

Information-capacity description of spin-chain correlations

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Information capacities achievable in the multi-parallel-use scenarios are employed to characterize the quantum correlations in unmodulated spin chains. By studying the qubit amplitude damping channel, we calculate the quantum capacity \mathcal{Q} , the entanglement assisted capacity C_E , and the classical capacity C_1 of a spin chain with ferromagnetic Heisenberg interactions.

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I. INTRODUCTION

Spin chains are gaining increasing attention as natural candidates of quantum channels [1–5]. An unknown quantum state can be prepared on one end of the chain and then transferred to the other end by simply employing the ability of the chain to propagate the state by means of its dynamical evolution. This procedure does not require any gating and therefore can be implemented without any need for a modulation of the couplings between the spins. Especially this last aspect may be of importance for solid-state quantum information. Several aspects of quantum communication using spin chains have been already obtained starting from the original proposal by Bose [1]. The fidelity of the transmitted state can be increased if the sender and the receiver can encode or decode the state using a finite number of spins [2]. Even perfect transmission can be achieved if the exchange couplings of the chain are chosen appropriately [3] or if it possible to perform measurements on the quantum spins of the chain [6]. Moreover, it was recently shown that a phase-covariant cloner can be realized using an unmodulated spin network [7] and that the spin chain approach to quantum communication seems to be realizable in solid-state devices with present day technology [8].

Another interesting aspect which is emerging recently is the interconnection between quantum information and condensed matter theory [9]. Examples are the study of nonlocal (quantum) correlations of spin systems in a variety of situations (see [10] and references therein) or reformulation of the density matrix renormalization group in the framework of quantum information theory [11]. In this paper we would like to further explore the use of concepts born in quantum information for the characterization of spin chains. Here we propose to use the information capacities of quantum channels as a tool to characterize some aspects of dynamical correlations in a spin chain. By calculating the capacities of the channel, obtained by identifying two separate sections of the chain as the extremes of a communication line, one can in fact get some information about the strength of the correlations among the interconnecting spins. In some sense this is equivalent to assigning to any couple of subsets of the spins in the sample the values of the corresponding capacities in order to create a “road map” of the information fluxes in the system.

Our results apply for the whole class of Hamiltonians for which the total magnetization along a fixed direction is conserved. In the present work we confine ourself to the case in which one bit is transferred through the chain. This situation corresponds to studying the sectors in which only a few spins are up. We believe, however, that the present approach can be further extended to other spin sectors.

The paper is organized in two distinct parts. In the first one we introduce the spin chain communication lines and we discuss some models in details (Sec. II). In the second part (Sec. III) instead we give a brief overview of channel capacities and, by studying the qubit amplitude damping channel [12], we calculate the quantum capacity \mathcal{Q} , the entanglement assisted capacity C_E , and the classical capacity C_1 of the spin chain models presented in Sec. II. As explained in detail throughout the paper the above quantities are computed in communication scenarios where many “copies” of the spin system are available and “not” for those scenarios where the communicating parties make multiple uses of the “same” spin chain. The paper ends in Sec. IV with the conclusions.

II. MODEL

Given a collection of N spins coupled by means of a time-independent Hamiltonian it is possible to define a quantum channel by identifying two set of spins of the sample (say, sets A and B) as two quantum registers. A first party (the sender of the message) encodes some information on A and a second party (the receiver) tries to recover such information from B some later time t [1,2,4]. Formally, one assumes that the encoding procedure takes place by initially decoupling the spins A from the remaining spins without disturbing them from their initial fiduciary state σ_0 , preparing A on some input message ρ_A , and finally allowing ρ_A and σ_0 to interact for a given time t through the Hamiltonian of the system. At this point we consider the state $\rho_B(t)$ of the spins B , obtained from the spin chain state $R(t) = U(t)(\rho_A \otimes \sigma_0)U^\dagger(t)$ by tracing away all the degrees of freedom of the system but those relative to B . [Here $U(t)$ is the unitary evolution of the system.] The resulting completely positive, trace preserving (CPT) mapping

$$\rho_A \rightarrow \mathcal{M}(\rho_A) \equiv \rho_B(t) = \text{Tr}^{(B)}[U(t)(\rho_A \otimes \sigma_0)U^\dagger(t)], \quad (1)$$

where $\text{Tr}^{(B)}$ means trace over all the spins but B , defines the quantum channel we are interested in. A caveat is in order.

Equation (1) does not provide a proper description of any realistic scenario where the communicating parties keep on operating on their quantum registers over an extended period of time. In fact, after a first “reading out” of the signal from the spin chain at time t , the state σ_0 of the spins not belonging to A will change, thus making Eq. (1) unable to describe the state of B for later times. As a matter of fact, in our system any repeated (in time) manipulation of the quantum registers will necessarily introduce memory effects in the communication for which a satisfying quantum information theory is still missing (some preliminary results on quantum channels with memory can be found in Ref. [13]).

Having the previous observations in mind, Eq. (1) can still be used to provide a quite complete characterization of the correlations between the registers A and B . Technically the map \mathcal{M} is defined in the Hilbert space \mathcal{H}_A of the configurations of A and depends on t , H_{int} , σ_0 , and on the choice of A and B . A common approach [1,2,7] used to quantify the correlations between the sets A and B of the chain is then to consider the fidelity associated with \mathcal{M} , by computing the average fidelity between the input states ρ_A and their output counterparts $\rho_B(t)$. Here we observe that a more detailed characterization of such correlations can be obtained by treating \mathcal{M} of Eq. (1) as the CPT map of a real memoryless channel. The correlations between A and B can then be analyzed by means of the information capacities [12,14–18] associated with \mathcal{M} . These quantities are related to the minimum “amount of redundancy” over multiple uses of the channel \mathcal{M} needed to achieve perfect message transmission—i.e., unitary fidelity: higher values of the capacities correspond to stronger correlations between input and output states. Depending on the nature (e.g., classical or quantum) of the information propagating through the channel one can define different capacities of \mathcal{M} and each of them are obtained by maximizing over all possible coding and decoding strategies which act over multiple channel uses (see Sec. III). From the discussion that follows Eq. (1) it should be evident that the capacities of the map \mathcal{M} might not provide a proper description of the communication performances of the spin chain. As a matter of fact, they only account for those scenarios where many parallel “copies” of the same system are simultaneously operated by the two communicating parties but fail to describe those scenarios where the communicating parties make successive multiple uses of the “same” spin chain.

A. Solvable models

Examples of a spin chain channel where the information capacities of \mathcal{M} can be solved exactly are provided by a chain of $1/2$ spins coupled through a ferromagnetic Heisenberg interaction [1]. The system Hamiltonian is

$$H = - \sum_{\langle i,j \rangle} \hbar J_{ij} (\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j + \gamma \sigma_z^i \sigma_z^j) - \sum_{i=1}^N \hbar B_i \sigma_z^i, \quad (2)$$

where the first summation is performed on the nearest-neighbor spins of the chain, $\sigma_{x,y,z}^i$ are the Pauli operators associated with the i th spin, J_{ij} are coupling constants, γ is

an anisotropy parameter, and B_i are associated with externally applied magnetic fields.

In the following we will assume the input set A and output set B of Eq. (1) to contain, respectively, the first k and last k spins of the chain, and that all the spins but A are initially prepared in the same eigenstate $|\downarrow\rangle$ of σ_z . [Here $|\downarrow\rangle$ and $|\uparrow\rangle$ are the eigenstates of σ_z associated, respectively, with the eigenvalues -1 and $+1$]. If the two communicating parties are allowed to use the entire Hilbert space of their quantum memory, Eq. (1) yields a k -qubit channel which, in general, is too complex to be analyzed on a complete basis. In the present paper we consider thus a simplification of this scenario where the sender and receiver use their k spins to encode a single logical qubit. This approach is clearly inefficient from an informational theoretical point of view, but on the positive side, the resulting maps can be treated analytically.

