Solution to the king's problem with observables that are not mutually complementary

Minoru Horibe, Akihisa Hayashi, and Takaaki Hashimoto
Department of Applied Physics, University of Fukui, Fukui 910, Japan
(Received 27 October 2004; published 23 March 2005)

We investigate the king's problem of the measurement of operators $\vec{n}_k \cdot \vec{\sigma}$ ($k=1, 2, 3$) instead of the three Cartesian components σ_x, σ_y, and σ_z of the spin operator $\vec{\sigma}$. Here, $\vec{n}_k$ are three-dimensional real unit vectors. We show the condition over three vectors $\vec{n}_k$ to ascertain the result for measurement of any one of these operators.

DOI: 10.1103/PhysRevA.71.032337 PACS number(s): 03.67.Hk, 03.65.Ud

I. INTRODUCTION

In the context of giving the method for inferring the outcome for measurement of any one of three Cartesian components of spin with certainty, Vaidman, Aharonov, and Albert [1] introduced the problem which became known later as the king's problem of a spin-$\frac{1}{2}$ particle.

Step 1. Alice sends a spin-$\frac{1}{2}$ particle to Bob.

Step 2. Bob chooses any observable of σ_x, σ_y, and σ_z, and measures it for the particle received to obtain the result $\beta(=\pm 1)$. After that, Bob sends the particle back to Alice.

Step 3. Alice carries out some measurements for the particle, before Bob tells her which observable was chosen. From this information and the result of the measurement, Alice infers the value β with certainty.

The solution using an entangled state of two particles was given in that paper.

Three operators σ_x, σ_y, and σ_z are complete in the sense that the density matrix under consideration is uniquely determined from the probabilities for finding the eigenstates of these operators. In addition to completeness, these operators are mutually complementary, namely, the eigenstates of each operator of them form mutually unbiased bases (MUB),

$$|\langle \beta, \vec{e}_x | \beta', \vec{e}_y \rangle| = |\langle \beta, \vec{e}_y | \beta', \vec{e}_z \rangle| = |\langle \beta, \vec{e}_z | \beta', \vec{e}_x \rangle| = \frac{1}{\sqrt{2}},$$

where $|\beta, \vec{e}_x\rangle, |\beta, \vec{e}_y\rangle$, and $|\beta, \vec{e}_z\rangle$ are eigenstates with eigenvalue $\beta(=\pm 1)$ for observables σ_x, σ_y, and σ_z, respectively.

When we try to extend this to the problem in D-dimensional Hilbert space, at least $D+1$ noncommuting observables are required for complete state determinations, so that we need $D+1$ mutually unbiased bases. However, only when the dimension of the Hilbert space is prime power are $D+1$ mutually unbiased bases obtained [2], and we have some evidence that the number of MUB is less than $D+1$ for the case where D is not equal to prime power [3–6]. For this reason, the solutions for the king's problem in prime power dimensional Hilbert spaces [7–9] are found.

Ben-Menahem [10] investigated a more general case for a spin-$\frac{1}{2}$ particle where three observables $\vec{n}_k \cdot \vec{\sigma}$ ($k=1, 2, 3$) are used in step 2, instead of σ_x, σ_y, and σ_z, and Alice makes a projective measurement in step 3. Here $\vec{n}_k$ is a real unit vector and is linearly independent of but not orthogonal to each other. These operators are complete, but a collection of orthonormal bases formed by the eigenstates of these operators is not MUB.

In this paper, we consider the same case as he did except that the POVM measurement is made at step 3. Our method is simpler than his, although we obtain the same results. Comparison will be made in Sec. III.

II. MODIFIED KING'S PROBLEM

We try to find the solution for the modified king's problem which is obtained by exchanging three observables σ_x, σ_y, and σ_z in the original king's problem for $\vec{n}_k \cdot \vec{\sigma}$ ($k=1$, 2, 3), following the procedure in the Introduction.

Step 1. Alice prepares the entangled state $|\Psi_0\rangle$ of two particles with spin $\frac{1}{2}$,

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|+1, \vec{e}_z\rangle \otimes |-1, \vec{e}_z\rangle - |-1, \vec{e}_z\rangle \otimes |+1, \vec{e}_z\rangle),$$

where $|\pm 1, \vec{e}_z\rangle$ is an eigenstate of the operator σ_z with eigenvalues ± 1. Since this state is a singlet state and invariant under the rotation, we can represent $|\Psi_0\rangle$ in the same form by using the eigenstates $|\pm 1, \vec{n}_k\rangle$ with eigenvalues ± 1 of the operator $\vec{n}_k \cdot \vec{\sigma}$,

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|+1, \vec{n}_k\rangle \otimes |-1, \vec{n}_k\rangle - |-1, \vec{n}_k\rangle \otimes |+1, \vec{n}_k\rangle).$$

Alice sends the second particle to Bob.

