Collective decoherence of the superpositional entangled states in the quantum Shor algorithm

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We consider collective decoherence for the quantum Shor algorithm. A quantum computer which interacts with its environment is modeled by a spin-1/2 chain interacting with harmonic oscillators at a given temperature. We calculate the nondiagonal matrix elements of the density matrix which are important for implementation of the quantum Shor algorithm, and study the decay rate and the Lamb phase shift for these elements. It is shown that the probability of superdecoherence in the quantum Shor algorithm is extremely small. The conditions for preserving quantum entanglement are formulated.

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I. INTRODUCTION

The problem of the decoherence of entangled states is well known as one of the major obstacles in quantum information processing [1,2]. In the simplest model of decoherence first considered in [3], every qubit independently interacts with its own environment. It was first pointed out in [4] that in many situations one faces the opposite case: all qubits interact with the same environment. This case is commonly called collective decoherence (CD). In the semiclassical approach, CD appears if the size of a quantum register is smaller than the correlation length of the bath: in this case all qubits experience the same fluctuation of the bath. In particular, it was shown in [4] that CD causes the phenomena of superdecoherence and subdecoherence: the decoherence sharply increases for one group of the nondiagonal elements of the density matrix and completely disappears for another group. Since then CD has been largely studied theoretically [5–11]. Experimental studies of CD in application to quantum-information processing have been reported for optical systems [12], ion traps [13], and nuclear magnetic resonance (NMR) in liquids [14–16]. The simplest experimental implementation of CD is NMR in a uniform magnetic field, which fluctuates in time in magnitude and direction [16]. There are mainly two factors that contribute to the decoherence of an entangled state. The first factor is a decrease of the magnitude of the nondiagonal density matrix elements [4]. This factor is referred as "decay," "damping," or "phase damping." The second factor is the relative phase shift between the nondiagonal matrix elements caused by the environment [7]. This factor does not affect the moduli of the matrix elements. However, it may destroy the quantum interference as well as the phase damping. This factor is referred to as the "Lamb shift" or "Lamb phase shift." Note that the collective Lamb shift is a unitary evolution, which may induce entanglement between the qubits.

The powerful error-correction codes first suggested in [17,18] are based on the assumption that the most probable errors occur independently to one or a few qubits. For the case of CD other approaches have been suggested. One of them, first proposed in [4] and later developed in [5–8], relies on the use of subdecoherence. A logical qubit is represented by a group of physical qubits in such a way that the states of the logical qubits are free from decoherence (decoherence-free subspaces). In particular, it was shown that universal quantum computation is possible within decoherence-free subspaces. The other more abstract approach first introduced in [9] proposes to represent information in terms of the conserved quantities of the quantum system, which are not affected by noise (noiseless subsystems). However, both decoherence-preventive schemes require additional computation resources and complexity. Consequently, it is important to discuss ways to preserve entanglement without application of complicated decoherence-preventive schemes.

In this paper, we consider the implementation of the quantum Shor algorithm (QSA) in the presence of CD. In the second section, we introduce an extremely simplified scheme of the QSA. We identify the nondiagonal elements of the density matrix that are important for QSA implementation in our scheme. In the third section, we consider both the decay and the Lamb shift of those matrix elements. Our statistical analysis demonstrates that the probability of superdecoherence in the QSA is extremely small. We formulate conditions for preserving quantum entanglement in our QSA scheme.

II. SIMPLE SCHEME FOR QSA IMPLEMENTATION

In this section, we describe a simple scheme for QSA implementation which we will use to analyze CD. As an example, we consider the decomposition of the number $N=15$ in a quantum computer (QC) with 4 qubits in the first n register, and 4 qubits in the second $f(n)$ register.

(1) One selects a coprime number, say $c=7$.