1. Encoding one logical qubit with one-spin-up vectors

As in Refs. [1,2] we introduce the one-spin-up vector

$$|j\rangle \equiv |\downarrow \cdots \downarrow \uparrow \downarrow \cdots \downarrow\rangle, \quad (3)$$

which for $j=1, \dots, N$ represents the state of the chain where the j th spin is prepared in the eigenstate $|\uparrow\rangle$ and the other $N-1$ ones in $|\downarrow\rangle$. Suppose that at time $t=0$ the sender prepares her or his spins in

$$|\Psi\rangle_A \equiv \alpha |\downarrow\rangle_A + \beta |\phi_1\rangle_A, \quad (4)$$

where α, β are complex amplitudes, $|\downarrow\rangle_A$ is the state of A with all spins down, and $|\phi_1\rangle_A$ is a given normalized superposition of $|j\rangle$ with j referring to spins of the input memory A —i.e.,

$$|\phi_1\rangle_A \equiv \sum_{j=1}^k c_j |j\rangle. \quad (5)$$

Since the Hamiltonian of Eq. (2) commutes with the total spin component along the z direction, one can show that at time t the whole chain is described by [2]

$$|\Psi(t)\rangle \equiv \alpha |\downarrow\rangle + \beta \sum_{j'=1}^N \sum_{j=1}^k c_j f_{j',j}(t) |j'\rangle, \quad (6)$$

with $|\downarrow\rangle$ the state of the chain with all spins down and with

$$f_{j,s}(t) \equiv \langle j | e^{-iHt/\hbar} | s \rangle \quad (7)$$

[see, for instance, Ref. [1] for the explicit functional dependence of $f_{j,s}(t)$ from the evolution time t]. According to Eq. (1) the state of B at time t is finally obtained from Eq. (6) by tracing over all the remaining $N-k$ spins—i.e.,

$$\begin{aligned} \rho_B(t) = & [|\alpha|^2 + (1-\eta)|\beta|^2] |\downarrow\rangle_B \langle \downarrow| + \eta |\beta|^2 |\phi'_1\rangle_B \langle \phi'_1| \\ & + \sqrt{\eta} \alpha \beta^* |\downarrow\rangle_B \langle \phi'_1| + \sqrt{\eta} \alpha^* \beta |\phi'_1\rangle_B \langle \downarrow|, \end{aligned} \quad (8)$$

with

$$\eta = \sum_{j'=N-k+1}^N \left| \sum_{j=1}^k c_j f_{j',j}(t) \right|^2. \quad (9)$$

In Eq. (8) the two orthonormal vectors $|\downarrow\rangle_B$ and $|\phi'_1\rangle_B$ are, respectively, the state of B with all spins down and

$$|\phi'_1\rangle_B \equiv \sum_{j'=N-k+1}^N \sum_{j=1}^k c_j f_{j',j}(t) |j'\rangle / \sqrt{\eta}. \quad (10)$$

Apart from an irrelevant unitary transformation [19] the map associated with $\rho_B(t)$ of Eq. (8) is a qubit amplitude damping channel [1,2] of efficiency η which acts on the orthonormal basis $\{|\downarrow\rangle_B, |\phi'_1\rangle_B\}$. For the sake of clarity let us identify $|\downarrow\rangle_A$ and $|\downarrow\rangle_B$ with the same logical qubit state $|0\rangle$ and $|\phi_1\rangle_A$, $|\phi'_1\rangle_B$ with $|1\rangle$. In this notation we can express the input state $|\Psi\rangle_A$ and the output state $\rho_B(t)$ as density matrices ρ and ρ' of the same qubit Hilbert space \mathcal{H}_A . Equation (8) becomes thus

$$\rho' = \mathcal{D}_\eta(\rho), \quad (11)$$

with \mathcal{D}_η the amplitude damping map characterized by the Kraus operators [12]

$$\begin{aligned} A_0 &= |0\rangle\langle 0| + \sqrt{\eta}|1\rangle\langle 1|, \\ A_1 &= \sqrt{1-\eta}|0\rangle\langle 1|. \end{aligned} \quad (12)$$

Equation (11) describes a quantum channel in which the logical information of A represented by the coefficients α and β of Eq. (4) is transferred to the output memory B with an accuracy which can be estimated by calculating the capacities of the map \mathcal{D}_η . The calculation of the capacities of \mathcal{D}_η is presented in Sec. III A.

2. Encoding one logical qubit with two-spin-up vectors

Consider the case where in Eq. (4) the vector $|\phi_1\rangle_A$ is replaced by a normalized superposition $|\phi_2\rangle_A$ of states $|j, \ell\rangle$ where the j th and ℓ th spins of the chain are in $|\uparrow\rangle$ while the remaining are in $|\downarrow\rangle$ —i.e.,

$$|\phi_2\rangle_A \equiv \sum_{j>\ell=1}^k d_{j,\ell} |j, \ell\rangle. \quad (13)$$

In other words, the sender still uses the state with no spin up to “transfer” α , but now a selected superposition of two-spin-up states is employed to “transfer” β . In this case one can show that the output state of the memory B is

$$\begin{aligned} \rho_B(t) &= (|\alpha|^2 + \eta_2|\beta|^2) |\downarrow\rangle_B\langle\downarrow| + \eta_1|\beta|^2 |\phi'_2\rangle_B\langle\phi'_2| \\ &+ \sqrt{\eta_1}\alpha\beta^* |\downarrow\rangle_B\langle\phi'_2| + \sqrt{\eta_1}\alpha^*\beta |\phi'_2\rangle_B\langle\downarrow| \\ &+ (1 - \eta_1 - \eta_2)|\beta|^2 \sigma_B, \end{aligned} \quad (14)$$

where $|\phi'_2\rangle_B$ is a superposition of two-spin-up states of B and σ_B is a density matrix of one-spin-up states of B whose eigenvectors are orthogonal with respect to $|\downarrow\rangle_B$ and $|\phi'_2\rangle_B$ (see Appendix A for details). For $\eta_1 + \eta_2 = 1$ the mapping (14) reduces to a qubit amplitude damping channel \mathcal{D}_η with quantum efficiency $\eta = \eta_1$. However, in the general case, Eq.

(14) is slightly more complex. In fact for $\eta_1 + \eta_2 < 1$ the component $|\phi_2\rangle_A$ of the input state undergoes three possible processes: with probability η_1 it is rotated into the output state $|\phi'_2\rangle_B$, with probability η_2 it is damped to $|\downarrow\rangle_B$, and finally with probability $\eta_3 = 1 - \eta_1 - \eta_2$ it is transformed in the density matrix σ_B . In this respect, the map (14) is similar (but *not* equal) to a channel that, with probability η_3 , decoheres the input and transforms $|\phi_2\rangle_A$ into σ_B while with probability $1 - \eta_3$ applies to it an amplitude damping channel transformation of quantum efficiency $\eta_1/(1 - \eta_3)$.

A compact description of Eq. (14) is obtained by identifying $|\downarrow\rangle_A, |\downarrow\rangle_B$ with the logical qubit state $|0\rangle$ and $|\phi_2\rangle_A, |\phi'_2\rangle_B$ with $|1\rangle$. With this notation, the transformation (14) can be expressed as a two-parameter CPT map,

$$\rho' = \mathcal{T}_{\eta_1, \eta_2}(\rho), \quad (15)$$

with Kraus operators given by

$$\begin{aligned} A_0 &= |0\rangle\langle 0| + \sqrt{\eta_1}|1\rangle\langle 1|, \\ A_1 &= \sqrt{\eta_2}|0\rangle\langle 1|, \\ A_{1+i} &= \sqrt{\eta_3 \xi_i} |\xi_i\rangle\langle 1|. \end{aligned} \quad (16)$$

Here $|\xi_i\rangle$ are the eigenvectors of σ_B associated with the eigenvalues $\xi_i > 0$: according to Eq. (A8) they are orthogonal with respect to $|0\rangle$ and $|1\rangle$ and there are at most k of them.

The capacity of the channel $\mathcal{T}_{\eta_1, \eta_2}$ is derived in Sec. III B.

