finally, in Sec. VI, we present some conclusions and miscellaneous comments.

II. GAUSSIAN STATES OF BOSONIC SYSTEMS

Let us consider a CV system, i.e., a system described by an infinite-dimensional Hilbert space $\mathcal{H}=\otimes_{i=1}^{n}\mathcal{H}_i$ resulting from the tensor product of infinite-dimensional Fock spaces $\mathcal{H}_i$'s. Let a_i be the annihilation operator acting on $\mathcal{H}_i$, and $\hat{x}_i=(a_i+a_i^\dagger)$ and $\hat{p}_i=(a_i-a_i^\dagger)/i$ be the related quadrature phase operators. The corresponding phase-space variables will be denoted by x_i and p_i. Let us group together the operators $\hat{x}_i$ and $\hat{p}_i$ in a vector of operators $\hat{X}=(\hat{x}_1,\hat{p}_1,\ldots,\hat{x}_n,\hat{p}_n)$. The canonical commutation relations (CCR) for the $\hat{X}_i$'s are encoded in the symplectic form $\mathbf{\Omega}$,

$$[\hat{X}_i,\hat{X}_j]=2i\Omega_{ij},$$

with

$$\mathbf{\Omega}\equiv\boldsymbol{\omega}^{\oplus n},\quad \boldsymbol{\omega}\equiv\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{1}$$

A complete description of a CV quantum state ϱ can be provided in terms of its symmetrically ordered characteristic function χ. If we define the displacement operator $D_\xi=\exp(i\hat{X}^{\mathsf{T}}\mathbf{\Omega}\xi)$, with $\xi\in\mathbb{R}^{2n}$, then the characteristic function χ associated to ϱ is given by $\chi(\xi)=\mathrm{Tr}[\varrho D_\xi]$. The set of Gaussian states is, by definition, the set of states with Gaussian characteristic functions. Therefore, a Gaussian state ϱ is completely characterized by its first and second statistical moments which form, respectively, the vector of first moments $\bar{X}\equiv(\langle\hat{X}_1\rangle,\langle\hat{X}_1\rangle,\ldots,\langle\hat{X}_n\rangle,\langle\hat{X}_n\rangle)$ and the covariance matrix (CM) $\boldsymbol{\sigma}$ of elements

$$\sigma_{ij}\equiv\frac{1}{2}\langle\hat{X}_i\hat{X}_j+\hat{X}_j\hat{X}_i\rangle-\langle\hat{X}_i\rangle\langle\hat{X}_j\rangle, \tag{2}$$

where, for any observable $\hat{o}$, $\langle\hat{o}\rangle\equiv\mathrm{Tr}(\varrho\hat{o})$. First statistical moments can be arbitrarily adjusted by local unitary operations, which do not affect any property related to correlations or entropies. Therefore, they will be unimportant to our aims and we will set them to 0 in the following, without any loss of generality. Throughout the paper, $\boldsymbol{\sigma}$ will stand for the covariance matrix of the Gaussian state ϱ.

The positivity of ϱ and the CCR entail the following relation on the CM $\boldsymbol{\sigma}$ of a quantum state ϱ ("Robertson-Schrödinger" uncertainty relation),

$$\boldsymbol{\sigma}+i\mathbf{\Omega}\geqslant 0. \tag{3}$$

Inequality (3) is the necessary and sufficient constraint $\boldsymbol{\sigma}$ has to fulfill to be a *bona fide* CM [16]. We mention that such a constraint implies $\boldsymbol{\sigma}\geqslant 0$.

The class of unitary transformations generated by second-order polynomials in the field operators ("second-order" operations) is especially relevant in manipulating Gaussian states. For an n-mode system, such operators may be mapped, through the so-called "*metaplectic*" representation, into the real symplectic group $Sp_{(2n,\mathbb{R})}$ [17], made up by linear operations acting on a linear space (called "phase space" in analogy with classical Hamiltonian dynamics), which preserves the symplectic form $\mathbf{\Omega}$ under congruence,

$$S\in Sp_{(2n,\mathbb{R})}\Leftrightarrow S^{\mathsf{T}}\mathbf{\Omega}S=\mathbf{\Omega}.$$

Symplectic operations preserve the Gaussian character of the input state, acting linearly on first moments and by congruence on second moments,

$$\boldsymbol{\sigma}\rightarrow S^{\mathsf{T}}\boldsymbol{\sigma}S.$$

Ideal squeezers and beam splitters are examples of (respectively, "active" and "passive") symplectic transformations.

A tensor product of Hilbert spaces (and of "second-order" unitary operations) is mapped into a direct sum of phase spaces (and of symplectic transformations). Under an $(m\times n)$-mode partition, resulting from the direct sum of phase spaces Γ_1 and Γ_2 with dimensions $2m$ and $2n$, respectively, we will refer to a transformation $S_l=S_1\oplus S_2$, with $S_1\in Sp_{(2m,\mathbb{R})}$ and $S_2\in Sp_{(2n,\mathbb{R})}$ acting on Γ_1 and Γ_2, as to a "local symplectic operation." The corresponding unitary transformation is the "local unitary transformation" $U_l=U_1\otimes U_2$.

Let us recall that, due to a theorem by Williamson [18], the CM of an n-mode Gaussian state can always be written as [16]

$$\boldsymbol{\sigma}=S^{\mathsf{T}}\boldsymbol{\nu}S, \tag{4}$$

where $S\in Sp_{(2n,\mathbb{R})}$ and $\boldsymbol{\nu}$ is the CM,

$$\boldsymbol{\nu}=\mathrm{diag}(\nu_1,\nu_1,\ldots,\nu_n,\nu_n), \tag{5}$$

corresponding to a tensor product of thermal states with diagonal density matrix $\varrho^{\otimes}$ given by

$$\varrho^{\otimes}=\bigotimes_i\frac{2}{\nu_i+1}\sum_{k=0}^{\infty}\left(\frac{\nu_i-1}{\nu_i+1}\right)^k|k\rangle_{ii}\langle k|,$$

$|k\rangle_i$ being the kth number state of the Fock space $\mathcal{H}_i$. The dual (Hilbert space) formulation of Eq. (4) then reads $\varrho=U^\dagger\varrho^{\otimes}U$, for some unitary U. The quantities ν_i's form the symplectic spectrum of the covariance matrix $\boldsymbol{\sigma}$ and can be computed as the eigenvalues of the matrix $|i\mathbf{\Omega}\boldsymbol{\sigma}|$ [19]. Such eigenvalues are in fact invariant under the action of symplectic transformations on the matrix $\boldsymbol{\sigma}$.

The symplectic eigenvalues ν_i encode essential information on the Gaussian state ϱ and provide powerful, simple ways to express its fundamental properties. For instance, provided that the CM $\boldsymbol{\sigma}$ satisfies $\boldsymbol{\sigma}\geqslant 0$, then

$$\nu_i\geqslant 1$$

is equivalent to the uncertainty relation (3). We remark that the full saturation of the uncertainty principle can only be achieved by pure n-mode Gaussian states, for which $\nu_i=1\ \forall\ i=1,\ldots,n$. Instead, mixed states such that $\nu_{i\leqslant k}=1$ and $\nu_{i>k}>1$, with $1\leqslant k\leqslant n$, only partially saturate the uncertainty principle, with partial saturation becoming weaker with decreasing k. Such states are minimum-uncertainty mixed Gaussian states in the sense that the phase quadrature operators of the first k modes satisfy the Heisenberg minimal uncertainty, while for the remaining $n-k$ modes, the state indeed contains some additional thermal and/or Schrödinger-like correlations which are responsible for the global mixedness of the state.