Step 2. Bob chooses any one of three observables $\vec{n}_k \cdot \vec{\sigma}$ ($k=1, 2, 3$). Bob gets the value β from the measurements of it and sends this particle back to Alice. Then, Alice has two particles which are in the state,

$$|-\beta, \vec{n}_k\rangle \otimes |\beta, \vec{n}_k\rangle = \frac{1}{\sqrt{2}}(|\vec{n}_k\rangle - \beta|\Psi_0\rangle), \tag{2.1}$$

where $|\vec{n}_k\rangle$ is given by a linear combination whose coefficients are equal to components of the vector $\vec{n}_k$,

$$|\vec{n}_k\rangle = (n_k)_x|X\rangle + (n_k)_y|Y\rangle + (n_k)_z|Z\rangle,$$

and $|X\rangle, |Y\rangle$, and $|Z\rangle$ are defined by

$$|X\rangle = \frac{i}{\sqrt{2}}(|+1, \vec{e}_z\rangle \otimes |+1, \vec{e}_z\rangle + |-1, \vec{e}_z\rangle \otimes |-1, \vec{e}_z\rangle),$$

TABLE I. The probability for the outcome related to $K(K=A,\dots,H)$.

Bob's choice	β	A	B	C	D	E	F	G	H
$\vec{n}_1\cdot\vec{\sigma}$	+1	0	0	nonzero	nonzero	nonzero	nonzero	0	0
$\vec{n}_1\cdot\vec{\sigma}$	−1	nonzero	nonzero	0	0	0	0	nonzero	nonzero
$\vec{n}_2\cdot\vec{\sigma}$	+1	0	nonzero	0	nonzero	nonzero	0	nonzero	0
$\vec{n}_2\cdot\vec{\sigma}$	−1	nonzero	0	nonzero	0	0	nonzero	0	nonzero
$\vec{n}_3\cdot\vec{\sigma}$	+1	0	nonzero	nonzero	0	nonzero	0	0	nonzero
$\vec{n}_3\cdot\vec{\sigma}$	−1	nonzero	0	0	nonzero	0	nonzero	nonzero	0

$$|Y\rangle = \frac{1}{\sqrt{2}}(|+1,\vec{e}_z\rangle \otimes |+1,\vec{e}_z\rangle - |-1,\vec{e}_z\rangle \otimes |-1,\vec{e}_z\rangle),$$

$$|Z\rangle = \frac{1}{\sqrt{2}}(|+1,\vec{e}_z\rangle \otimes |-1,\vec{e}_z\rangle + |-1,\vec{e}_z\rangle \otimes |+1,\vec{e}_z\rangle).$$

Step 3. In this step, it is assumed that Alice makes a POVM measurement, because a POVM measurement is more general than a projective measurement. We can adopt the same strategy as the original king's problem if there is a POVM set such that, for each k (k=1, 2, 3), the expectation value of an element of the POVM set in one of the two states $|-\beta,\vec{n}_k\rangle\otimes|\beta,\vec{n}_k\rangle(\beta=\pm1)$ is equal to zero and the expectation values of the same element in the other are not equal to zero. Since the most general POVM set like this needs $8=2^3$ elements, as is shown in Table I, we consider the king's problem in which Alice's measurement is described by the POVM set $\{E_K(K=A,B,\dots,H)\}$,

$$\sum_{K=A}^{H} E_K = \mathbf{1}_4, \tag{2.2}$$

$$E_K \geqslant 0(K=A,B,\dots,H), \tag{2.3}$$

where $\mathbf{1}_4$ is an identity operator on the Hilbert space of the two particles under consideration and the expectation values of the elements E_K for the state in Alice's hand are given in Table I. It is clear that Alice can infer the value β with certainty. For example, when Alice gets the outcome related to POVM element E_A and is told that Bob chose the observable $\vec{n}_1\cdot\vec{\sigma}$, Alice says that $\beta=-1$ since the probability for the outcome related to POVM element E_A of a measurement performed on the state $|-1,\vec{n}_1\rangle\otimes|+1,\vec{n}_1\rangle$ corresponding to $\beta=+1$ is zero. Similarly, for other cases, Alice can infer correct β.