III. CHANNEL CAPACITIES

The quantities we are interested in this paper are the quantum capacity Q , the classical capacity C , and the entanglement-assisted capacities C_E and Q_E . The quantum capacity Q measures the maximum amount of quantum information that can be reliably transmitted through the map \mathcal{M} per channel use [15]. Intuitively, this quantity is related to the dimension of the largest subspace of the multi-use-input Hilbert space which does not decohere during the communication process. The value of Q (in qubits per channel uses) can be computed as

$$Q \equiv \sup_n Q_n/n, \quad (17)$$

with

$$Q_n \equiv \max_{\rho \in \mathcal{H}^{\otimes n}} \{S(\mathcal{M}^{\otimes n}(\rho)) - S((\mathcal{M}^{\otimes n} \otimes \mathbb{1}_{anc})(\Phi))\}. \quad (18)$$

On the one hand, the “sup” in Eq. (17) is evaluated over n parallel channel uses [20], where the channel map is described by the superoperator which transforms the input states ρ of $\mathcal{H}^{\otimes n}$ into the output states $\mathcal{M}^{\otimes n}(\rho)$. On the other hand, for fixed values of n , the maximization in Eq. (18) is performed on all possible input density matrices $\rho \in \mathcal{H}^{\otimes n}$. The quantity in the brackets is the coherent information of the channel [15,21], $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, and Φ is a purification [12] of $\rho \in \mathcal{H}^{\otimes n}$ defined in the extended space obtained by adding an ancillary space \mathcal{H}_{anc} to $\mathcal{H}^{\otimes n}$.

The classical capacity C gives the maximum amount of classical information that can be reliably transmitted through the channel per channel use: here, in the multi-use scenario, the goal is to identify the largest set of orthogonal input messages which remain distinguishable (i.e., orthonormal) during the propagation. In this case, the system is not required to preserve the phases of superpositions of different messages: C characterizes thus the ability of the channel \mathcal{M} in preserving occupation numbers but not its decoherence effects on the transmitted signals.

As in the case of Eq. (17), in calculating the classical capacity it is necessary to perform a maximization over multiple uses of the channel \mathcal{M} —i.e. [16],

$$C \equiv \sup_n C_n/n, \quad (19)$$

where C_n is the classical capacity of the channel which can be achieved if the sender is allowed to encode the information on codewords which are entangled only up to n -parallel-channel uses. The value of C_n is obtained by maximizing the Holevo information [22] at the output of n parallel channel uses, over all possible ensembles $\{\xi_k, \rho_k\}$ —i.e.,

$$C_n \equiv \max_{\xi_k, \rho_k \in \mathcal{H}^{\otimes n}} \left\{ S(\mathcal{M}^{\otimes n}(\rho)) - \sum_k \xi_k S(\mathcal{M}^{\otimes n}(\rho_k)) \right\}, \quad (20)$$

with $\{\xi_k\}$ probabilities and $\rho \equiv \sum_k \xi_k \rho_k$ the average message transmitted. The optimization (19) could be avoided if the additivity conjecture of the Holevo information is true [14,23]: in this case in fact the optimal ensembles $\{\xi_k, \rho_k\}$ which achieve the maximum in Eq. (20) are separable with respect to the n parallel uses and C coincides with C_1 .

The last capacities we consider are the entanglement-assisted classical capacity C_E and its quantum counterpart Q_E [17,18]. These quantities give, respectively, the maximum amount of classical or quantum information that can be sent reliably through the channel per channel use, assuming that the sender and receiver share infinite prior entanglement. To calculate C_E it is not requested to perform a regularization over parallel channel uses as in the case of Eqs. (17) and (19). Here instead one has to maximize the quantum mutual information for the single-channel use [17,18,24]—i.e.,

$$C_E \equiv \max_{\rho \in \mathcal{H}} \{S(\rho) + S(\mathcal{M}(\rho)) - S((\mathcal{M} \otimes \mathbb{1}_{anc})(\Phi))\}, \quad (21)$$

where now Φ is a purification of the input message $\rho \in \mathcal{H}$. The entanglement-assisted quantum capacity can then be obtained as $Q_E = C_E/2$ by means of quantum teleportation [25] and superdense coding [26]. The relevance of C_E relies on the fact that this quantity gives a simple upper bound for the other capacities. Moreover, it is conjectured to provide an equivalence class for quantum channels [18].

In the next sections we will focus on the capacities of the CPT maps associated with spin chain models introduced in Secs. II A 1 and II A 2.

A. Capacities of the amplitude damping channel

In this section we analyze in detail the qubit amplitude damping channel. In particular we calculate its capacity Q ,

showing that in this case maximization (18) over parallel uses is not necessary. Moreover, we derive the capacities C_E and C_1 (classical capacity achieved with unentangled codewords) which were originally given in Ref. [18] without an explicit derivation.

The map \mathcal{D}_η is completely characterized by the parameter η and can be seen as an instance of the map \mathcal{E}_η associated with the lossy bosonic channel [27–29] (see Appendix B for details). In particular \mathcal{D}_η has the useful property that by concatenating two amplitude damping channel with quantum efficiencies η and η' , one obtains a new amplitude damping channel with efficiency $\eta\eta'$ —i.e.,

$$\mathcal{D}_{\eta'}(\mathcal{D}_\eta(\rho)) = \mathcal{D}_{\eta\eta'}(\rho), \quad (22)$$

which applies for any input state ρ .

Quantum capacity. An important simplification in calculating the quantity (17) derives by introducing the following representation of the CPT map \mathcal{D}_η :

$$\mathcal{D}_\eta(\rho) \equiv \text{Tr}_C[V(\rho \otimes |0\rangle_C\langle 0|)V^\dagger], \quad (23)$$

obtained by adding to the Hilbert space \mathcal{H}_A of the input logical qubit A an auxiliary Hilbert space \mathcal{H}_C and introducing the unitary operator V which in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_C$ is given by the 4×4 matrix

$$V \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\eta} & \sqrt{1-\eta} & 0 \\ 0 & -\sqrt{1-\eta} & \sqrt{\eta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (24)$$

In Eq. (23), $\text{Tr}_C[\dots]$ is the partial trace over the elements of the auxiliary space \mathcal{H}_C . The complementary channel $\tilde{\mathcal{D}}_\eta$ of \mathcal{D}_η is defined by replacing this operation with the partial trace over \mathcal{H}_A —i.e. [30],

$$\tilde{\mathcal{D}}_\eta(\rho) \equiv \text{Tr}_A[V(\rho \otimes |0\rangle_C\langle 0|)V^\dagger]. \quad (25)$$

Upon a swapping operation S which transforms A into C and vice versa, the transformation of Eq. (25) can be seen as a mapping from the Hilbert space \mathcal{H}_A onto itself. Moreover, by direct calculation one can verify that

$$\tilde{\mathcal{D}}_\eta(\rho) = S\mathcal{D}_{1-\eta}(\rho)S. \quad (26)$$

Using the composition rule of Eq. (22) it is thus possible to show that, for $\eta \geq 0.5$, one has [31]

$$\tilde{\mathcal{D}}_\eta(\rho) = S\mathcal{D}_{(1-\eta)/\eta}(\mathcal{D}_\eta(\rho))S. \quad (27)$$

(The quantum capacity in the case $\eta < 0.5$ is simple to compute and will be discussed at the end of this section.) This relation shows that for the channel \mathcal{D}_η is degradable; i.e., there exists a CPT map defined by the super operator $S\mathcal{D}_{(1-\eta)/\eta}(\dots)S$ which connects the output state $\mathcal{D}_\eta(\rho)$ with the output state $\tilde{\mathcal{D}}_\eta(\rho)$. According to a theorem proved by Devetak and Shor [30], this condition guarantees that the “sup” in Eq. (17) is achieved for $n=1$ (single-channel use): in other words, Eq. (27) guarantees the additivity of the coherent information for the channel \mathcal{D}_η . For $\eta \geq 0.5$ the quan-