The symplectic eigenvalues are clearly invariant under symplectic operations. Yet, it is often advantageous to introduce other symplectic invariants, which can be easily handled in terms of second statistical moments. In the present work, dealing with an n-mode Gaussian state with CM $\boldsymbol{\sigma}$, we will make use of the obvious invariant Det $\boldsymbol{\sigma}$ (whose invariance is a consequence of the fact that $\text{Det}\, S=1\ \forall S \in Sp_{(2n,\mathbb{R})}$) and of $\Delta_{\boldsymbol{\sigma}}=\sum_{i,j=1}^{n}\text{Det}\,\boldsymbol{\sigma}_{ij}$, where the $\boldsymbol{\sigma}_{ij}$ are 2×2 submatrices of $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma}=\begin{pmatrix}\boldsymbol{\sigma}_{11} & \cdot & \boldsymbol{\sigma}_{1n}\\ \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}_{n1} & \cdots & \boldsymbol{\sigma}_{nn}\end{pmatrix}. \tag{6}$$

The invariance of $\Delta_{\boldsymbol{\sigma}}$ in the multimode case follows from its invariance in the case of two-mode states, proved in Ref. [20], and from the fact that any symplectic transformation can be decomposed as the product of two-mode transformations [21]. The symplectic eigenvalues $\nu^{\mp}$ of a two-mode Gaussian state are simply determined by the invariants introduced above,

$$2(\nu^{\mp})^2=\Delta_{\boldsymbol{\sigma}} \mp \sqrt{\Delta_{\boldsymbol{\sigma}}^2-4\,\text{Det}\,\boldsymbol{\sigma}}. \tag{7}$$

Also the purity $\mu=\text{Tr}\,\varrho^2$ of a multimode Gaussian state ϱ, quantifying its degree of mixedness, is easily determined in terms of the symplectic invariant Det $\boldsymbol{\sigma}$, as [22]

$$\mu=1/\sqrt{\text{Det}\,\boldsymbol{\sigma}}. \tag{8}$$

Regarding the entanglement of Gaussian states, we recall that the positivity of the partial transpose is a necessary and sufficient criterion for two-mode states to be separable (PPT criterion) [4]. The validity of such a criterion has been later extended to generic Gaussian states of $(1\times n)$-mode systems [6] and to $(m+n)$-mode Gaussian states with a fully degenerate symplectic spectrum [23,24]. For a bipartite system with Hilbert space $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, made up of two subsystems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the operation of partial transposition is defined as the transposition of the degrees of freedom associated to only one of the two subsystems, i.e., to the transposition of only one of the reduced Hilbert spaces, say $\mathcal{H}_A$. Let us remark that the positivity of the partially transposed operator $\tilde{\varrho}$ does not depend on which subsystem is transposed nor on the basis chosen to perform the transposition. Therefore, the positivity of the partial transpose is invariant under local unitary transformations on the two subsystems. In particular, for two-mode Gaussian states, the PPT criterion reduces to a simple inequality on the smallest symplectic eigenvalue $\tilde{\nu}^-$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$ (partial transposition amounts to the mirror reflection of one of the four quadratures, see Ref. [4]). A two-mode Gaussian state is separable (i.e., not entangled) if and only if

$$\tilde{\nu}^- \geqslant 1. \tag{9}$$

A proper quantification of the entanglement, easily computable for two-mode Gaussian states, is provided by the negativity $\mathcal{N}$, thoroughly discussed and extended in Ref. [25] to CV systems (see also Refs. [26,27]). The negativity of a quantum state ϱ is defined as

$$\mathcal{N}(\varrho)=\frac{\|\tilde{\varrho}\|_1-1}{2}, \tag{10}$$

where $\tilde{\varrho}$ is the partially transposed density matrix and $\|\hat{o}\|_1=\text{Tr}|\hat{o}|$ stands for the trace norm of $\hat{o}$. The quantity $\mathcal{N}(\varrho)$ is equal to $|\sum_i \lambda_i|$, the modulus of the sum of the negative eigenvalues of $\tilde{\varrho}$, quantifying the extent to which $\tilde{\varrho}$ fails to be positive. Strictly related to $\mathcal{N}$ is the logarithmic negativity $E_{\mathcal{N}}$, defined as $E_{\mathcal{N}}\equiv\ln\|\tilde{\varrho}\|_1$, which constitutes an upper bound to the *distillable entanglement* of the quantum state ϱ and is related to the entanglement cost under PPT preserving operations [28]. It can be easily shown [29] that the logarithmic negativity of a two-mode Gaussian state is a simple function of the partially transposed symplectic eigenvalue $\tilde{\nu}^-$ alone,

$$E_{\mathcal{N}}=\max[0,-\ln\tilde{\nu}^-], \tag{11}$$

quantifying the extent to which inequality (9) is violated.

Let us recall that the bipartite entanglement of formation E_F [30] of a quantum state ϱ, shared by parties A and B, is defined as

$$E_F(\varrho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i E(|\psi_i\rangle), \tag{12}$$

where the minimum is taken over all the pure state realizations of ϱ,

$$\varrho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$$

and $E(|\psi_i\rangle)$ denotes the entropy of entanglement of the pure state $|\psi_i\rangle$, defined as the von Neumann entropy of the reduced state obtained by tracing over the variables of one of the two subsystems,

$$E(|\psi_i\rangle)=-\text{Tr}_A[\text{Tr}_B|\psi_i\rangle\langle\psi_i|\ln(\text{Tr}_B|\psi_i\rangle\langle\psi_i|)].$$

As far as symmetric [i.e., with $\text{Det}\,\boldsymbol{\sigma}_{11}=\text{Det}\,\boldsymbol{\sigma}_{22}$, with reference to the decomposition of Eq. (6)] two-mode Gaussian states are concerned, the entanglement of formation E_F can be computed [31]. The quantity E_F turns out to be, again, a decreasing function of $\tilde{\nu}^-$,

$$E_F=\max[0,h(\tilde{\nu}^-)], \tag{13}$$

with

$$h(x)=\frac{(1+x)^2}{4x}\ln\left(\frac{(1+x)^2}{4x}\right)-\frac{(1-x)^2}{4x}\ln\left(\frac{(1-x)^2}{4x}\right).$$

Therefore, the entanglement of formation provides, for two-mode symmetric Gaussian states, a quantification of entanglement fully equivalent to the one provided by the logarithmic negativity $E_{\mathcal{N}}$.