Now we find the POVM set $\{E_K(K=A,B,\dots,H)\}$. First, we consider the operator E_A. As the operator E_A is positive, we can have the operator a_A such that

$$E_A = a_A^\dagger a_A.$$

From Table I, this operator a_A should satisfy three conditions,

$$a_A(|-1,\vec{n}_1\rangle \otimes |+1,\vec{n}_1\rangle) = \frac{1}{\sqrt{2}}a_A[|\vec{n}_1\rangle - (+1)|\Psi_0\rangle] = 0,$$

$$a_A(|-1,\vec{n}_2\rangle \otimes |+1,\vec{n}_2\rangle) = \frac{1}{\sqrt{2}}a_A[|\vec{n}_2\rangle - (+1)|\Psi_0\rangle] = 0,$$

$$a_A(|-1,\vec{n}_3\rangle \otimes |+1,\vec{n}_3\rangle) = \frac{1}{\sqrt{2}}a_A[|\vec{n}_3\rangle - (+1)|\Psi_0\rangle] = 0,$$

where we used Eq. (2.1). As three states $|\vec{n}_k\rangle-(+1)|\Psi_0\rangle$ (k =1, 2, 3) are linearly independent of each other in four-dimensional Hilbert space owing to a linear independence of three vectors $\vec{n}_k$, using state $\langle\Psi_0|+\sum_{k=1}^3(S^{(A)}M^{-1})_k\langle\vec{n}_k|$ orthogonal to these three states, the operator a_A is written in the form

$$a_A = |\Phi_A\rangle\left(\langle\Psi_0| + \sum_{k=1}^{3}(S^{(A)}M^{-1})_k\langle\vec{n}_k|\right), \tag{2.4}$$

where $|\Phi_A\rangle$ is an undetermined state from these conditions, $S^{(A)}$ is a three-dimensional real vector

$$S^{(A)} = (+1,+1,+1),$$

and M is a 3×3 matrix whose (i,j) component is given by the inner product between $\vec{n}_i$ and $\vec{n}_j$ and is invertible because three vectors $\vec{n}_i$ are linearly independent of each other. Thus we get POVM element E_A,

$$E_A = a_A^\dagger a_A = C_A\left(|\Psi_0\rangle + \sum_{k=1}^{3}(S^{(A)}M^{-1})_k|\vec{n}_k\rangle\right) \times\left(\langle\Psi_0| + \sum_{k=1}^{3}(S^{(A)}M^{-1})_k\langle\vec{n}_k|\right),$$

where C_A is a non-negative constant,

$$C_A = \frac{1}{2}\langle\Phi_A|\Phi_A\rangle.$$

Similarly, E_K is restricted to the form

$$E_K = C_K\left(|\Psi_0\rangle + \sum_{k=1}^{3}(S^{(K)}M^{-1})_k|\vec{n}_k\rangle\right) \times\left(\langle\Psi_0| + \sum_{k.l=1}^{3}(S^{(K)}M^{-1})_k\langle\vec{n}_k|\right)\ (K=A,B,C,\dots,H), \tag{2.5}$$

where three-dimensional vectors $(S^{(K)})_k$ are given by

$$S^{(B)}=(+1,-1,-1),\quad S^{(C)}=(-1,+1,-1),$$

$$S^{(D)}=(-1,-1,+1),\quad S^{(E)}=(-1,-1,-1),$$

$$S^{(F)}=(-1,+1,+1),\quad S^{(G)}=(+1,-1,+1),$$

$$S^{(H)}=(+1,+1,-1).$$

From the condition (2.2) of the POVM set, the constants C_K satisfy the equations

$$\sum_{K=A}^{H} C_K = 1, \tag{2.6}$$

$$\sum_{K=A}^{H} C_K (S^{(K)})_l = 0, \tag{2.7}$$

$$\sum_{K=A}^{H} C_K (S^{(K)})_k (S^{(K)})_l = (M)_{kl}. \tag{2.8}$$

In order to get these equations, we used the expansion of an identity matrix in a set $\{|[\Psi_0\rangle, |\vec{n}_k\rangle\ (k=1,2,3)\}$,

$$\mathbf{1}_4 = |\Psi_0\rangle\langle\Psi_0| + \sum_{k,l}^{3} |\vec{n}_k\rangle (M^{-1})_{kl} \langle \vec{n}_l|.$$