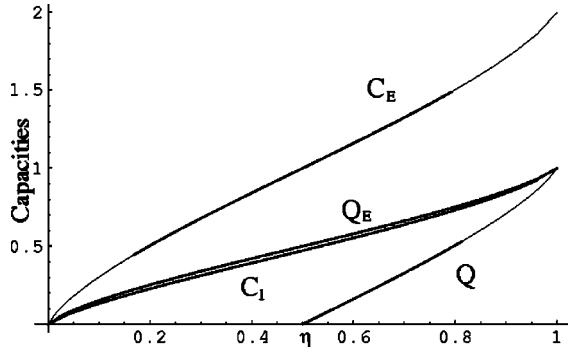


FIG. 1. Plot of the capacities (in bits per channel uses) of the qubit amplitude damping channel \mathcal{D}_η as a function of the noise parameter η . The curves Q , C_E , and C_1 represent, respectively, the quantum capacity, entanglement-assisted classical capacity, and the classical capacity achievable with unentangled encodings: they have been obtained by solving numerically the maximizations of Eqs. (36), (38), and (42). For $\eta=1$ (no-noise limit) both Q and C_1 give one qubit per bit for channel use, while as a consequence of the superdense coding effect [26] C_E gives two bits per channel uses. For $\eta=0.5$ we have, respectively, $Q=0$, $C_E=1$, and $C_1=0.4717$ (this result is in agreement with what found in Ref. [33]). As a consequence of the noncloning theorem [27,32] for $\eta<0.5$ the quantum capacity Q nullifies. The curve Q_E represents the entanglement-assisted quantum capacity and is obtained by simply dividing by 2 the values of C_E . The classical capacity C of the channel is lower bounded by C_1 and upper bounded by 1 (maximum entropy of a bit) and by C_E . If the additivity conjecture [23] applies to this channel, then $C=C_1$.

tum capacity of \mathcal{D}_η derives thus by solving the maximization (18) for $n=1$.

In the computational basis $\{|0\rangle, |1\rangle\}$ of \mathcal{H}_A , the most general input state of the map \mathcal{D}_η can be parametrized as follows:

$$\rho \equiv \begin{pmatrix} 1-p & \gamma^* \\ \gamma & p \end{pmatrix}, \quad (28)$$

where $p \in [0, 1]$ is the population associated with the state $|1\rangle$ and $|\gamma| \leq \sqrt{(1-p)p}$ is a coherence term. A purification $\Phi \equiv |\Phi\rangle\langle\Phi|$ of ρ is then obtained by introducing an ancillary qubit system \mathcal{H}_{anc} and considering the state

$$|\Phi\rangle \equiv \sqrt{1-p}|0\rangle \otimes |R_0\rangle + \sqrt{p}|1\rangle \otimes |R_1\rangle, \quad (29)$$

with $|R_{0,1}\rangle$ unit vectors of \mathcal{H}_{anc} such that

$$\langle R_0 | R_1 \rangle = \gamma / \sqrt{(1-p)p}. \quad (30)$$

From the Kraus decomposition (12) one can verify that the map \mathcal{D}_η transforms this state into the output

$$\mathcal{D}_\eta(\rho) = \begin{pmatrix} 1-\eta p & \sqrt{\eta}\gamma^* \\ \sqrt{\eta}\gamma & \eta p \end{pmatrix}. \quad (31)$$

The matrix (31) has eigenvalues

$$\lambda_{\pm}(\eta) \equiv [1 \pm \sqrt{(1-2\eta p)^2 + 4\eta|\gamma|^2}] / 2, \quad (32)$$

which gives an output von Neumann entropy equal to

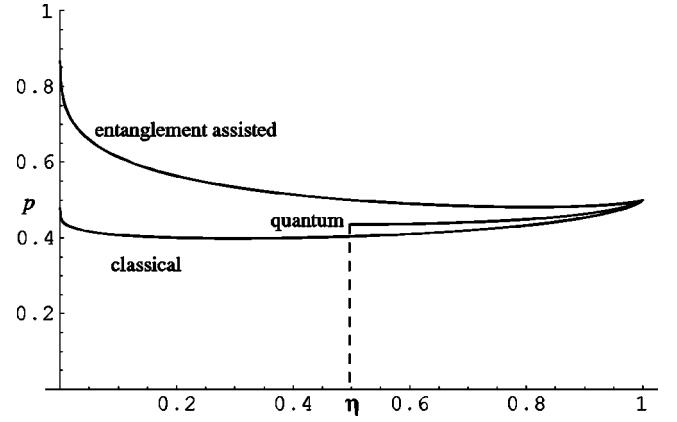


FIG. 2. Plot of the optimal populations p associated with the state $|1\rangle$ which provide the capacities of the amplitude damping channel \mathcal{D}_η as a function of the transmissivity η . The upper curve refers to the entanglement assisted capacities C_E and Q_E : for given η it has been obtained by finding that value of p which maximizes the function at the right-hand side of Eq. (38). The lower curve refers to the classical capacity C_1 through Eq. (42). Finally the intermediate curve refers to the quantum capacity Q through Eq. (36): this function is not defined for $\eta<0.5$ because for this value the Q is null (see Fig. 1). For $\eta \geq 0.5$ the optimal population p of the quantum capacity is bigger than the corresponding population of C_1 and lower than that of C_E .

$$S(\mathcal{D}_\eta(\rho)) = H_2(\lambda_+(\eta)), \quad (33)$$

with H_2 the binary entropy function defined in Eq. (C4). Analogously, by applying the map $(\mathcal{D}_\eta \otimes \mathbb{1}_{anc})$ to Φ we get the state described by the 4×4 complex matrix of Eq. (C1), which has entropy equal to

$$S((\mathcal{D}_\eta \otimes \mathbb{1}_{anc})(\Phi)) = H_2(\lambda_+(1-\eta)). \quad (34)$$

The quantity we need to maximize to obtain Q is hence given by

$$J(p, |\gamma|^2) \equiv H_2(\lambda_+(\eta)) - H_2(\lambda_+(1-\eta)), \quad (35)$$

which depends from ρ only through the parameters $p \in [0, 1]$ and $|\gamma|^2 \in [0, (1-p)p]$. As discussed in Appendix C the maximization (17) is achieved by choosing $\gamma=0$, which gives a quantum capacity equal to

$$Q \equiv \max_{p \in [0, 1]} \{H_2(\eta p) - H_2((1-\eta)p)\}. \quad (36)$$

For any given $\eta \geq 0.5$ this expression has been solved numerically and the results are plotted in Fig. 1. In Fig. 2, instead, we have reported, as a function of the parameter η , the optimal value of the population p which provides the maximum of the right-hand-side term of Eq. (36).

Let us now consider the low-transmissivity regime $\eta < 0.5$. In this case, the same noncloning argument given in Ref. [32] for the erasure channel and in Ref. [27] for the lossy bosonic channel can be used to prove that the quantum capacity Q of \mathcal{D}_η nullifies. An alternative proof of this fact can be obtained by noticing that the composition rule (22) implies that the quantum capacity of the channel \mathcal{D}_η is an increasing function of the transmissivity η . This thesis then

follows from the fact that for $\eta=0.5$ the right-hand-side term of Eq. (36) nullifies.

Entanglement-assisted capacity. To calculate this capacity we need to perform the maximization of Eq. (21) for all possible input states ρ . The quantum mutual information of the channel can be obtained by summing the coherent information $J(p, |\gamma|^2)$ of Eq. (35) to the input entropy of the message—i.e., using the parametrization introduced in the previous section:

$$I(p, |\gamma|^2) \equiv J(p, |\gamma|^2) + H_2\left(\frac{1 + \sqrt{(1-2p)^2 + 4|\gamma|^2}}{2}\right). \quad (37)$$

According to property (i) of Appendix C 1 the last term in the right-hand side of this expression is a decreasing function of $|\gamma|^2$; i.e., it is maximum for $\gamma=0$. From the previous section, we know that the same property applies also to $J(p, |\gamma|^2)$: we can thus conclude that, for any $p \in [0, 1]$, the function $I(p, |\gamma|^2)$ achieves its maximum value for $\gamma=0$. In other words, we can compute the entanglement assisted capacity C_E as

$$C_E \equiv \max_{p \in [0,1]} \{H_2(p) + H_2(\eta p) - H_2((1-\eta)p)\}. \quad (38)$$

This maximization can now be solved numerically: the resulting plot is given in Fig. 1. The optimal p 's that saturate the maximization of Eq. (38) are reported in Fig. 2.

