Detecting multipartite entanglement

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We discuss the problem of determining whether the state of several quantum mechanical subsystems is entangled. As in previous work on two subsystems we introduce a procedure for checking separability that is based on finding state extensions with appropriate properties and may be implemented as a semidefinite program. The main result of this work is to show that there is a series of tests of this kind such that if a multiparty state is entangled this will eventually be detected by one of the tests. The procedure also provides a means of constructing entanglement witnesses that could in principle be measured in order to demonstrate that the state is entangled.

I. INTRODUCTION

Entanglement has long been recognized as one of the central features of quantum mechanics and has been a primary focus of research in quantum information science over recent years because of its central role in phenomena such as teleportation, quantum cryptography and violation of Bell inequalities [1]. A common theme of theoretical research is the notion that entanglement is a resource that often makes it possible to acomplish tasks that cannot be performed in analogous classical scenarios. However, much of this intuition is based on our theoretical understanding of pure states of two separated systems. For mixed states and for states of many separated systems much less is known. In this paper we address the question of how to determine whether a given mixed state of several subsystems is entangled.

Entangled states of separated quantum systems, atoms or photons for example, are those that cannot be prepared by local operations and classical communication. In order to prepare entangled states it is necessary to have a non-trivial coherent interaction between the different subsystems. As a result a state ρ of N subsystems defined in $\otimes_{i=1}^{N} \mathcal{H}_{A_i}$, is said to be *fully separable* [2] or simply *separable*, that is not entangled, if it can be written as

$$\rho = \sum_i p_i \bigotimes_{j=1}^{N} |\psi_i^{(A_j)}\rangle\langle\psi_i^{(A_j)}|, \tag{1}$$

where the $|\psi_i^{(A_j)}\rangle$ are state-vectors on the spaces $\mathcal{H}_{A_j}$ and $p_i > 0$, $\Sigma_i p_i = 1$. If such a decomposition does not exist, the state cannot be prepared by local operations and classical communication between the parties and is termed entangled. The so-called *separability problem* arises from the fact that even for the case of two parties, and even given complete information about the matrix elements of the density operator of the system, it is difficult to determine whether such a decomposition as a mixture of product pure states exists. Much of the difficulty arises because density matrices can generally be decomposed into many different ensembles of pure states.

The separability problem for bipartite systems has received much attention and we refer the reader to one of the several reviews [3–5]. However, as a result of recent work by Gurvits on the computational complexity of the problem [6] it is extremely unlikely that any completely satisfactory solution can exist. Since Gurvits showed that the separability problem for a given bipartite mixed state is in the complexity class NP-HARD, it is extremely unlikely that any algorithm that checks whether a quantum state is entangled can be performed with an amount of computation that is polynomial in the dimension of the Hilbert spaces involved.

The worst case complexity of the problem is not the end of the story. There are simple, efficiently computable, tests that can establish the entanglement of a large subset of states. The most well known of these is the positive partial transpose or Peres-Horodecki criterion [7–9]. This simply requires making an appropriate rearrangement of the matrix elements of ρ, corresponding to transposing one of the parties, and checking that the resulting matrix is positive. In [10] we proposed a hierarchy of separability criteria that can be thought of as a generalization of this condition but which can only be checked by solving a semidefinite program. We subsequently showed, based on earlier work [11,12], that this series of tests was complete in the sense that any entangled state of two subsystems would eventually be detected by one of the tests in our hierarchy [13]. Another attractive feature of these conditions is that if a given test successfully identifies that the state of interest is entangled it also constructs an observable, known as an entanglement witness, that could in principle be measured in order to demonstrate this entanglement experimentally.

Semidefinite programs are members of a class of convex optimizations that may be solved with arbitrary accuracy in polynomial time [14,15]. By identifying the separability criteria in [10,13] as semidefinite programs it was possible to assess the computational difficulty of the criteria and to construct entanglement witnesses when the criteria successfully determine that a given state is entangled. Techniques from convex optimization are being applied increasingly fre-

quently in quantum information, notable examples include [16–20].

A. Overview of results and relation to other work

In this paper we study the problem of deciding whether a given state is fully separable or not, extending our results from the bipartite case to the case of an arbitrary number of parties. One might think that an approach to the problem of determining whether a given multiparty state is entangled would be to consider the different ways that the subsystems can be collected into two groups and determine whether the resulting bipartite states are entangled. In fact, all pure entangled states will result in a bipartite entangled state for some grouping into two parties. As a result, the reduced density matrix for some subset of the systems will have nonzero entropy thus showing that the state is entangled. Indeed checking the reduced density matrix for each party will suffice for any pure state. It is clearly possible to determine efficiently whether or not a pure state is entangled in this way. For mixed states, however, no such solution is possible. There are entangled states that are separable whenever the parties are arranged into two groups, as was first shown by an example constructed from unextendible product bases [21]. The multipartite separability problem cannot be reduced to a series of bipartite separability problems. In general it is possible to classify states based on their separability when the N particles are grouped into any number $k \leq N$ of groups. This classification was developed in detail by Dür and collaborators [22,23]. The full structure of multipartite entanglement is more complicated than in the bipartite case, including states such as the the GHZ and W states of three qubits [24] that cannot be interconverted by local operations and classical communication. In this work we will not consider we will not distinguish among these different kinds of entanglement only whether the state is entangled.