(2) A QC computes the function

$$f(n)=c^n(\mathrm{mod}\ N)=7^n(\mathrm{mod}\ 15). \qquad (1)$$

After this, a measurement on the $f(n)$ register is performed and some value f_0 is obtained, e.g., $f_0=7$. After the computation the state of the QC is

$$|\Psi\rangle = \frac{1}{\sqrt{D}}\sum_{n=0}^{D-1}|n,f(n)\rangle = \frac{1}{4}\{|0,1\rangle + |1,7\rangle + |2,4\rangle + |3,13\rangle + |4,1\rangle + \cdots + |15,13\rangle\}, \quad (2)$$

and after the measurement the state of the QC is

$$|\Psi\rangle = [(T_0/D)^{1/2}]\sum_n{}'|n\rangle = (1/2)\{|1\rangle + |5\rangle + |9\rangle + |13\rangle\}.$$

Here $D=2^4$ is the number of basis states in the n register; Σ_n' indicates that the sum is taken over the values n, $0\leqslant n\leqslant D-1$, which satisfy the equation $f(n)=f_0$, T_0 is the period of the function $f(n)$, and we use decimal notation to express the states of the n and $f(n)$ registers.

(3) The QC performs a discrete Fourier transform in the n register. The new state of the QC is

$$|\Psi\rangle = (T_0^{1/2}/D)\sum_{k=0}^{D-1}\sum_n{}' \exp(i2\pi kn/D)|k\rangle. \quad (3)$$

Quantum interference selects the values of k that are associated with the period of the function $f(n)$ and eliminates all other values of k. In our case, a QC selects the values $k=0,4,8,12$. As an example, for $k=1$ we have

$$\sum_n{}' \exp(2\pi ni/16) = 0. \quad (4)$$

(4) Measurement of the state of the n register reveals one of the selected values $k=0,4,8,12$.

(5) After multiple repetition of steps 1–4 one takes the fractions D/k for selected values of k (in the lowest terms, for nonzero values of k) and finds the period $T_0=4$ which is the maximum numerator in the fractions D/k.

(6) One computes the greatest common divisor (GCD)

$$\mathrm{GCD}(c^{T_0/2}\pm 1;N) = \mathrm{GCD}(49\pm 1;15), \quad (5)$$

which, in our case, provides both factors 3 and 5.

In order to describe the decoherence, we reformulate steps 2 and 3 in the QSA scheme in terms of the density matrix ρ. After the second step, the state of the QC is described by the density matrix

$$\rho = (T_0/D)\sum_{n,n'}{}'|n\rangle\langle n'|, \quad (6)$$

where $0\leqslant n,n'\leqslant D-1$, and $f(n)=f(n')=f_0$. After the third step, the density matrix of the QC becomes

$$\rho = (T_0/D^2)\sum_{k,k'=0}^{D-1}\sum_{n,n'}{}' \exp\{2\pi i(kn-k'n')/D\}|k\rangle\langle k'|. \quad (7)$$

In this equation, the sum of the diagonal matrix elements

$$(T_0/D^2)\sum_{k=0}^{D-1}\sum_{n,n'}{}' \exp[2\pi ik(n-n')/D]|k\rangle\langle k| \quad (8)$$

describes the probabilities of measurement at step 4. Equation (8) selects the same values of k as Eq. (3).

Next, we make a major simplification. We assume that the time for implementation of each step in the QSA scheme is negligible, i.e., each step is decoherence-free. In this case, the decoherence occurs between the steps 2 and 3, and between the steps 3 and 4. However, the decoherence between the steps 3 and 4 does not influence the results of measurement in step 4 which depends only on the values of the diagonal elements of the density matrix. Thus, only the decoherence between the steps 2 and 3 is important. Note that the nondiagonal terms in Eq. (6) can be written as

$$|n\rangle\langle n\pm pT_0|,\ p=1,2,\ldots\ . \quad (9)$$

To implement the QSA one only needs to preserve the nondiagonal terms (9) between steps 2 and 3.