III. STANDARD FORMS OF BISYMMETRIC MULTIMODE GAUSSIAN STATES

We shall say that a multimode Gaussian state ϱ is *fully symmetric* if it is invariant under the exchange of any two modes. In the following, we will consider the fully symmetric m-mode and n-mode Gaussian states $\varrho_{\boldsymbol{\alpha}^m}$ and $\varrho_{\boldsymbol{\beta}^n}$, with CMs $\boldsymbol{\sigma}_{\boldsymbol{\alpha}^m}$ and $\boldsymbol{\sigma}_{\boldsymbol{\beta}^n}$. Due to symmetry, we have that

$$\boldsymbol{\sigma}_{\alpha^m}=\begin{pmatrix}\boldsymbol{\alpha} & \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon} & \boldsymbol{\alpha} & \boldsymbol{\varepsilon} & \vdots\\ \vdots & \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \boldsymbol{\alpha}\end{pmatrix},\quad \boldsymbol{\sigma}_{\beta^n}=\begin{pmatrix}\boldsymbol{\beta} & \boldsymbol{\zeta} & \cdots & \boldsymbol{\zeta}\\ \boldsymbol{\zeta} & \boldsymbol{\beta} & \boldsymbol{\zeta} & \vdots\\ \vdots & \boldsymbol{\zeta} & \ddots & \boldsymbol{\zeta}\\ \boldsymbol{\zeta} & \cdots & \boldsymbol{\zeta} & \boldsymbol{\beta}\end{pmatrix}, \tag{14}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\beta}$, and $\boldsymbol{\zeta}$ are 2×2 real symmetric submatrices (the symmetry of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\zeta}$ stems again from the symmetry under the exchange of any two modes). All the properties related to correlations and entropic measures of multimode Gaussian states are invariant under local, single-mode symplectic operations. A first preliminary fact, analogous to the standard form reduction of two-mode states, will thus prove useful.

Standard form of fully symmetric states. Let $\boldsymbol{\sigma}_{\beta^n}$ be the CM of a fully symmetric n-mode Gaussian state. The 2×2 blocks $\boldsymbol{\beta}$ and $\boldsymbol{\zeta}$ of $\boldsymbol{\sigma}_{\beta^n}$, defined by Eq. (14), can be brought by means of local, single-mode symplectic operations $S \in Sp_{(2,\mathbb{R})}^{\oplus n}$ into the form $\boldsymbol{\beta}=\mathrm{diag}(b,b)$ and $\boldsymbol{\zeta}=\mathrm{diag}(z_1,z_2)$.

Proof. The blocks $\boldsymbol{\beta}$, being CM's of reduced single-mode Gaussian states, can be turned into their Williamson standard form by the same symplectic $S_l\in Sp_{(2,\mathbb{R})}$ acting on each mode. One is then left with the freedom of applying local, single-mode rotations that leave the blocks $\boldsymbol{\beta}$ invariant. The same rotation applied to each mode is sufficient to diagonalize $\boldsymbol{\zeta}$, since such a matrix is symmetric.

The coefficients b,z_1,z_2 of the standard form are determined by the local, single-mode invariant $\mathrm{Det}\,\boldsymbol{\beta}\equiv\mu_\beta^{-2}$, and by the symplectic invariants $\mathrm{Det}\,\boldsymbol{\sigma}_{\beta^2}\equiv\mu_{\beta^2}^{-2}$ and $\Delta_2\equiv\Delta(\boldsymbol{\sigma}_{\beta^2})$. Here μ_β (μ_{β^2}) is the marginal purity of the single-mode (two-mode) reduced states, while Δ_2 is the remaining invariant of the two-mode reduced states [32]. This parametrization is provided, in the present instance, by the following equations:

$$b=\frac{1}{\mu_\beta},\quad z_1=\frac{\mu_\beta}{4}(\epsilon_- - \epsilon_+),\quad z_2=\frac{\mu_\beta}{4}(\epsilon_- + \epsilon_+), \tag{15}$$

with

$$\epsilon_- = \sqrt{\Delta_2^2-\frac{4}{\mu_{\beta^2}^2}},$$

and

$$\epsilon_+ = \sqrt{\left(\Delta_2-\frac{4}{\mu_\beta^2}\right)^2-\frac{4}{\mu_{\beta^2}^2}}.$$

This parametrization has a straightforward interpretation, because μ_β and μ_{β^2} quantify the local mixednesses and Δ_2 regulates the entanglement of the two-mode blocks at fixed global and local purities [32].

Let us next determine and analyze the symplectic spectrum (symplectic eigenvalues) of $\boldsymbol{\sigma}_{\beta^n}$.

Symplectic degeneracy of fully symmetric states. The symplectic spectrum of $\boldsymbol{\sigma}_{\beta^n}$ is $(n-1)$-times degenerate. The two symplectic eigenvalues of $\boldsymbol{\sigma}_{\beta^n}\nu_\beta^-$ and $\nu_{\beta^n}^+$ read

$$\nu_\beta^- = \sqrt{(b-z_1)(b-z_2)},$$

$$\nu_{\beta^n}^+ = \sqrt{[b+(n-1)z_1][b+(n-1)z_2]}, \tag{16}$$

where ν_β^- is the $(n-1)$-times degenerate eigenvalue.

Proof. We recall that the symplectic eigenvalues of $\boldsymbol{\sigma}_{\beta^n}$ are the absolute values of the eigenvalues of $i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}$. Since the symplectic form $\boldsymbol{\Omega}$ is block diagonal, with 2×2 blocks $\boldsymbol{\omega}$ given by Eq. (1), the matrix $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ is just the matrix $\boldsymbol{\sigma}$ with $i\boldsymbol{\omega}$ multiplying on the left any 2×2 block. Let us now consider the set of vectors $\{v_i\}$, for $i=1,\ldots,n-1$,

$$v_i = (0,\ldots,0,\underbrace{v^{\mathsf{T}}}_{\text{mode } i},\underbrace{-v^{\mathsf{T}}}_{\text{mode } i+1},0,\ldots,0)^{\mathsf{T}}, \tag{17}$$

where, for convenience, we have introduced the two-dimensional vector

$$v=\left(i\frac{b-z_2}{\nu_\beta^-},1\right)^{\mathsf{T}}.$$

The v_i are $n-1$ linear independent vectors. One has

$$i\boldsymbol{\Omega}\boldsymbol{\sigma}v_i = i(0,\ldots,0,\underbrace{[\boldsymbol{\omega}(\boldsymbol{\beta}-\boldsymbol{\zeta})v]^{\mathsf{T}}}_{\text{mode } i},\underbrace{-[\boldsymbol{\omega}(\boldsymbol{\beta}-\boldsymbol{\zeta})v]^{\mathsf{T}}}_{\text{mode } i+1},0,\ldots,0)^{\mathsf{T}}. \tag{18}$$

A straightforward computation gives

$$i\boldsymbol{\omega}(\boldsymbol{\beta}-\boldsymbol{\zeta})v = i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}b-z_1 & 0\\ 0 & b-z_2\end{pmatrix}\begin{pmatrix} i\sqrt{\frac{b-z_2}{b-z_1}}\\ 1\end{pmatrix} = \nu_\beta^-\begin{pmatrix} i\sqrt{\frac{b-z_2}{b-z_1}}\\ 1\end{pmatrix}, \tag{19}$$

which recasts Eq. (18) into $i\boldsymbol{\Omega}\boldsymbol{\sigma}v_i=\nu_\beta^- v_i$, thus proving that the symplectic eigenvalue ν_β^- of $\boldsymbol{\sigma}$ is $(n-1)$-times degenerate. Note that, as one should expect, there exist also $n-1$ eigenvectors associated to the negative eigenvalue $-\nu_\beta^-$. To this end, it suffices to turn v into

$$\left(-i\frac{b-z_2}{\nu_\beta^-},1\right)^{\mathsf{T}}.$$

The remaining linearly independent eigenvector of $i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}$ is the vector

$$(w^{\mathsf{T}},\ldots,w^{\mathsf{T}})^{\mathsf{T}},$$

with

$$w^{\mathsf{T}} = (i\sqrt{b+(n-1)z_1}, \sqrt{b+(n-1)z_2}).$$

It is immediate to verify that such a vector is associated to the eigenvalue $\nu^{+}_{\beta^n}$, completing the proof.