Classical capacity with unentangled encodings. In the case of the lossy bosonic channel \mathcal{E}_η with constrained input average photon number the additivity property of the Holevo information has been proved [28]. Unfortunately, this derivation relies on some specific properties of the coherent input state of the bosonic channel: even though \mathcal{E}_η and \mathcal{D}_η are strongly related, it is hence difficult to use the result of [28] to establish the additivity conjecture for the qubit amplitude damping channel (see also Appendix B). Here we will not discuss further this problem, and simply we will focus on the capacity C_1 , which measures the maximum amount of classical information that can be reliably transmitted using only encodings that are not entangled over parallel channel uses [14,16,23]. The quantity C_1 is a lower bound for C and coincides with it provided the additivity conjecture holds. The classical capacity C_1 can be calculated by solving the maximization of Eq. (20) in the case of $n=1$. Consider the ensemble of messages where with probability ξ_k the channel is prepared in the input state,

$$\rho_k \equiv \begin{pmatrix} 1-p_k & \gamma_k^* \\ \gamma_k & p_k \end{pmatrix}, \quad (39)$$

with p_k and γ_k defined as in Eq. (28). Using the result of the previous section we can express the associated Holevo information as

$$\chi \equiv H_2\left(\frac{1 + \sqrt{(1-2\eta p)^2 + 4\eta|\gamma|^2}}{2}\right) - \sum_k \xi_k H_2\left(\frac{1 + \sqrt{(1-2\eta p_k)^2 + 4\eta|\gamma_k|^2}}{2}\right), \quad (40)$$

where now $p = \sum_k \xi_k p_k$ and $\gamma = \sum_k \xi_k \gamma_k$ are the parameters (28) associated with the average input message $\rho \equiv \sum_k \xi_k \rho_k$. According to Eq. (20) the capacity C_1 is obtained by maximizing χ over all possible choices of p_k , γ_k , and ξ_k . To solve this problem we first derive an upper bound for C_1 and then we show that there exist an encoding p_k , γ_k , and ξ_k which achieves such an upper bound.

From the property (i) of the binary entropy (C4) given in Appendix C 1, we can maximize the first term in the right-hand side of Eq. (40) by choosing $\gamma=0$. Moreover, one has

$$\begin{aligned} & \sum_k \xi_k H_2\left(\frac{1 + \sqrt{(1-2\eta p_k)^2 + 4\eta|\gamma_k|^2}}{2}\right) \\ & \geq \sum_k \xi_k H_2\left(\frac{1 + \sqrt{1-4\eta(1-\eta)p_k^2}}{2}\right) \\ & \geq H_2\left(\frac{1 + \sqrt{1-4\eta(1-\eta)(\sum_k \xi_k p_k)^2}}{2}\right), \end{aligned} \quad (41)$$

where the first inequality derives from the property (i) of $H_2(z)$ and from the fact that $|\gamma_k|^2 \leq (1-p_k)p_k$, while the second inequality is consequence of the property (ii). Replacing the above relation in Eq. (40) we obtain an upper bound for χ which does not depend on γ_k and which depends on ξ_k and p_k only through $p = \sum_k \xi_k p_k$. By maximizing this expression over all possible choices of the variable p we get the following upper bound of C_1 :

$$C_1 \leq \max_{p \in [0,1]} \left\{ H_2(\eta p) - H_2\left(\frac{1 + \sqrt{1-4\eta(1-\eta)p^2}}{2}\right) \right\}. \quad (42)$$

The right-hand-side term of this inequality is indeed the value of C_1 . This can be shown by noticing that for any $p \in [0, 1]$ and $d > 1$, the parameters

$$\begin{aligned} \xi_k &= 1/d, \quad p_k = p, \\ \gamma_k &= e^{2\pi i k/d} \sqrt{(1-p)p}, \end{aligned} \quad (43)$$

with $k=1, \dots, d$, produce a Holevo information χ of Eq. (40) which is coincident with the quantity in the brackets on the right-hand side of Eq. (42). The quantities (43) provide hence optimal encoding strategies for C_1 . On one hand, any ensemble element ρ_k of this encoding has a maximum absolute value of the coherence term γ_k : this minimizes the negative term of the Holevo information. On the other hand, the average message $\rho = \sum_k \xi_k \rho_k$ has minimum value $|\gamma|$ —i.e.,

$$\gamma = \sum_{k=1}^d \xi_k \gamma_k = \sqrt{(1-p)p} \sum_{k=1}^d e^{2\pi i k/d} / d = 0, \quad (44)$$

which maximizes the positive contribution to the Holevo information. This property of the channel \mathcal{D}_η is a common feature of many other channels whose classical capacity C has been solved [28,34]. The value of C_1 obtained by maximizing the right-hand-side term of Eq. (42) has been plotted in Fig. 1, while the optimal p 's are plotted in Fig. 2.

B. Capacities of the channel $\mathcal{T}_{\eta_1, \eta_2}$

In this section we analyze in detail the CPT map $\mathcal{T}_{\eta_1, \eta_2}$ of Eq. (15) associated with the spin chain communication line of Sec. II A 2. In this case we calculate the capacities Q , C_E and we provide a lower bound for C .

The map $\mathcal{T}_{\eta_1, \eta_2}$ is described by the positive parameters η_1 and η_2 of Eqs. (A4) and (A5) that satisfy the relation $\eta_1 + \eta_2 \leq 1$. In particular, for $\eta_2 = 1 - \eta_1$, $\mathcal{T}_{\eta_1, \eta_2}$ reduces to an amplitude damping map of transmissivity η_1 —i.e.,

$$\mathcal{T}_{\eta_1, 1-\eta_1}(\rho) = \mathcal{D}_{\eta_1}(\rho), \quad (45)$$

for any input state ρ . Moreover, the following composition rule applies:

$$\mathcal{T}_{\eta'_1, \eta'_2}(\mathcal{T}_{\eta_1, \eta_2}(\rho)) = \mathcal{T}_{\eta''_1, \eta''_2}(\rho), \quad (46)$$

with $\eta''_1 \equiv \eta_1 \eta'_1$ and $\eta''_2 \equiv \eta_2 + \eta_1 \eta'_2$. An interesting way to express $\mathcal{T}_{\eta_1, \eta_2}$ is finally provided by concatenating two amplitude damping channels with a CPT map \mathcal{P} that transforms one of the eigenvectors of the matrix σ_B (say, $|\zeta_1\rangle$) into σ_B itself—i.e.,

$$\mathcal{T}_{\eta_1, \eta_2}(\rho) = \mathcal{P}(\mathcal{D}'_{\eta_1/(1-\eta_2)}(\mathcal{D}_{1-\eta_2}(\rho))), \quad (47)$$

where the first amplitude damping channel $\mathcal{D}_{1-\eta_2}$ acts as usual on $|0\rangle$ and $|1\rangle$, while the second one $\mathcal{D}'_{\eta_1/(1-\eta_2)}$ is instead defined on the subspace generated by $|\zeta_1\rangle$ and $|1\rangle$. The main consequence of Eq. (47) is that any capacity of $\mathcal{T}_{\eta_1, \eta_2}$ cannot be greater than the corresponding capacity of $\mathcal{D}_{1-\eta_2}$. In fact, by applying the CPT transformations \mathcal{P} and $\mathcal{D}'_{\eta_1/(1-\eta_2)}$ to the output of an amplitude damping channel of transmissivity $1 - \eta_2$ one can simulate the corresponding output of $\mathcal{T}_{\eta_1, \eta_2}$.