III. CD IN THE QSA

We now consider the simple model of CD first introduced in [4]. The qubits are described by effective spin-1/2 operators. The environment is represented by a system of oscillators with continuously distributed frequencies. We analyze the interaction between the environment and the z component of the effective spin. Thus, our model does not include a spin relaxation, i.e., exchange of energy with the environment. We assume this system of environmental oscillators to be initially in thermal equilibrium. The Hamiltonian of the QC and environment is

$$\mathcal{H} = \sum_q \omega_q(a_q^\dagger a_q + 1/2) + \sum_j I_{jz}\sum_q(\lambda_q a_q^\dagger + \mathrm{H.c.}) + V_0 + V. \quad (10)$$

Here ω_q is the frequency of the qth oscillator, $a_q^\dagger$ and a_q are the creation and annihilation operators, I_{jz} is the operator of the z component of the jth spin, λ_q is the constant of interaction between spins and the qth oscillator, the operator V_0 describes the Zeeman interaction of spins with the permanent magnetic field which points in the z direction, and the permanent interactions between the z components of the spins (it can be, for example, the Ising interaction), the operator V describes the interactions that provide implementation of the QSA (it can be either the interaction between the spins of the QC and the pulses of the external field or short-time interaction between the spins caused by the action of the external field pulses), and we put $\hbar=1$. (Note that in the "ideal case" permanent interactions in the QC are absent, i.e., $V_0=0$.) In this model, the basis computational states $|n,f(k)\rangle$ are the eigenstates of the Hamiltonian when $V=0$. The interaction between the spins and the environment destroys the quantum superposition of the basis states.

First, we transfer to the interaction representation

$$\mathcal{H}_{\mathrm{int}}(t) = \exp(i\mathcal{H}_0 t)\mathcal{H}_1\exp(-i\mathcal{H}_0 t), \quad (11)$$

where $\mathcal{H}_0$ is the first term in Eq. (10) and $\mathcal{H}_1$ is the second term. Using the relations

$$\exp(\xi a^\dagger a)a^\dagger\exp(-\xi a^\dagger a) = a^\dagger e^{\xi},$$

$$\exp(\xi a^\dagger a)a\exp(-\xi a^\dagger a) = ae^{-\xi}, \quad (12)$$

we obtain from Eq. (11)

$$\mathcal{H}_{\rm int}(t) = \sum_j I_{jz} \sum_q (\lambda_q e^{i\omega_q t} a_q^\dagger + \text{H.c.}). \qquad (13)$$

Next, we compute the evolution operator

$$U = \hat{T} \exp\left\{ - i \int_0^t \mathcal{H}_{\rm int}(\tau) d\tau \right\}. \qquad (14)$$

Dividing the integration interval $(0,t)$ into small intervals $\Delta\tau$ and using the well-known expression for operators A and B with the commutator $[A,B]$,

$$e^{A+B} = e^{-[A,B]/2} e^A e^B, \qquad (15)$$

we obtain from Eqs. (14) and (15)

$$U = \prod_{m>k} \exp\left\{ - \frac{\Delta\tau^2}{2} [\mathcal{H}_{\rm int}(t_m), \mathcal{H}_{\rm int}(t_k)] \right\} \times \exp\left\{ - i \int_0^t \mathcal{H}_{\rm int}(\tau) d\tau \right\}, \qquad (16)$$

where

$$[\mathcal{H}_{\rm int}(t_m), \mathcal{H}_{\rm int}(t_k)] = - 2i \left(\sum_j I_{jz} \right)^2 \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k). \qquad (17)$$

Finally, we can write the expression for U in the form

$$U = \prod_{m>k} \exp\left\{ i\Delta\tau^2 \left(\sum_j I_{jz} \right)^2 \sum_q |\lambda_q|^2 \sin\omega_q (t_m - t_k) \right\} \times \exp\left\{ 2 \sum_j I_{jz} \sum_q [\lambda_q \eta_q(t) a_q^\dagger - \text{H.c.}] \right\}, \qquad (18)$$

$$\eta_q(t) = \frac{1 - \exp(i\omega_q t)}{2\omega_q}.$$

Note that the first factor in this expression does not contain the operators $a^\dagger$ and a.