The $(n-1)$-times degenerate eigenvalue ν^{-}_{β} is independent of n, while $\nu^{+}_{\beta^n}$ can be simply expressed as a function of the single-mode purity μ_β and the symplectic spectrum of the two-mode block with eigenvalues ν^{-}_{β} and $\nu^{+}_{\beta^2}$,

$$(\nu^{+}_{\beta^n})^2 = -\frac{n(n-2)}{\mu^2_\beta} + \frac{(n-1)}{2}[n(\nu^{+}_{\beta^2})^2 + (n-2)(\nu^{-}_{\beta})^2]. \tag{20}$$

In turn, the two-mode symplectic eigenvalues are determined by the two-mode invariants by the relation

$$2(\nu^{\mp}_{\beta})^2 = \Delta_{\beta^2} \mp \sqrt{\Delta^2_{\beta^2} - 4/\mu^2_{\beta^2}}. \tag{21}$$

The global purity Eq. (8) of a fully symmetric multimode Gaussian state is

$$\mu_{\beta^n} \equiv (\mathrm{Det}\, \boldsymbol{\sigma}_{\beta^n})^{-1/2} = \left[(\nu^{-}_{\beta})^{n-1}\nu^{+}_{\beta^n}\right]^{-1}, \tag{22}$$

and, through Eq. (20), can be fully determined in terms of the one- and two-mode parameters alone.

Obviously, analogous results hold for the m-mode CM $\boldsymbol{\sigma}_{\alpha^m}$ of Eq. (14), whose 2×2 submatrices can be brought to the form $\boldsymbol{\alpha}=\mathrm{diag}\,(a,a)$ and $\boldsymbol{\varepsilon}=\mathrm{diag}\,(e_1,e_2)$ and whose $(m-1)$-times degenerate symplectic spectrum reads

$$\nu^{-}_{\alpha} = (a-e_1)(a-e_2),$$

$$\nu^{+}_{\alpha^m} = [a+(m-1)e_1][a+(m-1)e_2]. \tag{23}$$

Let us now generalize this analysis to the $(m+n)$-mode Gaussian states with CM $\boldsymbol{\sigma}$, which results from a correlated combination of the fully symmetric blocks $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{\alpha^m} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^{\mathsf{T}} & \boldsymbol{\sigma}_{\beta^n} \end{pmatrix}, \tag{24}$$

where $\boldsymbol{\Gamma}$ is a $2m\times 2n$ real matrix formed by identical 2×2 blocks $\boldsymbol{\gamma}$. Clearly, $\boldsymbol{\Gamma}$ is responsible for the correlations existing between the m-mode and the n-mode parties. Once again, the identity of the submatrices $\boldsymbol{\gamma}$ is a consequence of the local invariance under mode exchange, internal to the m-mode and n-mode parties. States of the form of Eq. (24) will be henceforth referred to as *bisymmetric*. A significant insight into bisymmetric multimode Gaussian states can be gained by studying the symplectic spectrum of $\boldsymbol{\sigma}$ and comparing it to the ones of $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$.

Symplectic degeneracy of bisymmetric states. The symplectic spectrum of the CM $\boldsymbol{\sigma}$ Eq. (24) of a bisymmetric $(m+n)$-mode Gaussian state includes two degenerate eigenvalues, with multiplicities $m-1$ and $n-1$. Such eigenvalues coincide, respectively, with the degenerate eigenvalue ν^{-}_{α} of the reduced CM $\boldsymbol{\sigma}_{\alpha^m}$ and the degenerate eigenvalue ν^{-}_{β} of the reduced CM $\boldsymbol{\sigma}_{\beta^n}$.

Proof. One can proceed constructively, in analogy with the proof of the previous proposition. Let us consider the standard forms of the blocks $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$, while keeping the 2×2 submatrices $\boldsymbol{\gamma}$ in arbitrary, generally nonsymmetric, form. Let us next focus on the block $\boldsymbol{\sigma}_{\beta^n}$ and define the vectors $\overline{v_i}$ by

$$\overline{v_i} = (0,\ldots,0,v_i^{\mathsf{T}})^{\mathsf{T}}. \tag{25}$$

They are the vectors obtained from the vectors v_i's of Eq. (17) by appending to them $2m$ null entries on the left. Because of the identity of the blocks $\boldsymbol{\gamma}$, their contributions to the secular equation cancel out and it is straightforward to verify that the vectors $\overline{v_i}$'s are $n-1$ eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ with eigenvalue ν^{-}_{β}. The same argument holds considering the submatrix $\boldsymbol{\sigma}_{\alpha^m}$, thus completing the proof.

Equipped with these results, we are now in a position to determine the bipartite entanglement of bisymmetric multimode Gaussian states and prove that it can always be unitarily *localized* or *concentrated*.

Unitary localization of the entanglement of bisymmetric states. The bisymmetric $(m+n)$-mode Gaussian state with CM $\boldsymbol{\sigma}$ Eq. (24) can be brought, by means of a local unitary operation, with respect to the $(m\times n)$-mode bipartition with reduced CMs $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$, to a tensor product of single-mode uncorrelated states and of a two-mode Gaussian state.

Proof. Let us focus on the n-mode block $\boldsymbol{\sigma}_{\beta^n}$. The vectors $\overline{v_i}$ of Eq. (25), with the first $2m$ entries equal to 0, are, by construction, simultaneous eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}$ and $i\boldsymbol{\Omega}\boldsymbol{\sigma}$, with the same (degenerate) eigenvalue. This fact suggests that the phase-space modes corresponding to such eigenvectors are the same for $\boldsymbol{\sigma}$ and for $\boldsymbol{\sigma}_{\beta^n}$. Then, bringing by means of a local symplectic operation the CM $\boldsymbol{\sigma}_{\beta^n}$ in Williamson form, any $(2n-2)\times(2n-2)$ submatrix of $\boldsymbol{\sigma}$ will be diagonalized because the normal modes are common to the global and local CMs. In other words, no correlations between the m-mode party with reduced CM $\boldsymbol{\sigma}_{\alpha^m}$ and such modes will be left: all the correlations between the m-mode and n-mode parties will be concentrated in the two conjugate quadratures of a single mode of the n-mode block. Going through the same argument for the m-mode block with CM $\boldsymbol{\sigma}_{\alpha^m}$ would prove the proposition and show that the whole entanglement between the two multimode blocks can always be concentrated in only two modes, one for each of the two multimode parties.