Quantum capacity. As in the case of \mathcal{D}_η we can prove that $\mathcal{T}_{\eta_1, \eta_2}$ is degradable when its quantum capacity is not null. In fact, as in Eq. (23) define the unitary operator V of the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C$ such that, for any ρ of \mathcal{H}_A ,

$$\mathcal{T}_{\eta_1, \eta_2}(\rho) \equiv \text{Tr}_C[V(\rho \otimes |0\rangle_C \langle 0|)V^\dagger]. \quad (48)$$

An example of V can be obtained by introducing the following vector of $\mathcal{H}_A \otimes \mathcal{H}_C$:

$$|\Phi_\sigma\rangle \equiv \sum_i \sqrt{\xi_i} |\zeta_i\rangle_A \otimes |\zeta_i\rangle_C, \quad (49)$$

where $|\zeta_i\rangle_A \in \mathcal{H}_A$ are the eigenvectors of σ_B introduced in Eq. (16) while $|\zeta_i\rangle_C$ is an orthonormal set of states of \mathcal{H}_C that are orthogonal to $|0\rangle_C$ and $|1\rangle_C$. The state $|\Phi_\sigma\rangle$ is a purification of σ_B on $\mathcal{H}_A \otimes \mathcal{H}_C$. The unitary operator V can now be chosen

to be the identity everywhere but on the subspace of $\mathcal{H}_A \otimes \mathcal{H}_C$ generated by the orthonormal vectors $\{|00\rangle, |01\rangle, |10\rangle, |\Phi_\sigma\rangle\}$. On this subset we define V to have the matrix representation

$$V \equiv \begin{pmatrix} \frac{1 + \sqrt{\eta_1 - \eta_2}}{1 + \sqrt{\eta_1}} & \sqrt{\eta_2} & -\frac{\sqrt{\eta_2 \eta_3}}{1 + \sqrt{\eta_1}} \\ -\sqrt{\eta_2} & \sqrt{\eta_1} & -\sqrt{\eta_3} \\ -\frac{\sqrt{\eta_2 \eta_3}}{1 + \sqrt{\eta_1}} & \sqrt{\eta_3} & \frac{\eta_2 + \eta_3 \sqrt{\eta_1}}{1 - \eta_1} \end{pmatrix}, \quad (50)$$

with $\eta_3 = 1 - \eta_1 - \eta_2$. The complementary map $\tilde{\mathcal{T}}_{\eta_1, \eta_2}$ is finally obtained by substituting in Eq. (48) the trace over C with the trace on A [30]. From Eqs. (45) and (46) one can easily verify that for $\eta_1 \geq \eta_2$ the following relation applies for any input ρ :

$$\tilde{\mathcal{T}}_{\eta_1, \eta_2}(\rho) = S \mathcal{D}_{\eta_2/\eta_1}(\mathcal{T}_{\eta_1, \eta_2}(\rho)) S, \quad (51)$$

with S the swapping operation which transforms A in C . Since $\mathcal{D}_{\eta_2/\eta_1}$ is CPT, the above equation shows that for $\eta_1 \geq \eta_2$ the map $\mathcal{T}_{\eta_1, \eta_2}$ is degradable: the quantum capacity of this channel can be hence computed from Eq. (18) for $n=1$. A straightforward generalization of the qubit amplitude damping channel analysis shows that Eq. (35) still applies by replacing $\lambda_\pm(\eta)$ of Eq. (32) with

$$\lambda_\pm(\eta_1, \eta_2) \equiv \frac{1 - \eta_3 p}{2} \times \left[1 \pm \sqrt{\left(1 - \frac{2\eta_1 p}{1 - \eta_3 p}\right)^2 + \frac{4\eta_1 |\gamma|^2}{(1 - \eta_3 p)^2}} \right]. \quad (52)$$

The quantum capacity of $\mathcal{T}_{\eta_1, \eta_2}$ becomes hence

$$Q \equiv \max_{p \in [0,1]} \left\{ (1 - \eta_3 p) \left[H_2\left(\frac{1 - (1 - \eta_2)p}{1 - \eta_3 p}\right) - H_2\left(\frac{1 - (1 - \eta_1)p}{1 - \eta_3 p}\right) \right] \right\}. \quad (53)$$

Notice that as in the case of Eq. (36) the maximization over the input parameter γ of Eq. (28) has been saturated by setting $\gamma=0$ [the proof goes as in Eq. (36)]. A plot of Q as a function of η_1 and η_2 is reported in Fig. 3 by solving numerically the maximization on p . The above results do not apply for $\eta_1 \leq \eta_2$: in this case in fact $\mathcal{D}_{\eta_2/\eta_1}$ is not CPT. However, a noncloning argument can be used to prove that in this parameter region, the quantum capacity of $\mathcal{T}_{\eta_1, \eta_2}$ nullifies [35].

Entanglement-assisted capacity. The analysis of C_E proceeds as in the case of the amplitude damping channel. Here Eq. (38) is replaced by

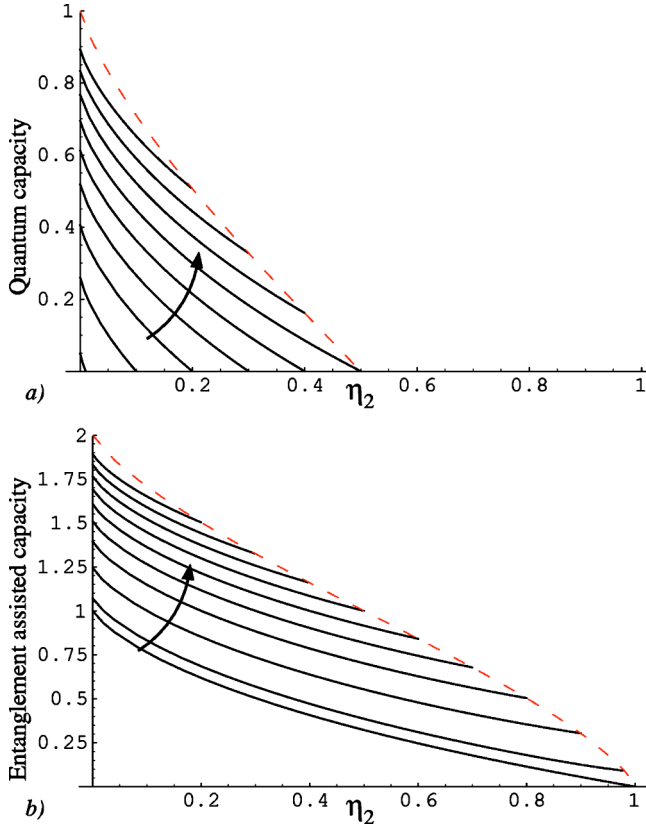


FIG. 3. Plots of the capacities (in bits per channel uses) of the channel $\mathcal{T}_{\eta_1, \eta_2}$ of Eq. (15) as a function of $\eta_2 \in [0, 1 - \eta_1]$ obtained for different η_1 (this parameter increases from 0 to 1 moving along the arrows). In both graphics the dashed curve is the capacity of an amplitude damping channel with efficiency $1 - \eta_2$ which according to Eq. (47) provides an upper bound for the corresponding capacity of $\mathcal{T}_{\eta_1, \eta_2}$. (a) Quantum capacity Q obtained by solving numerically the maximization over p of Eq. (53). Notice that for $\eta_2 > \eta_1$ Q nullifies [35]. (b) Entanglement-assisted classical capacity C_E of Eq. (54).

$$C_E \equiv \max_{p \in [0, 1]} \left\{ H_2(p) + (1 - \eta_3 p) \left[H_2 \left(\frac{1 - (1 - \eta_2)p}{1 - \eta_3 p} \right) - H_2 \left(\frac{1 - (1 - \eta_1)p}{1 - \eta_3 p} \right) \right] \right\}. \quad (54)$$

A numerical plot of this expression is given in Fig. 3.