Despite the extra difficulty of the multiparty case much of the structure of the bipartite separability problem is unchanged. There is a nice discussion of work on the multiparty separability problem in the review by Terhal [3]. A particularly important observation is that the set of fully separable states forms a compact convex set in the state space. In the bipartite case the separating hyperplane theorem of convex analysis guarantees that a state is entangled if and only if there is an observable known as an entanglement witness that detects this entanglement [8,9]. Entanglement witnesses are observables that have a positive expectation value for every separable state and a negative expectation value for some entangled state. Just as in the bipartite case the separating hyperplane theorem guarantees that if a multiparty state ρ is entangled then there is an observable W with a negative expectation value $\mathrm{Tr}[W\rho] < 0$ but a positive expectation value for all fully separable states [25]. This convexity structure and the resulting entanglement witnesses exactly mirror the bipartite separability problem.

Checking whether ρ is separable is equivalent to checking whether an entanglement witness exists. In the bipartite case this reduces to a problem that may be stated in terms of polynomial inequalities since entanglement witnesses map onto positive semidefinite bihermitian forms. The multipartite separability problem may still be phrased as quantified polynomial inequalities:

$$\forall \; W[\forall \; P_{\mathrm{prod}} \; \mathrm{Tr}[P_{prod}W] \geqslant 0 \Rightarrow \mathrm{Tr}[\rho W] \geqslant 0], \qquad (2)$$

where $P_{prod} = \otimes_j P_j$ is a pure product state and $P_j = |\psi^{(A_j)}\rangle\langle\psi^{(A_j)}|$ a rank one projector on $\mathcal{H}_{A_j}$. By writing the condition $\mathrm{Tr}[P_{prod}W] \geqslant 0$ in terms of the components of the various $|\psi^{(A_j)}\rangle$ it is clear that the polynomials that arise in the multipartite case are no longer bi-Hermitian but multihermitian; that is Hermitian in the sets of variables corresponding to each of the subsystems. If this proposition is satisfied then ρ is fully separable.

Problems that can be written in terms of quantified polynomials inequalities of a finite number of variables (the components of $W, |\psi^{(A_j)}\rangle$) are known as semialgebraic problems. Semialgebraic problems are known to be *decidable* by the Tarski-Seidenberg decision procedure [26] which provides an explicit algorithm to solve the separability problem in all cases and therefore to decide whether ρ is entangled. Exactly the same is true of the bipartite problem but as we noted in [13], exact techniques in algebraic geometry that could be used to solve the separability problem scale very poorly with the number of variables and tend not to perform well in practice except for very small problem instances. Such general methods of algebraic geometry have, however, been applied to the separability problem [27] and related problems [28].

As we noted above there are efficient procedures that, like the PPT test, demonstrate that a state is entangled in many cases. In general, algorithms that are able to solve in polynomial time many but not all problem instances of a computationally hard problem are not excluded by complexity theory (even presuming that $P \neq NP$). In fact in [29] one of us showed that for all semialgebraic problems it is possible to construct a series of semidefinite programs that are able to solve large classes of problem instances. A direct application of those techniques would lead to a complete hierarchy of efficiently computable separability criteria such that every entangled state would be detected at some level in the hierarchy. However, the most obvious version of this would result from writing all the variables and parameters (the state, the coefficients of W and so on) in terms of their real and imaginary parts and treating the resulting problem involving real polynomials as a question in real algebraic geometry, to which the methods of [29] apply directly. The resulting sequences of criteria would be difficult to interpret in terms of the original quantum mechanical problem structure. In this paper we show how to construct a complete series of multiparty separability criteria that, while falling in the general scheme of [29], may be phrased directly in terms of quantum mechanical states and observables.

A recent series of papers has considered a slightly different setting for both the bipartite and multipartite separability problems [30–32]. Brandão and Vianna point out that the separability problem is an example of a class of convex optimizations known as robust semidefinite programs. Although robust semidefinite programs, just as semialgebraic problems, are computationally difficult there are also well

studied relaxations that are able to address certain problem instances. Brandão and Vianna show that both deterministic algorithms that are able to solve some problem instances [30] and probabilistic algorithms that give correct answers with some probability [31] can provide tractable approaches to the separability problem, at least in low dimensions.

The tests we consider are an obvious generalization of [13] to the multiparty case. As such they revolve around the question of whether certain symmetric state extensions exist for ρ. The general problem of when a global state is consistent with a given set of reduced density matrices for various overlapping subsystems of a multipartite quantum system has a long history. The importance of this general state extension problem was emphasised by Werner in [33,34]. A simple example is to specify that two systems A and B are in some entangled pure state and that B and a third system C have a reduced density matrix that is also this same pure entangled state. That this specification of reduced states is inconsistent with any quantum state for the whole system A, B, C is known in quantum information as the monogamy of pure state entanglement [35]; given that two quantum systems are in a pure entangled state it is not possible for either one to be entangled with a third system. Mixed entangled states also tend to be monogamous; Werner used the violation of Bell inequalities for certain mixed bipartite entangled states ρ to rule out the existence of a state of A, B, C where the reduced states of both A, B and C, B are ρ [34]. This logic can be reversed; the existence of such a global state on A, B, C implies that there is a local hidden variable description for certain Bell experiments on ρ [36] and this construction can readily be extended to multiparty cases [36,37]. The connection between this consistency problem for reduced states and the bipartite separability problem which is central to [10,13] is in fact made in a brief comment in [2]. Using the techniques of [10,13] all of the state extension problems resulting from specifying sets of reduced density matrices and asking if this specification is consistent with a global mixed state can be phrased as semidefinite programs, a fact has important implications for practical calculations.