After the second step in the QSA scheme the density matrix of the QC and environment can be represented as

$$\rho(t) = (T_0/D) \sum_{n,n'} {}' U \rho_e(0) |n\rangle\langle n'| U^\dagger. \qquad (19)$$

Here $\rho_e(0)$ is the initial density matrix of the environment

$$\rho_e(0) = \prod_q [1 - \exp(- \omega_q/T)] \exp[- (\omega_q/T) a_q^\dagger a_q], \qquad (20)$$

where we put $k_B = 1$.

The process of decoherence in the QC is caused by entanglement between the thermal environment and the QC. The reduced density matrix of the QC can be represented as

$$(T_0/D) \sum_{n,n'}{}' \rho_e^{nn'} |n\rangle\langle n'|, \qquad (21)$$

where $\rho_e^{nn'}$ is the environmental factor,

$$\begin{aligned}\rho_e^{nn'} = \text{Tr}\Bigg\{ & \prod_{m>k} \exp\left[- i\Delta\tau^2 I_z^2(n) \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k) \right] \\ & \times \exp\left[2 I_z(n) \sum_q [\lambda_q \eta_q(t) a_q^\dagger - \text{H.c.}] \right] \rho_e(0) \\ & \times \prod_{m>k} \exp\left[i\Delta\tau^2 I_z^2(n') \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k) \right] \\ & \times \exp\left[- 2 I_z(n') \sum_q [\lambda_q \eta_q(t) a_q^\dagger - \text{H.c.}] \right] \Bigg\}. \end{aligned} \qquad (22)$$

Here $I_z(n)$ is the spin z component in the state $|n\rangle$, and $I_z^2(n) = [I_z(n)]^2$. Direct computation of the trace in Eq. (22) results in the following expression [4,7]:

$$\rho_e^{nn'} = \exp\{ i\alpha(t) [I_z^2(n) - I_z^2(n')] \} \exp\{ - \beta(t) [I_z(n) - I_z(n')]^2 \}, \qquad (23)$$

where

$$\begin{aligned}\alpha(t) &= \Delta\tau^2 \sum_{m>k} \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k) \\ &= \sum_q |\lambda_q|^2 \int_0^t dt' \int_0^{t'} dt'' \sin \omega_q (t' - t'') \\ &= \sum_q \frac{|\lambda_q|^2}{\omega_q^2} (\omega_q t - \sin \omega_q t), \end{aligned} \qquad (24)$$

$$\beta(t) = 2 \sum_q |\lambda_q \eta_q(t)|^2 \coth(\omega_q/2T).$$

Here we take the limit $\Delta\tau \to 0$.

The first factor in the expression for $\rho_e^{nn'}$ describes the Lamb phase shift, and the second factor describes the decay of the off-diagonal matrix elements. The Lamb phase shift is zero for a single spin and is equal to $\alpha(t)$ for two spins. The decay of the nondiagonal matrix element for a single spin is determined by the value $\beta(t)$. Expression (23) describes the phenomenon of superdecoherence for the decay rate if the difference $[I_z(n) - I_z(n')]^2$ is close to its maximum possible value L^2, where L is the number of qubits in the n register. Correspondingly, it describes a superdecoherence for the Lamb phase shift if the difference $|I_z^2(n) - I_z^2(n')|$ is close to $L^4/4$.