To prove this property, we proceed first by investigating the relationship between the transformations which diagonalize $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ and the symplectic operations that bring $\boldsymbol{\sigma}$ in Williamson normal form $\boldsymbol{\nu}$ [33]. The problem one is immediately faced with is that these transformations are not unique because the normal form associated to $\boldsymbol{\sigma}$ is invariant under local rotations (this local freedom is always present in the selection of normal modes) *and*, due to degeneracy, also under global symplectic rotations of the modes associated to the degenerate eigenvalue ν^{-}_{β}. Thus there is an ambiguity in selecting the eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ and therefore in determining the transformation that diagonalizes it. Moreover, if $\{w_i\}$ is a set of $2(m+n)$ column-vectors normalized eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$, then any matrix T of the form

$$T = (\xi_1 w_1, \ldots, \xi_k w_k) \tag{26}$$

diagonalizes $i\boldsymbol{\Omega\sigma}$: $T^{-1}(i\boldsymbol{\Omega\sigma})T=D$ (with the ξ_i's arbitrary complex coefficients). However, we can proceed by observing that the 2×2 matrix $i\boldsymbol{\omega}$ is diagonalized by the unitary transformation $\overline{U}$, with

$$\overline{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix},$$

so that $\overline{U}^{\dagger} ia\boldsymbol{\omega}\overline{U}=\mathrm{diag}(a,-a)$ (where a is any complex number). We can then define the matrix $U=\overline{U}^{\oplus(m+n)}$, which is local in the sense that it is block diagonal and acts on each mode separately, such that for any normal form $\boldsymbol{\nu}$,

$$U^{-1} i\boldsymbol{\Omega\nu} U = D, \tag{27}$$

where $D=T^{-1}i\boldsymbol{\Omega\sigma}T$ is a diagonal matrix with entries $\{\mp\nu_i\}$ (in terms of the symplectic eigenvalues). Let us next denote by S one of the symplectic transformations that bring $\boldsymbol{\sigma}$ in normal form: $S^{\mathsf{T}}\boldsymbol{\sigma}S=\boldsymbol{\nu}$. It is then easy to see that

$$\begin{aligned} D &= T^{-1}(i\boldsymbol{\Omega\sigma})T = U^{-1}(i\boldsymbol{\Omega\nu})U \\ &= U^{-1}(i\boldsymbol{\Omega}S^{\mathsf{T}}\boldsymbol{\sigma}S)U = U^{-1}S^{-1}(i\boldsymbol{\Omega\sigma})SU, \end{aligned} \tag{28}$$

and therefore

$$S = TU^{-1} = TU^{\dagger}, \tag{29}$$

where in Eq. (28) we have exploited the fundamental property of symplectic transformations: $S^{-1\mathsf{T}}\boldsymbol{\Omega}S^{-1}=\boldsymbol{\Omega}$. Equation (29) shows that there must exist *some* symplectic transformation that diagonalizes $i\boldsymbol{\Omega\sigma}$ and satisfies the further condition given by Eq. (29). In fact, it is obvious that not every T diagonalizing $i\boldsymbol{\Omega\sigma}$ is a symplectic transformation when multiplied on the right by $U^{\dagger}$. Vice versa, if this last condition holds, the symplectic operation that brings $\boldsymbol{\sigma}$ in normal form is given by Eq. (29). The modes that diagonalize the quadratic form $\boldsymbol{\sigma}$ in phase space can be reconstructed in terms of S: since they are linear combinations of the original modes and $S^{\mathsf{T}}\boldsymbol{\sigma}S$ is diagonal, they can be expressed by real column vectors identified by the columns of S.

We can now go back to our original problem: leaving aside the involved task of exactly determining which choice of the eigenvectors of $i\boldsymbol{\Omega\sigma}$ leads to a symplectic transformation of the form Eq. (29), we are anyway assured that in the subspace associated to the eigenvalues $\mp\nu_\beta^-$, such eigenvectors must be linear combinations of the $\overline{v}_i$'s defined in Eq. (25) and their counterparts associated to the eigenvalue $-\nu_\beta^-$ (with their first $2m$ entries, related to the m-mode party, set equal to 0). Therefore, the transformation T reads, in general,

$$T = \begin{pmatrix} T_{1,1} & \cdots & T_{1,m} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & \cdots & T_{m,m} & \mathbf{0} & \cdots & \mathbf{0} \\ T_{m+1,1} & \cdots & T_{m+1,m} & T_{m+1,m+1} & \cdots & T_{m+1,m+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ T_{m+n,1} & \cdots & T_{m+n,m} & T_{m+n,m+1} & \cdots & T_{m+n,m+n} \end{pmatrix}, \tag{30}$$

where $\mathbf{0}$ stands for 2×2 null matrices and $T_{i,j}$ are 2×2 blocks, whose exact form is unessential to our aims. Exploiting Eq. (29), for the last $2(n-1)$ columns of S we obtain, in terms of 2×2 matrices,

$$(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{\text{first } m \text{ modes}}, \overline{U}^{*}T_{2,i}^{\mathsf{T}}, \ldots, \overline{U}^{*}T_{n,i}^{\mathsf{T}})^{\mathsf{T}}. \tag{31}$$

Due to the presence of the first m null entries, the $n-1$ modes determined by Eq. (31) are normal modes of both the *global* CM $\boldsymbol{\sigma}$ and the *local* CM $\boldsymbol{\sigma}_{\beta^n}$. An analogous proof, going along the same lines of reasoning, holds for the reduced CM $\boldsymbol{\sigma}_{\alpha^m}$: it can be reduced to a local normal form that shares $m-1$ normal modes with the global CM $\boldsymbol{\sigma}$. These results imply that the form in which all the correlations between the two parties are shared only by a single mode of the n-mode party and by a single mode of the m-mode party can be obtained by means of local symplectic (unitary) operations, namely by the symplectic operations bringing the block $\boldsymbol{\sigma}_{\beta^n}$ and the block $\boldsymbol{\sigma}_{\alpha^m}$ in Williamson form.

For the reader's ease and for the sake of pictorial clarity, we can supplement the proof by explicitly writing down the different forms of the CM $\boldsymbol{\sigma}$ at each step; such matrix representations allow an immediate visualization of the process of unitary concentration of the entanglement between a single pair of modes, one for each multimode party. The CM $\boldsymbol{\sigma}$ of a bisymmetric $(m+n)$-mode Gaussian state reads [see Eq. (24)]

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \boldsymbol{\gamma} & \cdots & \cdots & \boldsymbol{\gamma} \\ \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon} & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon} & \vdots & & \ddots & \vdots \\ \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \alpha & \boldsymbol{\gamma} & \cdots & \cdots & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \cdots & \cdots & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} & \boldsymbol{\zeta} & \cdots & \boldsymbol{\zeta} \\ \vdots & \ddots & & \vdots & \boldsymbol{\zeta} & \ddots & \boldsymbol{\zeta} & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \boldsymbol{\zeta} & \ddots & \boldsymbol{\zeta} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \cdots & \cdots & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\zeta} & \cdots & \boldsymbol{\zeta} & \boldsymbol{\beta} \end{pmatrix}. \tag{32}$$