The question of when a specification of reduced states for a quantum system is consistent with a global state for the system was raised again in [38]. Subsequent work has focussed on when a set of one-party reduced density matrices is consistent with a *pure* state of the joint system for some number of qubits or qutrits [39–42]. The situation when two-party reduced density matrices are specified for mixed states of three quantum systems is considered in [42]. In each of these cases it is possible to derive necessary conditions for compatibility based on the eigenvalues of the reduced density matrices. Very recent work by Jones and Linden [43] shows that the general question of when a set of reduced states is consistent with a pure quantum state for the whole system is expressible as a specific problem in real algebraic geometry. This seems to be a very significant difference to the version of the problem in which the joint state is allowed to be mixed since most interesting classes of problems in real algebraic geometry prove to be computationally hard while semidefinite programs may be solved in polynomial time. See [29] for a discussion of this point and algorithms that solve problems in real algebraic geometry using semidefinite

programming. In other important recent progress on state extension problems, Linden and Wootters [44] have shown that the reduced density matrices of a certain fraction of the parties of a generic multiparty pure state completely determine the state; the bounds on this fraction have been significantly improved in [43].

Another very important instance of this state extension problem, termed by Coleman the N-representability problem [45], has been much studied in physical chemistry over a long period (for recent discussions and references see [46–48]). The N-representability problem poses the question of which two-body reduced density matrices are consistent with a valid global state of N fermions. The antisymmetrization of the fermion wavefunction requires that all two-particle reduced density matrices be the same and the global state be antisymmetric to swapping particles. The reason for interest in this problem is that the ground state energy of an interacting fermion system can be written in terms of the two-body reduced density matrix if only two-body interactions occur in the Hamiltonian. A lot of information about the ground states of molecular systems could be found if tractable conditions for N-representability existed. A similar connection between state extension problems and the ground states of spin systems with local interactions was also noted by Werner [34]. In the tradition of work on this problem in physical chemistry necessary conditions for N-representability are often found in terms of conditions on the particle and hole correlations and it has recently been realized that these in turn may be able to be expressed as semidefinite programs [48–50].

The key idea of this paper is to propose a sequence of state extensions that must exist if a given multiparty quantum state ρ is separable. Like all state extension problems these may be expressed as semidefinite programs. The key result is the determination that this sequence of tests is complete in the sense that it can in principle detect all entangled states. This is achieved by an inductive argument in the number of parties. Like [13] this argument depends on the strengthened version of the quantum de Finetti theorem proven in [11,12].

The rest of the paper is structured as follows. In Sec. II we introduce the separability criteria we will consider. As discussed in Sec. III these can be checked by solving a semidefinite program and we show how to use the theory of semidefinite programming to construct entanglement witnesses for ρ whenever one of the criteria shows ρ to be entangled. The central result that a given series of separability criteria is complete in the sense that any entangled state will be detected by some test in the series is proven in Sec. IV. In Sec. V we explicitly consider the example of Bennett *et al.* [21] of a completely bound entangled state where no PPT test or bipartite separability test would suffice to demonstrate that the state is entangled. Finally, we conclude in Sec. VI.

II. MULTIPARTITE SEPARABILITY CRITERIA

Let ρ be a N-partite state defined in $\otimes_{i=1}^{N} \mathcal{H}_{A_i}$, where the different parties A_i are represented by Hilbert spaces $\mathcal{H}_{A_i}$ of dimension d_{A_i} respectively. Let $\vec{n} = (n_1, \ldots, n_N)$ be a vector of

positive integers greater than or equal to one. We will say that a state $\rho_{\vec{n}}$ defined in $\otimes_{i=1}^{N} \mathcal{H}_{A_i}^{\otimes n_i}$, which can be viewed as the original space supplemented by (n_i-1) copies of party A_i, is a *locally symmetric extension (LSE) of ρ*, if it satisfies the following two properties.

(1) $\rho_{\vec{n}} = V_{i,\tau(i)} \rho_{\vec{n}} V_{i,\tau(i)}$ $\forall i, 1 \leq i \leq N$, and $\forall$ $\tau(i) \in S_{n_i}$, with

$$V_{i,\tau(i)} = \left(\overset{i-1}{\underset{j=1}{\otimes}} \mathbb{1}_{A_j}^{\otimes n_j} \right) \otimes \Pi_{\tau(i)} \otimes \left(\overset{N}{\underset{j=i+1}{\otimes}} \mathbb{1}_{A_j}^{\otimes n_j} \right), \qquad (3)$$

where S_{n_i} is the group of permutations of n_i objects and $\Pi_{\tau(i)}$ is the operator that applies the permutation $\tau(i) \in S_{n_i}$ to the n_i copies of party A_i.

(2) $\rho = \mathrm{Tr}_{\{A_1^{\otimes n_1-1}\cdots A_N^{\otimes n_N-1}\}}[\rho_{\vec{n}}]$.

The first property means that $\rho_{\vec{n}}$ remains invariant whenever we permute the copies of a certain party. Due to this symmetry, we do not need to specify which copies of A_i are we tracing over in the second property. Furthermore, we can define a PPT locally symmetric extension (PPTLSE), by requiring $\rho_{\vec{n}}$ to remain positive semidefinite under any possible partial transposition.

We will now show that we can use this definition to generate a family of separability criteria. It is very easy to see that any fully separable state has LSE for any vector $\vec{n}$. This can be seen from (1), since the state

$$\rho_{\vec{n}} = \sum p_i \overset{N}{\underset{j=1}{\otimes}} (|\psi_i^{(A_j)}\rangle\langle\psi_i^{(A_j)}|)^{\otimes n_i} \qquad (4)$$

clearly has the required properties. Moreover, the state in (4) is obviously PPT, since it is fully separable. We have then the property that any fully separable state has PPTLSE to any number of copies of its parties. This observation can be used to generate a family of separability criteria. Any state that fails to have a PPTLSE for some number of copies *must be entangled*.