From these equations (2.6)–(2.8), we have seven independent equations for eight variables C_K,

$$\begin{aligned}
&(C_A+C_E)+(C_B+C_F)+(C_C+C_G)+(C_D+C_H)=1,\\
&(C_A+C_E)-(C_B+C_F)-(C_C+C_G)+(C_D+C_H)=\vec{n}_1\cdot\vec{n}_2,\\
&(C_A+C_E)-(C_B+C_F)+(C_C+C_G)-(C_D+C_H)=\vec{n}_1\cdot\vec{n}_3,\\
&(C_A+C_E)+(C_B+C_F)-(C_C+C_G)-(C_D+C_H)=\vec{n}_2\cdot\vec{n}_3,\\
&(C_A-C_E)+(C_B-C_F)-(C_C-C_G)-(C_D-C_H)=0,\\
&(C_A-C_E)-(C_B-C_F)+(C_C-C_G)-(C_D-C_H)=0,\\
&(C_A-C_E)-(C_B-C_F)-(C_C-C_G)+(C_D-C_H)=0,
\end{aligned} \tag{2.9}$$

and we get a solution with one parameter r,

$$C_A = \frac{1}{8}(1+r+\vec{n}_1\cdot\vec{n}_2+\vec{n}_1\cdot\vec{n}_3+\vec{n}_2\cdot\vec{n}_3),$$

$$C_B = \frac{1}{8}(1+r-\vec{n}_1\cdot\vec{n}_2-\vec{n}_1\cdot\vec{n}_3+\vec{n}_2\cdot\vec{n}_3),$$

$$C_C = \frac{1}{8}(1+r-\vec{n}_1\cdot\vec{n}_2+\vec{n}_1\cdot\vec{n}_3-\vec{n}_2\cdot\vec{n}_3),$$

$$C_D = \frac{1}{8}(1+r+\vec{n}_1\cdot\vec{n}_2-\vec{n}_1\cdot\vec{n}_3-\vec{n}_2\cdot\vec{n}_3),$$

$$C_E = \frac{1}{8}(1-r+\vec{n}_1\cdot\vec{n}_2+\vec{n}_1\cdot\vec{n}_3+\vec{n}_2\cdot\vec{n}_3),$$

$$C_F = \frac{1}{8}(1-r-\vec{n}_1\cdot\vec{n}_2-\vec{n}_1\cdot\vec{n}_3+\vec{n}_2\cdot\vec{n}_3),$$

$$C_G = \frac{1}{8}(1-r-\vec{n}_1\cdot\vec{n}_2+\vec{n}_1\cdot\vec{n}_3-\vec{n}_2\cdot\vec{n}_3),$$

$$C_H = \frac{1}{8}(1-r+\vec{n}_1\cdot\vec{n}_2-\vec{n}_1\cdot\vec{n}_3-\vec{n}_2\cdot\vec{n}_3).$$

Unfortunately, since the coefficient C_K is non-negative, all three unit vectors $\vec{n}_k$ which are linearly independent of each other are not permitted. However, we can easily see that we get this POVM set if linearly independent unit vectors $\vec{n}_k,(k=1,2,3)$ satisfy the following inequality:

$$1 > |\vec{n}_1\cdot\vec{n}_2| + |\vec{n}_2\cdot\vec{n}_3| + |\vec{n}_3\cdot\vec{n}_1|, \tag{2.10}$$

and it is clear that there are three vectors $\vec{n}_k$ satisfying this inequality. When we express the solution C_K with different forms,

$$C_A = \frac{1}{16}\{|\vec{n}_1+\vec{n}_2+\vec{n}_3|^2+2r-1\},$$

$$C_B = \frac{1}{16}\{|-\vec{n}_1+\vec{n}_2+\vec{n}_3|^2+2r-1\},$$

$$C_C = \frac{1}{16}\{|\vec{n}_1-\vec{n}_2+\vec{n}_3|^2+2r-1\},$$

$$C_D = \frac{1}{16}\{|\vec{n}_1-\vec{n}_2-\vec{n}_3|^2+2r-1\},$$

$$C_E = \frac{1}{16}\{|\vec{n}_1+\vec{n}_2+\vec{n}_3|^2-2r-1\},$$

$$C_F = \frac{1}{16}\{|-\vec{n}_1+\vec{n}_2+\vec{n}_3|^2-2r-1\},$$

$$C_G = \frac{1}{16}\{|\vec{n}_1-\vec{n}_2+\vec{n}_3|^2-2r-1\},$$

$$C_H = \frac{1}{16}\{|\vec{n}_1-\vec{n}_2-\vec{n}_3|^2-2r-1\},$$

we can find the necessary and sufficient condition which guarantees that these variables C_K are non-negative,

$$|\vec{n}_1\pm\vec{n}_2\pm\vec{n}_3| \geqslant 1, \tag{2.11}$$

for all combinations of signs in front of the second and the third terms of the left-hand side.