According to what we have just shown, reducing to normal form the block $\boldsymbol{\sigma}_{\beta^n}$ brings the global CM $\boldsymbol{\sigma}$ in the form CM $\boldsymbol{\sigma}'$,

$$\boldsymbol{\sigma}' = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \boldsymbol{\gamma}' & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon} & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \boldsymbol{\alpha} & \boldsymbol{\gamma}' & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{\gamma}'^{\mathsf{T}} & \cdots & \cdots & \boldsymbol{\gamma}'^{\mathsf{T}} & \boldsymbol{\nu}_{\beta^n}^{+} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{\nu}_{\beta}^{-} & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\nu}_{\beta}^{-} \end{pmatrix},$$

where the 2×2 blocks $\boldsymbol{\nu}_{\beta^n}^{+}=\nu_{\beta^n}^{+}\mathbb{1}_2$ and $\boldsymbol{\nu}_{\beta}^{-}=\nu_{\beta}^{-}\mathbb{1}_2$ are the Williamson normal blocks associated to the two symplectic eigenvalues of $\boldsymbol{\sigma}_{\beta^n}$. The identity of the submatrices γ' is due to the invariance under permutation of the first m modes, which are left unaffected. The subsequent symplectic diagonalization of $\boldsymbol{\sigma}_{\alpha^m}$ puts the global CM $\boldsymbol{\sigma}$ in the following form [notice that the first $(m+1)$-mode reduced CM is again a matrix of the same form of $\boldsymbol{\sigma}$, with $n=1$],

$$\boldsymbol{\sigma}'' = \begin{pmatrix} \boldsymbol{\nu}_\alpha^- & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \mathbf{0} & \boldsymbol{\nu}_\alpha^- & \mathbf{0} & \mathbf{0} & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\nu}_{\alpha^m}^+ & \boldsymbol{\gamma}'' & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\gamma}''^{\mathsf{T}} & \boldsymbol{\nu}_{\beta^n}^+ & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{\nu}_\beta^- & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\nu}_\beta^- \end{pmatrix}, \qquad (33)$$

with $\boldsymbol{\nu}_{\alpha^m}^+ = \nu_{\alpha^m}^+ \mathbb{1}_2$ and $\boldsymbol{\nu}_\alpha^- = \nu_\alpha^- \mathbb{1}_2$. Equation (33) shows explicitly that the state with CM $\boldsymbol{\sigma}''$, obtained from the original state with CM $\boldsymbol{\sigma}$ by exploiting local unitary operations, is the tensor product of $m+n-2$ uncorrelated single-mode states and of a correlated two-mode Gaussian state. The proof is therefore complete, and shows that the amount of entanglement (quantum correlations) present in any bisymmetric multimode Gaussian state can be localized (concentrated) in a two-mode Gaussian state (i.e., shared only by a single pair of modes), via local unitary operations. These results and their consequences will be discussed in detail in the following sections. □

IV. BLOCK ENTANGLEMENT OF MULTIMODE GAUSSIAN STATES

In the previous section, the study of the multimode CM $\boldsymbol{\sigma}$ of Eq. (32) has been reduced to a two-mode problem by means of local unitary operations. This finding allows for an exhaustive analysis of the bipartite entanglement between the m- and n-mode blocks of a multimode Gaussian state, resorting to the powerful results available for two-mode Gaussian states. For any multimode Gaussian state with CM $\boldsymbol{\sigma}$, let us define the associated *equivalent* two-mode Gaussian state ϱ_{eq}, with CM $\boldsymbol{\sigma}_{eq}$ given by

$$\boldsymbol{\sigma}_{eq} = \begin{pmatrix} \boldsymbol{\nu}_{\alpha^m}^+ & \boldsymbol{\gamma}'' \\ \boldsymbol{\gamma}''^{\mathsf{T}} & \boldsymbol{\nu}_{\beta^n}^+ \end{pmatrix}, \qquad (34)$$

where the 2×2 blocks have been implicitly defined in the CM (33). As already mentioned, the entanglement of the bisymmetric state with CM $\boldsymbol{\sigma}$, originally shared among all the $m+n$ modes, can be *completely* concentrated by local unitary (symplectic) operations on a single pair of modes in the state with CM $\boldsymbol{\sigma}_{eq}$. Such an entanglement is, in this sense, localizable. Obviously, this kind of localization of entanglement by local unitaries is conceptually very different from the localization of entanglement by local measurements first discussed by Verstraete, Popp, and Cirac for qubit systems [15]. We now move on to describe some consequences of this result.

A first qualificative remark is in order. It is known that the PPT criterion is necessary and sufficient for the separability of Gaussian states of (1×1)-mode and $(1\times n)$-mode bipartitions. In view of the invariance of such a criterion under local unitary transformations, which can be appreciated by the definition of partial transpose at the Hilbert space level, and considering the results proved in the previous section, it is immediate to verify that the following property holds.

PPT criterion for bisymmetric multimode Gaussian states. For generic $(m\times n)$-mode bipartitions, the positivity of the partial transpose (PPT) is a necessary and sufficient condition for the separability of bisymmetric $(m+n)$-mode Gaussian states.

This statement is a first important generalization to $m\times n$ bipartitions of the result proved by Werner and Wolf for the case of $1\times n$ bipartitions [6]. In particular, it implies that no bisymmetric bound entangled Gaussian states may exist [6,34] and all the $m\times n$ block entanglement of such states is distillable. Moreover, it justifies the use of the negativity and the logarithmic negativity as measures of entanglement for these multimode Gaussian states.

As for the quantification of the entanglement, exploiting some recent results on two-mode Gaussian states [29,32] we can select the relevant quantities that, by determining the correlation properties of the two-mode Gaussian state with CM $\boldsymbol{\sigma}_{eq}$, also determine the entanglement and correlations of the multimode Gaussian state with CM $\boldsymbol{\sigma}$. These quantities are, clearly, the equivalent marginal purities $\mu_{\alpha eq}$ and $\mu_{\beta eq}$, the global purity μ_{eq}, and the equivalent two-mode invariant Δ_{eq}. Let us remind that, by exploiting Eqs. (16), (23), and (15), the symplectic spectra of the CMs $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$ may be recovered by means of the local two-mode invariants μ_β, μ_α, μ_{β^2}, μ_{α^2}, Δ_{β^2}, and Δ_{α^2}. The quantities $\mu_{\alpha eq}$ and $\mu_{\beta eq}$ are easily determined in terms of local invariants alone,

$$\mu_{\alpha eq} = 1/\nu_{\alpha^m}^+, \quad \mu_{\beta eq} = 1/\nu_{\beta^n}^+. \qquad (35)$$

On the other hand, the determination of μ_{eq} and Δ_{eq} requires the additional knowledge of two global symplectic invariants of the CM $\boldsymbol{\sigma}$; this should be expected, because they are susceptible of quantifying the correlations between the two parties. The natural choices for the global invariants are the global purity $\mu=1/\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}$ and the invariant Δ, given by

$$\Delta = m\,\mathrm{Det}\,\boldsymbol{\alpha} + m(m-1)\mathrm{Det}\,\boldsymbol{\varepsilon} + n\mathrm{Det}\,\boldsymbol{\beta} + n(n-1)\mathrm{Det}\,\boldsymbol{\zeta} + 2mn\,\mathrm{Det}\,\boldsymbol{\gamma}.$$

One has

$$\mu_{eq} = (\nu_\alpha^-)^{m-1}(\nu_\beta^-)^{n-1}\mu, \qquad (36)$$

$$\Delta_{eq} = \Delta - (m-1)(\nu_\alpha^-)^2 - (n-1)(\nu_\beta^-)^2. \qquad (37)$$