For any vector $\vec{n}$ that represents the number of copies of the different parties, we can construct a separability criterion by just asking the question of whether the state ρ has a PPTLSE to that particular number of copies. Thus, we can construct a countably infinite family of separability criteria. This is similar to the situation in the bipartite case discussed in [13]. However, in the multipartite case, these criteria cannot be all ordered in a hierarchical structure, although they have a natural *partial order*. For example, if a state has a PPTLSE to $\vec{n}$ copies, then it clearly has PPTLSE to $\vec{k}$ copies, for all $\vec{k}$ that satisfy $k_i \leq n_i, \forall i$, since we can construct such an extension by tracing (n_i-k_i) copies of party A_i, $1 \leq i \leq N$. This property of the extensions is mapped into the partial order of N-tuples given by

$$\vec{k} < \vec{n} \Leftrightarrow k_i \leq n_i, \quad \forall \ i, 1 \leq i \leq N. \qquad (5)$$

Conversely, if a state does not have a PPTLSE to $\vec{k}$ copies, which means it is entangled, then it cannot have PPTLSE to $\vec{n}$ copies, for any $\vec{n}$ satisfying $\vec{k} < \vec{n}$. However, there does not seem to be any relationship between the existence of PPTLSE to number of copies whose vectors are not related by the partial order (5).

In the following section we will discuss the semidefinite programs that determine whether a state has a PPTLSE. By using the duality theory of semidefinite programs we will show how to construct entanglement witnesses in cases where a PPTLSE fails to exist.

III. SEPARABILITY CRITERIA AS SEMIDEFINITE PROGRAMS AND ENTANGLEMENT WITNESSES

The techniques of [10,13] allow us to determine whether a given PPTSE exists by solving a semidefinite programming feasibility problem. Such problems amount to deciding whether there exists a positive matrix subject to given affine constraints. We will not dwell on the details here which are essentially identical to [10,13].

We begin by noting that the state extension (4) satisfies a stronger property than invariance under swapping the copies of the different Hilbert spaces. Let us denote the symmetric subspace of k copies of $\mathcal{H}_{A_i}$ by $\mathrm{Sym}^k(A_i)$. Let $\pi_k^{(i)}$ be the projectors onto these subspaces. Then the PPTLSE of Eq. (4) has support on the tensor product of these symmetric subspaces $\mathrm{Sym}^{n_1}(A_1) \otimes \cdots \otimes \mathrm{Sym}^{n_N}(A_N)$. For all i the PPTLSE of Eq. (4) satisfies $\pi_{n_i}^{(i)} \rho_{\vec{n}} \pi_{n_i}^{(i)} = \rho_{\vec{n}}$. More economically we may define a projector $\pi_{\vec{n}} = \Pi_i \pi_{n_i}^{(i)}$ onto the subspace $\otimes_i \mathrm{Sym}^{n_i}(A_i)$.

Since the extension must remain positive under all possible partial transpositions, we need to impose a whole set of positivity constraints on $\rho_{\vec{n}}$. We will write then

$$\rho_{\vec{n}}^{T_{\mathcal{S}}} \geq 0, \qquad (6)$$

where we use $\mathcal{S}$ to represent any subset of the tensor factors in $\otimes_{i=1}^{N} \mathcal{H}_{A_i}^{\otimes n_i}$ that yields an independent partial transpose, including the empty set, which we will associate with not applying any partial transposition.

To summarize the conditions on $\rho_{\vec{n}}$, for a given $\vec{n}$ we must

$$\begin{aligned} &\text{find} && \rho_{\vec{n}} \\ &\text{subject to} && \rho_{\vec{n}}^{T_{\mathcal{S}}} \geq 0 \ \ \forall \ \mathcal{S} \\ & && \pi_{\vec{n}} \rho_{\vec{n}} \pi_{\vec{n}} = \rho_{\vec{n}} \\ & && \mathrm{Tr}_{\{A_1^{\otimes n_1-1}\cdots A_N^{\otimes n_N-1}\}}[\rho_{\vec{n}}] = \rho. \end{aligned} \qquad (7)$$

Both of the equalities above can be written in terms of a finite number of trace constraints by writing them in terms of an explicit basis for Hermitian matrices as in [13]. So the partial trace conditions on $\rho_{\vec{n}}$ define an affine subset of matrices on $\otimes_{i=1}^{N} \mathcal{H}_{A_i}^{\otimes n_i}$ and if a positive symmetric state extension exists this subset will intersect with the cone of positive semidefinite matrices. Determining whether the intersection is empty is a semidefinite programming feasibility problem.

We may now apply the duality theory of semidefinite programs to find the dual optimization [14]. This optimization proves to be a search for an entanglement witness Z. The dual optimization is written in terms of matrix variables $Z_{\mathcal{S}}$ acting on $\otimes_{i=1}^{N} \mathcal{H}_{A_i}^{\otimes n_i}$ (one matrix for each distinct partial transpose) and Z is defined in terms of these. The optimization is then

$$\text{minimize} \qquad \text{Tr}[Z\rho]$$

$$\text{subject to} \qquad Z_{\mathcal{S}} \geqslant 0 \ \forall \ \mathcal{S}$$

$$\pi_{\vec{n}}(Z \otimes \mathbb{1})\pi_{\vec{n}} = \pi_{\vec{n}}\left(\sum_{\mathcal{S}} Z_{\mathcal{S}}^{T_{\mathcal{S}}}\right)\pi_{\vec{n}}. \tag{8}$$

Note that Z is an observable on the physical Hilbert space $\otimes_i \mathcal{H}_{A_i}$ and the identity $\mathbb{1}$ acts on the duplicate copies of different parties $\otimes_{i=1}^{N} \mathcal{H}_{A_i}^{\otimes(n_i-1)}$. Thus the different $Z_{\mathcal{S}}$ are observables on the same space as the state extensions $\rho_{\vec{n}}$, $\otimes_{i=1}^{N}\mathcal{H}_{A_i}^{\otimes n_i}$. We will show that this dual optimization answers the question of the existence of a PPTLSE $\rho_{\vec{n}}$ equally well and has the added benefit that when such an extension does not exist the optimal solution is an entanglement witness.