Lower bound for C_1 . The analysis of C_1 for the channel $\mathcal{T}_{\eta_1, \eta_2}$ is slightly more difficult than that for \mathcal{D}_η as the convexity properties used in Eq. (41) do not hold in this case. Here thus we give a lower bound for C_1 obtained by assuming the encoding Eq. (43):

$$C_1 \geq \max_{p \in [0, 1]} \left\{ (1 - \eta_3 p) \left[H_2 \left(\frac{1 - (1 - \eta_2)p}{1 - \eta_3 p} \right) - H_2 \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\eta_1 \eta_2 p^2}{(1 - \eta_3 p)^2}} \right) \right] \right\}. \quad (55)$$

IV. CONCLUSIONS

In the previous section we calculated the capacities Q , C_E , and C_1 for the qubit amplitude damping channel \mathcal{D}_η and for the channel $\mathcal{T}_{\eta_1, \eta_2}$. As discussed in Sec. II A, by identifying the parameters η , η_1 , and η_2 with the quantities of Eqs. (9), (A4), and (A5) we can use these results to analyze the correlations between distant points of the chain. A detailed analysis of the fidelity for state transfer (9) in the case of $k=1$ and uniform coupling $J_{i,j}$ is given in Ref. [1]. Such a paper solves the dependence of η from the evolution time and from the length of the chain. In particular in [1] it is shown that the maximum value of η drops with the distance between the encoding spin A (at the site r) and the decoding spin B (at site s) as $|r-s|^{-2/3}$. Since according to Sec. III A the quantum capacity Q vanishes when the transmissivity is ≤ 0.5 , this implies that long chains are not suitable to directly transmit quantum information using the scheme of Sec. II A 1 with $k=1$. Even though the communication scenarios described by means of Eq. (1) are incomplete (see the discussion of Sec. II), it is interesting to see in which way the above difficulties can be overcome. One possibility is to optimize the values of η by tailoring the interaction $J_{i,j}$ between the spins as proposed in Ref. [3]. Alternatively one can use registers with $k > 1$ and then optimizing the value of η of Eq. (9) by means an appropriate choice [2] of the coefficients c_j of Eq. (5).

Yet a different approach would require the partition of the chain in smaller segments so to transfer information faithfully along the chain through swaps between neighboring segments analogously to the ideas of quantum repeaters introduced in [36]. Breaking the chain can be achieved by applying locally external magnetic fields. A combination of a time-dependent control of part of the chain together with perfect transmission of small segments may lead to an improvement of the performances of spin chains to transport quantum information. On the other hand, the quantum capacity of the chain can be boosted by giving access the sender and receiver to a free two-way classical communication line [18,37]. In this case, the ability of the channel in transmitting quantum signal can be increased by means of entanglement distillation protocols and teleportation. For instance (as noticed also in [1]), since any two-qubit entanglement state is distillable [38], the two-way quantum capacity $Q_{2\text{-way}}$ of the chain is strictly greater than zero also when Q is null. To compute the exact value of $Q_{2\text{-way}}$ one needs to find the optimal distillation protocol for \mathcal{M} : unfortunately this is quite a challenging task.

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APPENDIX A: DERIVATION OF Eq. (14)

The Hamiltonian (2) preserves the total spin component along the z axis. The set of two-spin-up states $|j, \ell\rangle$ is hence transformed into itself with unitary matrix elements

$$f_{j\ell,j'\ell'}(t) \equiv \langle j, \ell | e^{-iHt/\hbar} | j', \ell' \rangle. \quad (\text{A1})$$

Using this transformation it is possible to show that at time t the spin chain state becomes

$$|\Psi(t)\rangle \equiv \alpha |\Downarrow\rangle + \beta \sum_{j>\ell=1}^N d_{j,\ell}(t) |j, \ell\rangle, \quad (\text{A2})$$

where, for $j > \ell = 1, \dots, N$,

$$d_{j,\ell}(t) \equiv \sum_{j'>\ell'=1}^k d_{j'\ell'} f_{j\ell,j'\ell'}(t) \quad (\text{A3})$$

are the time evolved of the coefficient $d_{j,\ell}$ of Eq. (13). Equation (A2) shows that only the fraction

$$\eta_1 \equiv \sum_{j>\ell=N-k+1}^N |d_{j,\ell}(t)|^2 \quad (\text{A4})$$

of $|\Psi(t)\rangle$ has two spins up in B . Analogously,

$$\eta_2 \equiv \sum_{j>\ell=1}^{N-k} |d_{j,\ell}(t)|^2 \quad (\text{A5})$$

is the probability of $|\Psi(t)\rangle$ having both two spins up outside of B , while

$$\eta_3 \equiv 1 - \eta_1 - \eta_2 = \sum_{\ell=1}^{N-k} \sum_{j=N-k+1}^N |d_{j,\ell}(t)|^2 \quad (\text{A6})$$

is the probability of having one spin up in B and the other outside of B . By taking the partial trace of Eq. (A2) over the first $N-k$ spins of the chain we obtain the state of the quantum memory B of Eq. (14). In such an expression,

$$|\phi_2'\rangle_B \equiv \sum_{j>\ell=N-k+1}^N d_{j,\ell}(t) |j, \ell\rangle / \sqrt{\eta_1} \quad (\text{A7})$$

is a rotation of $|\phi_2\rangle_A$ that can be compensated for at the decoding stage. On the other hand, σ_B of Eq. (14) is a density matrix of B formed by states with one-spin up vectors—i.e.,

$$\sigma_B \equiv \sum_{\ell=1}^{N-k} \eta_3(\ell) |\phi_1(\ell)\rangle \langle \phi_1(\ell)| / \eta_3, \quad (\text{A8})$$

with

$$|\phi_1(\ell)\rangle_B \equiv \sum_{j=N-k+1}^N d_{j,\ell}(t) |j, \ell\rangle / \sqrt{\eta_3(\ell)},$$

$$\eta_3(\ell) \equiv \sum_{j=N-k+1}^N |d_{j,\ell}(t)|^2. \quad (\text{A9})$$

Notice that the rank of σ_B is at most equal to the maximum number of orthogonal one-spin-up states of B —that is, k . Moreover, the support of σ_B is clearly orthogonal to $|\Downarrow\rangle_B$ and $|\phi_2'\rangle_B$.

APPENDIX B: RELATION WITH THE BOSONIC LOSSY CHANNEL

In the lossy bosonic channel \mathcal{E}_η [27–29] an input bosonic mode described by the annihilation operator a interacts, through a beam splitter of transmissivity η , with the vacuum state $|\emptyset\rangle_b$ of an external bosonic mode described by the annihilation operator b . Any input state ρ of the mode a is hence transformed by this map according to the equation

$$\mathcal{E}_\eta(\rho) \equiv \text{Tr}_b[U(\rho \otimes |\emptyset\rangle_b \langle \emptyset|) U^\dagger], \quad (\text{B1})$$

where the trace is performed over the external mode b and where U is the beam splitter unitary operator defined by

$$U^\dagger a U = \sqrt{\eta} a + \sqrt{1-\eta} b, \quad (\text{B2})$$

$$U^\dagger b U = \sqrt{\eta} b - \sqrt{1-\eta} a. \quad (\text{B3})$$

By restricting the inputs ρ to the Hilbert space spanned by the vacuum state and the one-photon Fock state the map (23) has the same Kraus decomposition (12) of the qubit map \mathcal{D}_η . Some capacities of the channel \mathcal{E}_η have been solved under constrained average input photon number: the classical capacity C is given in Ref. [28] while the entanglement-assisted capacity C_E and a lower bound for Q which is supposed to be tight are given in Ref. [39]. Unfortunately, since an average input photon number constraint cannot prevent the average message of \mathcal{E}_η from being supported on Fock states with more than one photon (apart from the trivial case of zero average photon number), the results obtained in [28,39] provide only trivial upper bounds for the corresponding capacities of \mathcal{D}_η . For instance, consider the entanglement-assisted case where the capacities of both the channels can be computed. In the input Hilbert space we are considering here, the average photon number of the transmitted message is provided by the average population associated with the one-photon Fock state. A fair comparison between the capacities of \mathcal{D}_η and \mathcal{E}_η can be hence obtained by taking the value of C_E associated with a lossy bosonic \mathcal{E}_η channel where the average input photon number is given by the population p of Eq. (38) which maximizes the entanglement-assisted capacity of \mathcal{D}_η . According to [39] the capacity of \mathcal{E}_η is then given by $C_E \equiv g(p) + g(\eta p) - g((1-\eta)p)$, where $g(x) = (x+1)\log_2(x+1) - x\log_2 x$. A simple numerical analysis can be used to verify that this quantity is always bigger than the corresponding value (38) of \mathcal{D}_η .