The entanglement, quantified by the logarithmic negativity, and the mutual information between the m-mode and the n-mode subsystems can thus be easily determined, as is the case for two-mode states. In particular, the smallest symplectic eigenvalue $\tilde{\nu}_{eq}$ of the matrix $\tilde{\boldsymbol{\sigma}}_{eq}$, derived from $\boldsymbol{\sigma}_{eq}$ by partial transposition, fully quantifies the entanglement between the m-mode and n-mode partitions. Recalling the results known for two-mode states [29,32], the quantity $\tilde{\nu}_{eq}$ reads

$$2\tilde{\nu}_{eq}^2 = \tilde{\Delta}_{eq} - \sqrt{\tilde{\Delta}_{eq}^2 - \frac{4}{\mu_{eq}^2}},$$

with

$$\tilde{\Delta}_{eq} = \frac{2}{\mu_{\alpha eq}^2} + \frac{2}{\mu_{\beta eq}^2} - \Delta_{eq}.$$

The logarithmic negativity $E_{\mathcal{N}}^{\alpha^m|\beta^n}$ measuring the bipartite entanglement between the m-mode and n-mode subsystems is then

$$E_{\mathcal{N}}^{\alpha^m/\beta^n} = \max[-\ln \tilde{\nu}_{eq}, 0]. \quad (38)$$

In the case $\nu_{\alpha^m}^{+} = \nu_{\beta^n}^{+}$, corresponding to the condition

$$[a + (m-1)e_1][a + (m-1)e_2] = [b + (n-1)z_1][b + (n-1)z_2], \quad (39)$$

the equivalent two-mode state is symmetric and we can determine also the entanglement of formation, using Eq. (13). Let us note that the possibility of exactly determining the entanglement of formation of a multimode Gaussian state of an $(m\times n)$-mode bipartition is a rather remarkable consequence, even under the symmetry constraints obeyed by the CM $\boldsymbol{\sigma}$. Another relevant fact to point out is that, since both the logarithmic negativity and the entanglement of formation are decreasing functions of the quantity $\tilde{\nu}_{eq}$, the two measures induce the same entanglement hierarchy on such a subset of equivalently symmetric states (i.e., states whose equivalent two-mode CM $\boldsymbol{\sigma}_{eq}$ is symmetric).

From Eq. (36) it follows that, if the $(m+n)$-mode bisymmetric state is pure $(\mu = \nu_{\alpha^m}^{-} = \nu_{\beta^n}^{-} = 1)$, then the equivalent two-mode state is pure as well $(\mu_{eq}=1)$ and, up to local symplectic operations, it is a two-mode squeezed vacuum. Therefore, *any pure bisymmetric multimode Gaussian state is equivalent, under local unitary (symplectic) operations, to a tensor product of a pure two-mode squeezed vacuum and of* $m+n-2$ *uncorrelated vacua*.

More generally, if both the reduced m-mode and n-mode CMs $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^m}$ of a bisymmetric, mixed multimode Gaussian state $\boldsymbol{\sigma}$ of the form Eq. (24) correspond to Gaussian mixed states of partial minimum uncertainty, i.e., if $\nu_{\alpha^m}^{-} = \nu_{\beta^n}^{-} = 1$, then Eq. (36) implies $\mu_{eq} = \mu$. Therefore, the equivalent two-mode state has the same entanglement and the same degree of mixedness of the original multimode state. In all other cases of bisymmetric multimode states, one has that $\mu_{eq} > \mu$ and the process of localization produces a two-mode state with higher purity than the original multimode state. In this specific sense, we see that the process of localization implies a process of purification as well. We can understand this key point observing that the entanglement is localized by performing local unitary transformations which are reversible by definition. Then, in principle, by only using passive and active linear optics elements such as beam splitters, phase shifters, and squeezers [9], one can implement a reversible machine that, from mixed, bisymmetric multimode states with strong quantum correlations between all the modes (and consequently between the m-mode and the n-mode partial blocks) but weak couplewise entanglement, is able to extract a highly pure, highly entangled two-mode state (with no entanglement lost, all the $m\times n$ entanglement can be localized). If needed, the same machine would be able, starting from a two-mode squeezed state and a collection of uncorrelated thermal or squeezed states, to distribute the two-mode entanglement between all modes, converting the two-mode into multimode, multipartite quantum correlations, again with no loss of entanglement. The bipartite or multipartite entanglement can then be used on demand, the first for instance in a CV quantum teleportation protocol, the latter to secure quantum key distribution or to perform multimode entanglement swapping.

V. QUANTITATIVE LOCALIZATION OF THE BLOCK ENTANGLEMENT

In this section, we will explicitly compute the block entanglement (i.e., the entanglement between different blocks of modes) for some instances of multimode Gaussian states. We will study its scaling behavior as a function of the number of modes and explore in deeper detail the localizability of the multimode entanglement. We focus our attention on fully symmetric $2n$-mode Gaussian states described by a $2n\times 2n$ CM $\boldsymbol{\sigma}_{\beta^{2n}}$ given by Eq. (14). These states are trivially bisymmetric under any bipartition of the modes, so that their block entanglement is always localizable by means of local symplectic operations. Let us recall that concerning the covariances in normal forms of fully symmetric states (see Sec. III), pure states are characterized by

$$z_i = \{1 + b^2(2n-2) - (2n-1) - (-1)^i \times \sqrt{(b^2-1)[(2bn)^2 - (2n-2)^2]}\}/[2b(2n-1)], \quad (40)$$

and belong to the class of CV GHZ-type states discussed in Refs. [9,11]. These multipartite entangled states are generated as the outputs of the application of a sequence of $2n-1$ beam splitters to $2n$ single-mode squeezed inputs [9]. In the limit of infinite squeezing, these states reduce to the simultaneous eigenstates of the relative positions and the total momentum, which define the proper GHZ states of CV systems [9]. The CM $\boldsymbol{\sigma}_{\beta^{2n}}^{p}$ of this class of pure states, for a given number of modes, depends only on the parameter $b \equiv 1/\mu_\beta \geqslant 1$, which is an increasing function of the single-mode squeezing. Correlations between the modes are induced according to the above expression for the covariances z_i. Exploiting our previous analysis, we can compute the entanglement between a block of k modes and the remaining $2n-k$ modes, both for pure states (in this case the block entanglement is simply the Von Neumann entropy of each of the reduced blocks) and, remarkably, also for mixed states.

We can in fact consider a generic $2n$-mode fully symmetric mixed state with CM $\boldsymbol{\sigma}_{\beta^{2n}}^{p\backslash q}$, obtained from a pure fully symmetric $(2n+q)$-mode state by tracing out q modes. For any q, for any dimension k of the block $(k\leqslant n)$, and for any nonzero squeezing (i.e., for $b>1$), one has that $\tilde{\nu}_k < 1$, meaning that the state exhibits genuine multipartite entanglement,

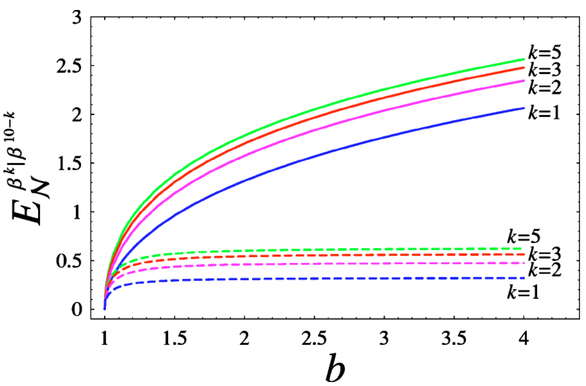

FIG. 1. (Color online) Hierarchy of block entanglements of fully symmetric $2n$-mode Gaussian states of $k\times(2n-k)$ bipartitions ($n=10$) as a function of the single-mode squeezing b. The block entanglements are depicted both for pure states (solid lines) and for mixed states obtained from fully symmetric $(2n+4)$-mode pure Gaussian states by tracing out four modes (dashed lines). All the quantities plotted are dimensionless.

as first remarked in Ref. [9] for pure states: each k-mode party is entangled with the remaining $(2n-k)$-mode block. Furthermore, the genuine multipartite nature of the entanglement can be precisely quantified by observing that $E_{\mathcal{N}}^{\beta^k|\beta^{2n-k}}$ is an increasing function of the integer $k\leq n$, as shown in Fig. 1. Moreover, we note that the multimode entanglement of mixed states remains finite also in the limit of infinite squeezing, while the multimode entanglement of pure states diverges with respect to any bipartition, as shown in Fig. 1.