Suppose that some $Z_\star$ satisfying these constraints exists and has $\text{Tr}[Z_\star \rho] < 0$ and yet there is also a PPTLSE $\rho_{\vec{n}}$. Then

$$\text{Tr}[Z_\star \rho] = \text{Tr}[(Z_\star \otimes \mathbb{1})\rho_{\vec{n}}]$$

$$= \text{Tr}\left[\pi_{\vec{n}}\left(\sum_{\mathcal{S}} Z_{\mathcal{S}}^{T_{\mathcal{S}}}\right)\pi_{\vec{n}}\rho_{\vec{n}} \right]$$

$$= \sum_{\mathcal{S}} \text{Tr}[Z_{\mathcal{S}}\rho_{\vec{n}}^{T_{\mathcal{S}}}] \geqslant 0,$$

which is a contradiction. The first line follows from the fact that $\rho_{\vec{n}}$ is an extension for ρ, the second line from the symmetry of ρ and the constraints on Z. The third again uses the symmetry of ρ and the property of partial transposes that $\text{Tr}[X^{T_{\mathcal{S}}}Y] = \text{Tr}[XY^{T_{\mathcal{S}}}]$. Finally positivity results from the requirement that both $\rho_{\vec{n}}$ and the different $Z_{\mathcal{S}}$ are positive semidefinite. If such an observable $Z_\star$ exists then ρ cannot have a PPTLSE $\rho_{\vec{n}}$ and thus ρ must be entangled. Equally all separable states σ do have have a PPTLSE $\sigma_{\vec{n}}$ given by Eq. (4) and as a result $\text{Tr}[Z_\star \sigma] \geqslant 0$. Therefore $Z_\star$ is an entanglement witness. As in the bipartite case discussed in [13] these entanglement witnesses have interesting algebraic properties that relate them to the general methods of [29]. Since the details are essentially identical to the bipartite case we refer the interested reader to these two references.

This leaves the possibility that the optimum of the dual semidefinite program (8) is positive and yet no PPTLSE $\rho_{\vec{n}}$ exists. As in the bipartite case this possibility must be excluded by appealing to strong duality [13]. Broadly speaking when no PPTLSE exists the existence of an entanglement witness of the form $Z_\star$ is guaranteed by the separating hyperplane theorem of convex analysis applied to an appropriate convex set associated with the feasibility problem (7). However, in order to apply this theorem we must check that this set is in fact closed. In our case this may be determined by checking that $Z=\mathbb{1}>0$ satisfies the constraints of the dual semidefinite program (8). For full details of this argument see Appendix B of [13]. We may conclude that when no PPTLSE exists we may use the dual semidefinite program to construct an entanglement witness and equally that the optimum of the dual program can only be positive if a PPTLSE exists.

These two equivalent semidefinite programs can be implemented numerically using exactly the techniques described in [13] and we will not dwell on these details here. Once again it is important to implement the optimizations in a way that preserves the symmetries, making use of the fact that $\rho_{\vec{n}}$ can be restricted to lie on the symmetric subspace $\otimes_i \text{Sym}^{n_i}(A_i)$. For a fixed number of parties and a fixed $\vec{n}$ the computation required to solve the two semidefinite programs will scale polynomially with the Hilbert space dimensions involved. Also for a fixed number of parties and fixed Hilbert space dimensions the computation required to perform the tests will scale as some polynomial of the components of $\vec{n}$. In this case the number of inequivalent partial transposes will be limited very greatly by the symmetry between the different copies of the subspaces A_i. Unfortunately as the number of parties increases the number of inequivalent partial transpose tests will increase very rapidly. However the tests will be of use even if only a restricted subset of the possible partial transposes (a restricted subset of the possible $\mathcal{S}$ in the above formulas) are actually used. The number of inequivalent partial transposes is related to the number of possible partitions of N quantum systems and is discussed in [23].

IV. COMPLETENESS OF THE FAMILY OF TESTS

Each test described in the previous section gives a necessary condition for separability of a multipartite state. We have discussed how these tests can be stated as semidefinite programs, which implies that there are efficient algorithms to solve them. In this section we will show that this family of criteria is also complete, in the sense that any mutipartite entangled state will be detected by some test. We will actually prove a stronger result; a weaker family of tests is already complete. The proof is based on the completeness of the bipartite hierarchy of tests [11–13], and the properties of the Quantum de Finetti representation [51].

Theorem 1 (multipartite completeness). Let ρ be a multipartite mixed state in $\otimes_{i=1}^{N}\mathcal{H}_{A_i}$, such that ρ has locally symmetric extensions (LSE) $\rho_{\vec{n}_k}$ for its first $(N-1)$ parties, associated with the vectors $\vec{n}_k = (k, k, \ldots, k, 1)$, $\forall \ k \geqslant 1$. Then ρ is fully separable.