The channels \mathcal{D}_η and \mathcal{E}_η share many common features. In particular \mathcal{E}_η obeys the same composition rule of \mathcal{D}_η given in Eq. (22) and it is degradable, since for any input ρ one has

$$\tilde{\mathcal{E}}_\eta(\rho) = P S \mathcal{E}_{1-\eta}(\rho) S P, \quad (\text{B4})$$

where now $P = e^{i\pi b^\dagger b}$ and S is the swap operator which transforms a in b and vice versa. This relation was used in [29] without explicitly proving it and applies to all Gaussian channels of the form (23) where the external bosonic mode b is prepared in a circularly symmetric input. For the sake of completeness, here we give an explicit derivation of Eq. (B4) in the case of the purely lossy bosonic channel.

Consider a generic input state ρ of the bosonic channel, with characteristic function $\Gamma(\mu) \equiv \text{Tr}_a[\rho D_a(\mu)]$ —i.e.,

$$\rho \equiv \int \frac{d^2\mu}{\pi} \Gamma(\mu) D_a(-\mu), \quad (\text{B5})$$

where $D_a(\mu) \equiv \exp[\mu a^\dagger - \mu^* a]$ is the displacement operator of the input mode a [40]. As shown in [29], the channel \mathcal{E}_η transforms ρ into

$$\mathcal{E}_\eta(\rho) = \int \frac{d^2\mu}{\pi} \Gamma'(\mu) D_a(-\mu), \quad (\text{B6})$$

with

$$\Gamma'(\mu) \equiv \Gamma(\sqrt{\eta}\mu) e^{-(1-\eta)|\mu|^2/2}. \quad (\text{B7})$$

Analogously it is possible to verify that the complementary map $\tilde{\mathcal{E}}_\eta$ [defined as in Eq. (25) by replacing the partial trace over b with the partial trace over a in the Eq. (B1)] produces the transformation

$$\tilde{\mathcal{E}}_\eta(\rho) = \int \frac{d^2\mu}{\pi} \tilde{\Gamma}'(\mu) D_b(-\mu), \quad (\text{B8})$$

where $D_b(\mu)$ is the displacement operator of the mode b and where

$$\tilde{\Gamma}'(\mu) \equiv \Gamma(-\sqrt{1-\eta}\mu) e^{-\eta|\mu|^2/2}. \quad (\text{B9})$$

Suppose now $\eta \geq 0.5$ and apply the lossy map $\mathcal{E}_{1-\eta/\eta}$ to the state of Eq. (B6): according to the composition rule (22) this will transform its symmetric characteristic function to

$$\Gamma''(\mu) \equiv \Gamma(\sqrt{1-\eta}\mu) e^{-\eta|\mu|^2/2} = \tilde{\Gamma}'(-\mu), \quad (\text{B10})$$

producing the state

$$\mathcal{E}_{(1-\eta)/\eta}(\mathcal{E}_\eta(\rho)) = \int \frac{d^2\mu}{\pi} \tilde{\Gamma}'(\mu) D_a(\mu). \quad (\text{B11})$$

This proves the identity (B4) since under the unitary transformation PS the annihilation operator a is transformed into $-b$ and the state (B11) becomes equal to the output (B8) of the composite map.

APPENDIX C: SOME USEFUL RELATIONS

In this appendix we provide some relations used in Sec. III to derive the capacities of the qubit amplitude damping channel.

1. Entropy of exchange

The output state $(\mathcal{D}_\eta \otimes \mathbb{1}_{anc})(\Phi)$ associated with the purification $|\Phi\rangle$ of Eq. (29) can be expressed in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_{anc}$ (here $|0\rangle_{anc} \equiv |R_0\rangle$ while $|1\rangle_{anc}$ is proportional to the component of $|R_1\rangle$ orthogonal to $|R_0\rangle$). This gives the 4×4 matrix

$$\begin{pmatrix} 1-p + (1-\eta)\frac{|\gamma|^2}{1-p} & (1-\eta)\gamma\frac{\sqrt{(1-p)p-|\gamma|^2}}{1-p} & \sqrt{\eta}\gamma^* & \sqrt{\eta}\sqrt{(1-p)p-|\gamma|^2} \\ (1-\eta)\gamma^*\frac{\sqrt{(1-p)p-|\gamma|^2}}{1-p} & (1-\eta)\left(p - \frac{|\gamma|^2}{1-p}\right) & 0 & 0 \\ \sqrt{\eta}\gamma & 0 & \eta\frac{|\gamma|^2}{1-p} & \eta\gamma\frac{\sqrt{(1-p)p-|\gamma|^2}}{1-p} \\ \sqrt{\eta}\sqrt{(1-p)p-|\gamma|^2} & 0 & \eta\gamma^*\frac{\sqrt{(1-p)p-|\gamma|^2}}{1-p} & \eta\left(p - \frac{|\gamma|^2}{1-p}\right) \end{pmatrix} \quad (\text{C1})$$

The matrix (C1) has eigenvalues 0 (2-times degenerate) and

$$\Lambda_\pm \equiv \{1 \pm \sqrt{[1-2(1-\eta)p]^2 + 4(1-\eta)|\gamma|^2}\}/2 \equiv \lambda_\pm(1-\eta), \quad (\text{C2})$$

with λ_\pm the eigenvalues of $\mathcal{D}_\eta(\rho)$ given in Eq. (32). The exchange entropy associated with ρ is hence given by

$$S((\mathcal{D}_\eta \otimes \mathbb{1}_{anc})(\Phi)) = -\Lambda_+ \ln \Lambda_+ - \Lambda_- \ln \Lambda_- \equiv H_2(\Lambda_+), \quad (\text{C3})$$

where, for $x \in [0, 1]$,

$$H_2(x) \equiv -x \log_2 x - (1-x) \log_2 (1-x) \quad (\text{C4})$$

is the binary entropy [41]. Some useful relations of $H_2(z)$ are the following.

(i) The function $H_2(z)$ is decreasing with respect to the variable $|1/2+z|$. This property is a consequence of the fact that the entropy associated with a binary stochastic variable is maximum when the probabilities associated with different outcomes are equal.

(ii) The function $H_2((1+\sqrt{1-z^2})/2)$ is convex with respect to z . This property can be easily verified and is related with the convexity of the entanglement of formation with respect to concurrence in qubit systems [42].

2. Dependence on γ of the coherent information

As discussed in the text, the quantum capacity Q of \mathcal{D}_η is obtained by maximizing the function $J(p, |\gamma|^2)$ of Eq. (35) over all values of $p \in [0, 1]$ and $|\gamma|^2 \in [0, (1-p)p]$. For $\eta \geq 1/2$, this expression is decreasing in $|\gamma|^2$; i.e., for any $|\beta| \in [0, 1]$ it achieves the maximum value for $\gamma=0$. The proof of this result is quite tedious, but can be obtained analytically by studying the partial derivative of Eq. (35) with respect to the parameter $|\gamma|^2$. We skip all the details of this analysis which is not of fundamental interest and simply observe that

the problem can be reduced to studying the properties of the function

$$f_y(x) \equiv \left(\frac{1 - \sqrt{(1-2x)^2 + 4xy}}{1 + \sqrt{(1-2x)^2 + 4xy}} \right)^{x/\sqrt{(1-2x)^2 + 4xy}} \quad (C5)$$

on the domain $x \in [0, 1-y]$, for any $y \in [0, 1]$. One can then verify that $f_y(x)$ is decreasing in x : this guarantees that $J(p, |\gamma|^2)$ is monotonically decreasing in $|\gamma|^2$, yielding the thesis.

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 [20] In the context of memoryless channel the parameter n is often presented as the number of “successive” uses of the channel [12,14]. However, in the spin chain model we are considering here this definition is misleading since repetitive uses of the same chain introduce memory effects that are not taken into account by Eq. (1) (see also Sec. II). For this reason we refer to n as the number “parallel” uses of the channel to stress the fact that the sender is operating at the same time on n identical “copies” of the original spin chain.
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