In fully symmetric Gaussian states, the block entanglement is localizable with respect to any $k\times(2n-k)$ bipartition. Since in this instance *all* the entanglement can be concentrated on a single pair of modes, after the partition has been decided, no strategy could grant a better yield than the local symplectic operations bringing the reduced CMs in Williamson form (because of the monotonicity of the entanglement under general LOCC). However, the amount of block entanglement, which is the amount of concentrated two-mode entanglement after unitary localization has taken place, actually depends on the choice of a particular $k\times(2n-k)$ bipartition, giving rise to a hierarchy of localizable entanglements.

Let us suppose that a given Gaussian multimode state (say, for simplicity, a fully symmetric state) is available and its entanglement is meant to serve as a resource for a given protocol. Let us further suppose that the protocol is optimally implemented if the entanglement is concentrated between only two modes of the global systems, as is the case, e.g., in a CV teleportation protocol between two single-mode parties. Which choice of the bipartition between the modes allows for the best entanglement concentration by a succession of local unitary operations? In this framework, for an even number of modes, the worst localization strategy consists in assigning 1 mode (i.e., setting $k=1$) at one party and $2n-1$ modes to the other. Conversely, the best option for localization is an equal $k=n$ splitting of the $2n$ modes between the two parties. The logarithmic negativity $E_{\mathcal{N}}^{\beta^n|\beta^n}$, concentrated into two modes by local operations, represents the optimal

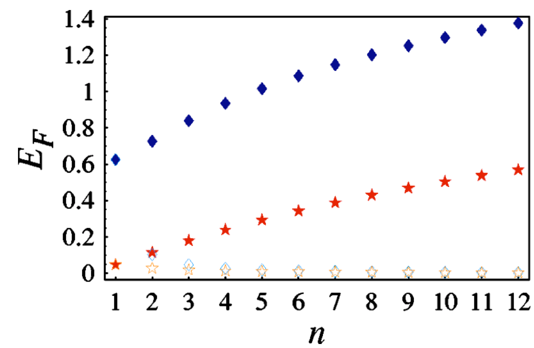

FIG. 2. (Color online) Scaling, with half the number of modes, of the entanglement of formation in two families of fully symmetric $2n$-mode Gaussian states. Diamonds denote pure states, while mixed states (denoted by stars) are obtained from $(2n+4)$-mode pure states by tracing out four modes. For each class of states, two sets of points are plotted, one referring to $n\times n$ entanglement (filled symbols), and the other to 1×1 entanglement (empty symbols). Notice how the $n\times n$ entanglement, equal to the optimal localizable entanglement (OLE) and estimator of genuine multipartite quantum correlations among all the $2n$ modes, increases to the detriment of the bipartite 1×1 entanglement between any pair of modes. The single-mode squeezing parameter is fixed at $b=1.5$. All the quantities plotted are dimensionless.

localizable entanglement (OLE) of the state $\boldsymbol{\sigma}_{\beta^{2n}}$, where "optimal" refers to the choice of the bipartition. Clearly, the OLE of a state with $2n+1$ modes is given by $E_{\mathcal{N}}^{\beta^{n+1}|\beta^n}$. These results may be applied to arbitrary, pure or mixed, fully symmetric Gaussian states.

We now turn to the study of the scaling behavior with n of the OLE of $2n$-mode states, to understand how the number of local cooperating parties can improve the maximal entanglement that can be shared between two parties. For generic (mixed) fully symmetric $2n$-mode states of $n\times n$ bipartitions, the OLE can be quantified also by the entanglement of formation E_F, as the equivalent two-mode state is symmetric. It is then useful to compare, as a function of n, the 1×1 entanglement of formation between a pair of modes (all pairs are equivalent due to the global symmetry of the state) before the localization, and the $n\times n$ entanglement of formation, which is equal to the optimal entanglement concentrated in a specific pair of modes after performing the local unitary operations. The results of this study are shown in Fig. 2. The two quantities are plotted at fixed squeezing b as a function of n both for a pure $2n$-mode state with CM $\boldsymbol{\sigma}_{\beta^{2n}}^{p}$ and a mixed $2n$-mode state with CM $\boldsymbol{\sigma}_{\beta^{2n}}^{p\setminus4}$. As the number of modes increases, any pair of modes becomes steadily less entangled, but the total multimode entanglement of the state grows and, as a consequence, the OLE increases with n. In the limit $n\rightarrow\infty$, the $n\times n$ entanglement diverges while the 1×1 one vanishes. This holds both for pure and mixed states, although the global degree of mixedness produces the typical behavior that tends to reduce the total entanglement of the state.

VI. CONCLUDING REMARKS

We have shown that bisymmetric multimode Gaussian states (pure or mixed) can be reduced, by local symplectic operations, to the tensor product of a correlated two-mode Gaussian state and of uncorrelated thermal states (the latter being obviously irrelevant as far as the correlation properties of the multimode Gaussian state are concerned). As a consequence, *all* the entanglement of bisymmetric multimode Gaussian states of arbitrary $m \times n$ bipartitions is *unitarily localizable* in a single (arbitrary) pair of modes shared by the two parties. Such a useful reduction to two-mode Gaussian states is somehow similar to the one holding for states with fully degenerate symplectic spectra [23,24], encompassing the relevant instance of pure states, for which all the symplectic eigenvalues are equal to 1. The present result allows us to extend the PPT criterion as a necessary and sufficient condition for separability for all bisymmetric multimode Gaussian states of arbitrary $m \times n$ bipartitions, and to quantify their entanglement.

Notice that, in the general bisymmetric instance addressed in this work, the possibility of performing a two-mode reduction is crucially partition-dependent. However, as we have explicitly shown, in the case of fully symmetric states all the possible bipartitions can be analyzed and compared, yielding remarkable insight into the structure of the multimode block entanglement of Gaussian states. This leads finally to the determination of the maximum, or optimal localizable entanglement that can be concentrated on a single pair of modes.

It is important to notice that the multipartite entanglement in the considered class of multimode Gaussian states can be produced and detected [9,35], and also, by virtue of the present analysis, reversibly localized by all-optical means. Moreover, the multipartite entanglement allows for a reliable (i.e., with fidelity $\mathcal{F} > \mathcal{F}_c$, where $\mathcal{F}_c = 1/2$ is the classical threshold) quantum teleportation between any two parties with the assistance of the remaining others [35]. This quantum teleportation network has been recently demonstrated experimentally with the use of fully symmetric three-mode Gaussian states [36].

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