Moreover, there are unique conditional probability densities $P_l(\omega_{A_l}|\omega_{A_{l-1}}, \ldots, \omega_{A_1})$, $1 \leqslant l \leqslant N-1$, and a unique function $\lambda : D_{A_1} \times \cdots \times D_{A_{N-1}} \to D_{A_N}$, where D_{A_i} is the space of states in $\mathcal{H}_{A_i}$, such that

$$\rho = \int_{D_1^{N-1}} \left(\overset{N-1}{\underset{i=1}{\otimes}} \omega_{A_i}\right) \otimes \lambda_{A_N}(\omega_{A_1}, \ldots, \omega_{A_{N-1}})$$

$$\times \Pi_{i=1}^{N-1}P_i(\omega_{A_i}|\omega_{A_{i-1}}, \ldots, \omega_{A_1})d\omega_{A_i}, \tag{9}$$

(with $\int_{D_1^{N-1}}$ meaning $\int_{D_{A_1}}\cdots\int_{D_{A_{N-1}}}$).

Proof. The proof is by induction in the number of parties. The proof of the case $N=2$ is theorem 1 in [13].

Let us assume the result holds for $N-1$. Let ρ in $\otimes_{i=1}^{N}\mathcal{H}_{A_i}$ have the LSE mentioned in the statement of the theorem. Consider the split $A_1 - (A_2, \ldots, A_N)$ of the N parties and regard ρ as a bipartite state. Consider the LSE of ρ associated with the vector $\vec{n}_k$. Then, by tracing out $(k-1)$ copies of $\mathcal{H}_{A_i}$, $2 \leqslant i \leqslant N$, we obtain a state in $\mathcal{H}_{A_1}^k \otimes (\otimes_{i=2}^{N}\mathcal{H}_{A_i})$ that is invariant under permutations of the copies of $\mathcal{H}_{A_1}$ and yields ρ when we trace $(k-1)$ copies of $\mathcal{H}_{A_1}$. Hence, ρ has SE to any

number of copies of A_1, and applying the result of the bipartite case, we can write

$$\rho = \int_{D_{A_1}} \omega_{A_1} \otimes \sigma(\omega_{A_1}) P_1(\omega_{A_1}) d\omega_{A_1}, \qquad (10)$$

where $\sigma(\omega_{A_1})$ is a *unique* state in $\otimes_{i=2}^{N} \mathcal{H}_{A_i}$, and $P_1(\omega_{A_1})$ is a *unique* probability density on the space of states $\mathcal{D}_{A_1}$. Our strategy will be to construct a family of LSEs for the states $\sigma(\omega_{A_1})$ and invoke the inductive hypothesis to conclude that they are separable.

Now, consider the state in $\mathcal{H}_{A_1} \otimes (\otimes_{i=2}^{N-1} \mathcal{H}_{A_i}^{\otimes k}) \otimes \mathcal{H}_{A_N}$ defined by

$$\rho_{(1,k,\ldots,k,1)} = \text{Tr}_{A_1^{\otimes(k-1)}}[\rho_{\vec{n}_k}]. \qquad (11)$$

Note that $\rho_{\vec{n}_k}$ exists by hypothesis but need not be unique. We will impose the further condition on $\rho_{\vec{n}_k}$ that for all $m > k$ there is an LSE $\rho_{\vec{n}_m}$ for ρ such that

$$\text{Tr}_{\{A_1^{\otimes m-k} \ldots A_{N-1}^{\otimes m-k}\}}[\rho_{\vec{n}_m}] = \rho_{\vec{n}_k}. \qquad (12)$$

Thus $\rho_{\vec{n}_k}$ itself has symmetric extensions to larger numbers of copies of the different parties.[1]

It is not difficult to see that if we consider again the bipartite split $A_1 - (A_2, \ldots, A_N)$, the state $\rho_{(1,k,\ldots,k,1)}$ has symmetric extensions to any number of copies of $\mathcal{H}_{A_1}$. For example, if we want a symmetric extension to m copies, $m \leq k$, we can take

$$\text{Tr}_{A_1^{\otimes(k-m)}}[\rho_{\vec{n}_k}] \qquad (13)$$

(where $\text{Tr}_{A_1^{\otimes 0}}$ means not taking any trace), and if $m > k$ we take

$$\text{Tr}_{\{A_2^{\otimes m-k} \ldots A_{N-1}^{\otimes m-k}\}}[\rho_{\vec{n}_m}]. \qquad (14)$$

Thus we have that $\rho_{(1,k,\ldots,k,1)}$ has symmetric extensions to any number of copies of $\mathcal{H}_{A_1}$, so applying the bipartite result again we can write

$$\rho_{(1,k,\ldots,k,1)} = \int_{D_{A_1}} \omega_{A_1} \otimes \sigma_{(k,\ldots,k,1)}(\omega_{A_1}) P_{(k,\ldots,k,1)}(\omega_{A_1}) d\omega_{A_1}, \qquad (15)$$

where both the state $\sigma_{(k,\ldots,k,1)}(\omega_{A_1})$ in $(\otimes_{i=2}^{N-1} \mathcal{H}_{A_i}^{\otimes k}) \otimes \mathcal{H}_{A_N}$ and the probability density $P_{(n_2,\ldots,n_{N-1},1)}(\omega_{A_1})$ defined on D_{A_1} are *unique*.