The expressions for $\alpha(t)$ and $\beta(t)$ can be rewritten in terms of the dispersion relation $dq/d\omega$ and the density of states $G(\omega)$:

$$\alpha(t) = \int d\omega\, \kappa^2(\omega) \frac{\omega t - \sin(\omega t)}{\omega^2},$$

$$\beta(t) = 2 \int d\omega\, \kappa^2(\omega) \frac{\sin^2(\omega t/2)}{\omega^2} \coth\left(\frac{\omega}{2T} \right),$$

$$\kappa^2(\omega) = |\lambda(\omega)|^2 G(\omega) \frac{dq}{d\omega}.$$

Now we should recall that the only nondiagonal matrix elements that are important for the QSA are given by Eq. (9).

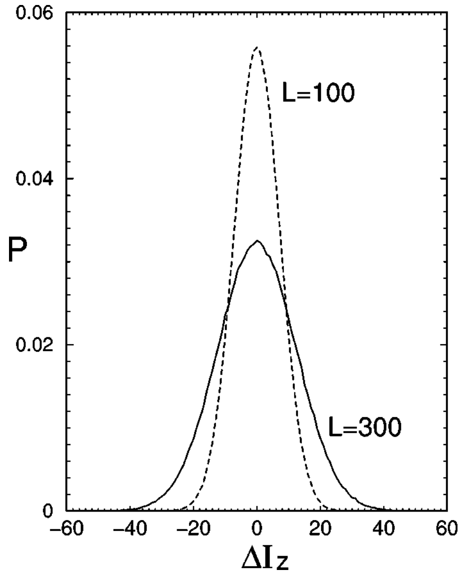

FIG. 1. The probability distribution $P(\Delta I_z)$ for $L=100$ (dashed line), and for $L=300$ (solid line) with 10 000 trials.

Thus, we should find the probability distribution for the values ΔI_z and ΔI_z^2,

$$\Delta I_z = I_z(n_0 + pT_0) - I_z(n_0 + p'T_0),$$

$$\Delta I_z^2 = I_z^2(n_0 + pT_0) - I_z^2(n_0 + p'T_0), \quad p < p', \qquad (25)$$

where n_0 is the minimum value of n; (n_0+pT_0) and $(n_0+p'T_0)$ count all other values of n that correspond to a given value of the function $f(n)$.

Suppose that a QC is to factor a large number N. Typically, the period T_0 of the function $f(n)$ is large. We choose randomly a number T_0 and a number $n_0<T_0$. Then, we compute the values in (25) for the states $|n_0+pT_0\rangle$ and $|n_0+p'T_0\rangle$. In our computer simulations we consider two cases: the number of qubits in the n register is $L=100$ and 300. For the first case, we randomly choose the value $T_0<7.9\times10^{28}$ and for the second case $T_0<1.3\times10^{89}$. For each value of T_0 we choose an arbitrary value of n_0 $(n_0<T_0)$ and compute ΔI_z and ΔI_z^2 for $p<p'\leqslant 10$. Then we repeat the computations for other values of T_0 and n_0. Figure 1 shows the probability distribution for ΔI_z. Figure 2 shows the probability distribution for ΔI_z^2. One can see that the probability of superdecoherence in the QSA is extremely small. As an example, for $L=300$ the ratio $(\Delta I_z/L)^2$ is smaller than 0.0256, and the ratio $4\Delta I_z^2/L^2$ is smaller than 0.019 with the probability $(1-10^{-4})$.

Next, we discuss the conditions for preserving quantum entanglement in the QSA. We denote the time interval between steps 2 and 3 of our QSA scheme (the characteristic time of computation) by t_0. Suppose that P_0 is the acceptable probability of the QSA failing. From the probability distributions we can estimate the boundary values ΔI_{z0} and ΔI_{z0}^2 which satisfy the conditions