III. DISCUSSION AND SUMMARY

In this paper, we considered the modified king's problem that Bob chooses any one from three observables $\vec{n}_k \cdot \vec{\sigma}$, instead of σ_x, σ_y, and σ_z, and that he makes measurement of it in step 2. We showed that, if linearly independent unit vectors $\vec{n}_k$ satisfy the inequality (2.11), Alice can infer the result of Bob's measurement from the outcome for the measurement of POVM $E_K(K=A,B,\ldots,H)$ and the information of Bob's choice with certainty.

Ben-Menahem [10] considered the same model with projective measurement in step 3 and concluded that no inequalities were imposed on $\vec{n}_l \cdot \vec{n}_k$ beyond the geometric ones. He derived the equations for the coefficients b_A of expansion for an initially prepared state in eigenstates of the observable Alice measures in the final step,

$$\sum_A d_A = 1,$$

$$\sum_A \epsilon_A^{(l)} \epsilon_A^{(k)} d_A = \vec{n}_l \cdot \vec{n}_k,$$

where $d_A = |b_A|^2$ and $\epsilon_A^{(l)}$, which is a factor related to Alice's strategy, takes ± 1. The first equation is the normalization condition for an initially prepared state. Replacing d_A and $\epsilon_A^{(l)}$ with C_K and $S_l^{(K)}$, respectively, we can see that these equations become the same as Eqs. (2.6) and (2.8) we solved in Sec. II, although the physical meaning of d_A is different from that of C_K. It is shown that the vectors $\vec{n}_l$ have to satisfy our inequalities (2.11) from these equations and the positivity condition for d_A. For $|\Sigma_{k=1}^3 \vec{n}_k|$, we have

$$\left(\sum_{k=1}^{3} \vec{n}_k\right) \cdot \left(\sum_{l=1}^{3} \vec{n}_l\right) = \sum_{k,l=1}^{3} \sum_A \epsilon_A^{(l)} \epsilon_A^{(k)} d_A,$$

$$= \sum_A \left(\sum_{l=1}^{3} \epsilon_A^{(l)}\right)\left(\sum_{k=1}^{3} \epsilon_A^{(k)}\right) d_A,$$

$$\geqslant \sum_A d_A = 1.$$

Here we used the inequalities

$$\left|\sum_{l=1}^{3} \epsilon_A^{(l)}\right| \geqslant 1, \quad d_A \geqslant 0.$$

Similarly, other inequalities are obtained. Indeed, after tedious but not difficult calculation, we can see that our inequalities (2.11) for vectors $\vec{n}_k$ are equivalent to positivity conditions for d_A. If the conclusion in his paper [10] were right, any three vectors $\vec{n}_k$ would satisfy our inequalities (2.11). However, it is not difficult for us to find configuration of three vectors $\vec{n}_l$ which do not satisfy our inequalities. When, as these vectors $\vec{n}_k$, we choose vectors obtained by rotating three vectors on the x-y plane, such that the angles between each other are equal to $2\pi/3$, by a small angle toward the z axis, the inner products become

$$\vec{n}_1 \cdot \vec{n}_2 = -\frac{1}{2} + \delta_{12},$$

$$\vec{n}_2 \cdot \vec{n}_3 = -\frac{1}{2} + \delta_{23},$$

$$\vec{n}_3 \cdot \vec{n}_1 = -\frac{1}{2} + \delta_{31}(\delta_{12} + \delta_{23} + \delta_{31} < 1)$$

and we get

$$(\vec{n}_1 + \vec{n}_2 + \vec{n}_3) \cdot (\vec{n}_1 + \vec{n}_2 + \vec{n}_3) = \delta_{12} + \delta_{23} + \delta_{31} < 1.$$

Owing to the completeness of the observable $\vec{n}_k \cdot \vec{\sigma}$, we considered the case where three unit vectors $\vec{n}_k$ are linearly independent. We investigate the king's problem defined by three vectors $\vec{n}_k$ that are linearly dependent but that are not parallel. Without losing the generality, it is supposed that the vectors $\vec{n}_k$ satisfy

$$\vec{n}_3 = x\vec{n}_1 + y\vec{n}_2 (x,y \neq 0). \tag{3.1}$$

As these vectors are unit vectors, we have

$$(x \pm y)^2 = 1 - 2xy(\vec{n}_1 \cdot \vec{n}_2 \mp 1) \text{ or } (x \pm y)^2 \neq 1. \tag{3.2}$$