If we trace out $(k-1)$ copies of $\mathcal{H}_{A_i}$, $2 \leq i \leq N-1$, in (15), we obtain

$$\rho = \int_{D_{A_1}} \omega_{A_1} \otimes \text{Tr}_{\{A_2^{\otimes k-1} \ldots A_{N-1}^{\otimes k-1}\}}[\sigma_{(k,\ldots,k,1)}(\omega_{A_1})]$$
$$\times P_{(k,\ldots,k,1)}(\omega_{A_1}) d\omega_{A_1}. \qquad (16)$$

If we compare (10) and (16), we can use the uniqueness of the decomposition to conclude that

$$\sigma(\omega_{A_1}) = \text{Tr}_{\{A_2^{\otimes k-1} \ldots A_{N-1}^{\otimes k-1}\}}[\sigma_{(k,\ldots,k,1)}(\omega_{A_1})], \quad \forall \ k \geq 1, \qquad (17)$$

and

$$P_{(k,\ldots,k,1)}(\omega_{A_1}) = P_1(\omega_{A_1}). \qquad (18)$$

For each ω_{A_1}, the state $\sigma(\omega_{A_1})$ is a state in $\otimes_{i=2}^{N} \mathcal{H}_{A_i}$. We claim that this state has locally symmetric extensions for the first $(N-2)$ parties that are associated with vectors of $N-1$ components of the form $\vec{n}_k = (k,k,\ldots,k,1), \ \forall \ k \geq 1$.

Equation (17) proves the existence of the extensions. To prove the symmetry, we use Eq. (15) and uniqueness of the decomposition. First note that, by hypothesis, we can state that

$$\rho_{(1,k,\ldots,k,1)} = V_{i,\tau(i)} \rho_{(1,k,\ldots,k,1)} V_{i,\tau(i)}, \qquad (19)$$

which holds $\forall \ i$, $2 \leq i \leq (N-1)$, and $\forall \ \tau(i) \in S_k$, since these symmetry requirements are implied by the symmetry properties of $\rho_{\vec{n}_k}$. Note that the permutation operators in (19) act only on parties A_2 through A_{N-1}. If we apply (19) to both sides of (15), we obtain

$$\rho_{(1,k,\ldots,k,1)} = \int_{D_{A_1}} \omega_{A_1} \otimes [V_{i,\tau(i)} \sigma_{(k,\ldots,k,1)}(\omega_{A_1}) V_{i,\tau(i)}]$$
$$\times P_1(\omega_{A_1}) d\omega_{A_1}. \qquad (20)$$

But comparing (20) with (15), and using again the uniqueness of the decomposition, we have

$$\sigma_{(k,\ldots,k,1)}(\omega_{A_1}) = V_{i,\tau(i)} \sigma_{(k,\ldots,k,1)}(\omega_{A_1}) V_{i,\tau(i)}. \qquad (21)$$

So the extensions of $\sigma(\omega_{A_1})$ have the required symmetry.

We can now apply the inductive hypothesis to $\sigma(\omega_{A_1})$ and conclude that this state must be fully separable and in fact

$$\sigma(\omega_{A_1}) = \int_{D_2^{N-1}} \left(\overset{N-1}{\underset{i=2}{\otimes}} \omega_{A_i} \right) \otimes \lambda_{A_N}(\omega_{A_1}, \ldots, \omega_{A_{N-1}})$$
$$\times \Pi_{i=2}^{N-1} P_i(\omega_{A_i} | \omega_{A_{i-1}}, \ldots, \omega_{A_1}) d\omega_{A_i}. \qquad (22)$$

Combining (10) with (22) we finally get

$$\rho = \int_{D_1^{N-1}} \left(\overset{N-1}{\underset{i=1}{\otimes}} \omega_{A_i} \right) \otimes \lambda_{A_N}(\omega_{A_1}, \ldots, \omega_{A_{N-1}})$$
$$\times \Pi_{i=1}^{N-1} P_i(\omega_{A_i} | \omega_{A_{i-1}}, \ldots, \omega_{A_1}) d\omega_{A_i}, \qquad (23)$$

showing that the state ρ is fully separable. $\square$.

This result generates a sequence of separability criteria labeled by the integer k. Since the existence of a LSE for some k_1 implies the existence of a LSE for all k_2, $k_2 \leq k_1$, then we have that this sequence has a hierarchical structure,

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[1] It may not be immediately obvious that this is possible. Suppose not; then for all $\vec{n}_k$ LSE $\rho_{\vec{n}_k}$ there is some $m > k$ such that there is no $\vec{n}_m$ LSE satisfying (12). Take any $\vec{n}_m$ LSE $\rho_{\vec{n}_m}$ and consider the state $\text{Tr}_{\{A_1^{\otimes m-k} \ldots A_{N-1}^{\otimes m-k}\}}[\rho_{\vec{n}_m}]$; it is clear that this state is a $\vec{n}_k$ LSE for ρ which is a contradiction.

similar to the one introduced for the bipartite case in [13]. We note that this sequence of state extensions is exactly the one considered in [37] in context of finding local hidden variable theories for multipartite states ρ. This shows that, exactly as in [36], these local hidden variable theories can only give a local realistic description of Bell experiments having an arbitrary number of detector settings for the two observers when the states of interest are separable. However, applying this particular hierarchy of tests is not the best practical tool to detect entanglement of multipartite states.

From theorem 1 we have the following corollary.

Corollary 1. A multipartite mixed state ρ in $\otimes_{i=1}^{N} \mathcal{H}_{A_i}$ has PPTLSE to any number of copies of its first $(N-1)$ parties, if and only if ρ is fully separable.

Proof. If ρ is fully separable, it has a decomposition of the form (1) and hence we can construct the PPTLSE given by (4). On the other hand, if ρ has PPTLSE to any number of copies of its first $(N-1)$ parties, in particular it has PPTLSE to extensions associated with the vectors $\vec{n}_k = (k, k, \ldots, k, 1)$, $\forall\, k \geqslant 1$. Since any PPTLSE is also a LSE, according to theorem 1 ρ must be fully separable. $\square$ (Note that we could replace PPTLSE by LSE in the statement of corollary 1 and still recover the same result.)