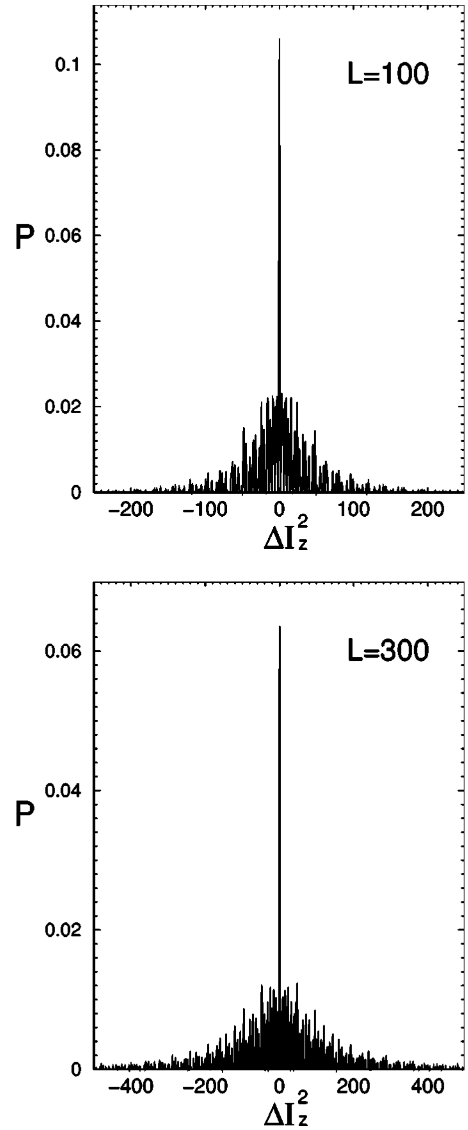

FIG. 2. The probability distribution $P(\Delta I_z^2)$ for $L=100$ and 300 with 10 000 trials.

$$P(|\Delta I_z| > \Delta I_{z0}) < P_0, \quad P(|\Delta I_z^2| > \Delta I_{z0}^2) < P_0. \qquad (26)$$

Then the conditions for preserving the quantum entanglement in our QSA scheme can be formulated as

$$\beta(t_0)(\Delta I_{z0})^2 < P_0, \quad \alpha(t_0)\Delta I_{z0}^2 < P_0. \qquad (27)$$

From these conditions we can derive the requirements for the values $\beta(t_0)$ and $\alpha(t_0)$. As an example, for $L=300$ and $P_0=10^{-4}$ we obtain

$$\beta(t_0) < 4\times10^{-8}, \quad \alpha(t_0) < 2\times10^{-7}. \qquad (28)$$

As an example of a simple estimation let us assume that about $M=10$ low-frequency phonon modes cause the collective decoherence in a solid-state spin QC. If the size of the sample $L\sim1$ cm and the speed of elastic waves $c\sim500$ m/s, then the phonon's frequency is $\omega_q/2\pi\approx c/L\approx50$ kHz. We assume that the distribution of the phonon frequencies has a width $\Delta\omega$, which is small compared to the average frequency $\langle\omega_q\rangle=\omega$. Let us take a temperature $T \gg \omega$, e.g., $T=1$ mK, the interaction constant $\lambda_q \ll \omega$, e.g., $\lambda_q \approx \lambda = 10^{-6}\omega$, and consider a relatively large computation time ωt_0, $\Delta\omega t_0 \gg 1$. From Eq. (24) we estimate the functions $\alpha(t_0)$ and $\beta(t_0)$:

$$\alpha(t_0) \approx 5 \times 10^{-6} t_0, \quad \beta(t_0) \approx 8 \times 10^{-9}. \tag{29}$$

One can see that the parameter β does not depend on t_0 and satisfies the first inequality (28). From the second inequality (28) we obtain the requirement for the computation time t_0: $t_0 < 0.04$ s.

IV. CONCLUSION

We discuss CD in the QSA without application of decoherence-preventive methods. We analyze a simple QSA scheme and estimate those nondiagonal density matrix elements that are important for QSA implementation. Next, we consider CD for these matrix elements taking into account both the decay and the Lamb phase shift. We show that the probability of the superdecoherence in the QSA is extremely small. Finally, we formulate the conditions required for preserving quantum entanglement in our QSA scheme.

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