We return to the condition for a_K defined by $E_K = a_K^\dagger a_K$,

$$a_K(|-(S^{(K)})_k, \vec{n}_k\rangle \otimes |(S^{(K)})_k, \vec{n}_k\rangle)$$

$$= \frac{1}{\sqrt{2}} a_K\big[|\vec{n}_k\rangle - (S^{(K)})_k |\Psi_0\rangle\big] = 0. \tag{3.3}$$

Using Eq. (3.1), we have

$$\{(S^{(K)})_3 - x(S^{(K)})_1 - y(S^{(K)})_2\} a_K |\Psi_0\rangle = 0 \text{ or } a_K|\Psi_0\rangle = 0.$$

Here we used Eq. (3.2). The condition (3.3) is rewritten in the equations

$$a_K|\Psi_0\rangle = 0,$$

$$a_K|\vec{n}_1\rangle = a_K|\vec{n}_1\rangle = 0.$$

Therefore, we can express all E_K in the state $|\vec{n}_1 \times \vec{n}_2\rangle$ orthogonal to states $|\Psi_0\rangle, |\vec{n}_1\rangle$, and $|\vec{n}_2\rangle$,

$$E_K = C_K |\vec{n}_1 \times \vec{n}_2\rangle\langle \vec{n}_1 \times \vec{n}_2|.$$

However, the set $\{E_K\}$ is not POVM because $\Sigma_{K=A,B,\ldots,H} E_K \neq \mathbf{1}_4$. Thus there is no solution to the king's problem in this case.

Can we reduce the number of elements of the POVM set from eight to four like the original problem? In Eq. (2.9) of the previous section, substituting zero into the variables C_E, C_F, C_G, and C_H, we have the equations

$$C_A + C_B + C_C + C_D = 1,$$

$$C_A - C_B - C_C + C_D = \vec{n}_1 \cdot \vec{n}_2,$$

$$C_A - C_B + C_C - C_D = \vec{n}_1 \cdot \vec{n}_3,$$

$$C_A + C_B - C_C - C_D = \vec{n}_2 \cdot \vec{n}_3,$$

$$C_A + C_B - C_C - C_D = 0,$$

$$C_A - C_B + C_C - C_D = 0,$$

$$C_A - C_B - C_C + C_D = 0 .$$

If and only if these unit vectors $\vec{n}_k$ are orthogonal to each other, the solution exists to the above equations and a different element E_K of the POVM set in this solution extracts a different state of an orthonormal basis in four-dimensional Hilbert space of two spin-$\frac{1}{2}$ particles. Thus the modified king's problem results in the original one. When C_A, C_B, C_C, and C_D are zero, we can get the same result. We can show that there is no solution to the equations for other cases, when three vectors $\vec{n}_k$ are linearly independent, after tedious but not difficult calculations.

We had the solution for the king's problem using three observables which are complete but not mutually complementary for spin-$\frac{1}{2}$ particle. However, we have not discussed the problems such that (i) Bob chooses any one of two observables, (ii) Alice uses another entangled state, and (iii) the dimension of the Hilbert space is larger than 2.

We will discuss the king's problem for these cases elsewhere.

[1] Lev Vaidman, Yakir Aharonov, and David Z. Albert, Phys. Rev. Lett. **58**, 1385 (1987).
[2] W. K. Wootters and B. C. Fields, Ann. Phys. (N.Y.) **191**, 363 (1989).
[3] G. Zauner, dissertation, Universität Wien (1999).
[4] A. Klappenecker and M. Rötteler, e-print quant-ph/0309120.
[5] C. Archer, e-print quant-ph/0312204.
[6] W. K. Wootters, e-print quant-ph/0406032.
[7] Yakir Aharonov and Berthold-Georg Englert, Z. Naturforsch., A: Phys. Sci. **56a**, 16 (2001).
[8] Berthold-Georg Englert and Yakir Aharonov, Phys. Lett. A **284**, 1 (2001),
[9] P. K. Aravind, Z. Naturforsch., A: Phys. Sci. **58a**, 85 (2003).
[10] Shahar Ben-Menahem, Phys. Rev. A **39**, 1621 (1989).