Corollary 1, although equivalent to theorem 1, seems to be less practical, since we require the existence of many more PPTLSE. However, since the existence of any PPTLSE is a *necessary* condition for separability, its nonexistence is a *sufficient* condition for entanglement. The advantage of an application of these results based on corollary 1 rather than on theorem 1, lies in the fact that we might be able to show entanglement by searching for a PPTLSE to one extra copy of *one* of the parties instead of one extra copy of *all* parties. In terms of the resources needed to implement this might amount to a huge saving. For example, if we have a state in $2 \otimes 4 \otimes 4$, it is much easier to search for a PPTLSE to 3 copies of the first party, than it would be to search for a PPTLSE to one copy of each of the parties. Corollary 1 gives us the chance of choosing a more economical way of testing for entanglement. We will see later on, when we discuss a particular example, that this approach can be very useful.

In [25] the multipartite separability problem was discussed in terms of linear maps positive on products states. Every multipartite entanglement witness can be transformed into such a linear map and our result has implications for the characterization of these maps. In [13] we characterized strictly positive maps as those that are completely positive when composed with one of a class of maps onto the symmetric subspace of some number of copies of the output space of the linear map. An exactly similar characterization of the adjoint of a linear map strictly positive on product states is possible based on theorem 1. Since the only difference is extending notation of [13] to the multipartite case we will not give an explicit discussion.

V. EXAMPLE

Here we consider the example of a complete bound entangled three qubit state constructed by Bennett *et al.* from an unextendible product basis [21].

In the example, we look for one-copy extensions of one of the parties, i.e., the case where $\vec{n} = (2, 1, 1)$. Equivalently, from the dual viewpoint, we look for witnesses Z satisfying the conditions of the dual semidefinite program (8).

A. A $2 \otimes 2 \otimes 2$ state from UPBs

We apply the results to a $2 \otimes 2 \otimes 2$ tripartite state, first proposed in [21]. This entangled state is constructed using unextendible product bases (UPBs), and has the very interesting property of being separable for every possible bipartition of the three parties. The state has the following expression:

$$\rho = \frac{1}{4}\left(\mathbb{1} - \sum_{j=1}^{4} |\psi_j\rangle\langle\psi_j|\right), \quad (24)$$

where

$$\psi_1 = |0, 1, +\rangle, \quad \psi_2 = |1, +, 0\rangle,$$

$$\psi_3 = |+, 0, 1\rangle, \quad \psi_4 = |-, -, -\rangle,$$

and $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. After solving the SDP, we easily arrive at a witness whose matrix representation is given below:

$$Z = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 4 & 1 & 0 & 1 & 3 & -1 & 1 \\ -1 & 1 & 4 & 3 & 1 & -1 & 0 & 1 \\ 1 & 0 & 3 & 4 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 4 & 0 & 3 & 1 \\ 1 & 3 & -1 & 1 & 0 & 4 & 1 & -1 \\ 1 & -1 & 0 & 1 & 3 & 1 & 4 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}. \quad (25)$$

It can be verified that $\mathrm{Tr}[Z\rho] = -\frac{3}{8} < 0$, but Z is nonnegative in all product states. This is certified by an identity, obtained from the solution of the SDP, that expresses Z in terms of a set of matrices Z_S and shows that the conditions of the dual program (8) are satisfied.

We see that it is possible to demonstrate that ρ is entangled by looking for an extension to only a single extra copy of one of the subsystems $[\vec{n} = (2, 1, 1)]$ so long as we require that the extension has positive partial transposes. This highlights the fact that the hierarchy involving PPTLSEs can be much more powerful than the hierarchy that looks for LSEs without requiring positivity of the partial transposes of the state extension. In our example, since ρ is separable when considered as a bipartite state resulting from any bipartition of the three parties, locally symmetric extensions of ρ exist for $\vec{n} = (k, 1, 1)$ with any k, although we know from the completeness theorem that LSEs for ρ cannot exist for $\vec{n} = (k, k, 1)$ for all k.

VI. CONCLUSIONS

In this paper we have discussed separability criteria for multipartite quantum states based on the existence of exten-

sions of the state to a larger space consisting of several copies of each of the subsystems. The symmetric extensions we consider always exist if the state is separable but do not necessarily exist for entangled states. We showed that multipartite entangled states will eventually fail one of these tests and in this case we constructed an entanglement witness using the duality theory of semidefinite programming.

It would be enlightening to better understand the physical significance of these symmetric extensions. It can be said that they highlight a version of the monogamy of entanglement for mixed states; if a group of entangled systems are in a strongly entangled state it is hard for them to share the same entanglement with other systems. Another interpretation carries over from the bipartite case, the symmetric state extensions, if they exist, provide local hidden variable descriptions for large classes of possible multiparty Bell experiments. This is discussed much more fully in [36,37].

Other questions for further study include the behavior of our tests under local operations and classical communication. Unlike the positive partial transpose test it is not clear that the property of having a symmetric extension to a given number of copies of the subsystems is preserved under local operations and classical communication. Certainly the tests we construct are invariant under local unitary operations but, just as in the bipartite case [13], there are state transformations that may be achieved with some probability by local operations and classical communication that can convert a state having a symmetric state extension into one that does

not. A sequence of tests for entanglement that was invariant under local operations and classical communication would point to the existence of many sets of states, other than the positive partial transpose states and the separable states, that are closed under local operations and classical communication and this could have interesting consequences for quantum information theory. Another important open question is the problem of finding explicit product state decompositions for separable states. As they exist at the moment our tests only provide definitive answers when the state of interest turns out to be entangled. A more powerful procedure would be able to detect separable states and construct product state decompositions when this is possible.

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