Perfect transfer of arbitrary states in quantum spin networks

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We propose a class of qubit networks that admit perfect state transfer of any two-dimensional quantum state in a fixed period of time. We further show that such networks can distribute arbitrary entangled states between two distant parties, and can, by using such systems in parallel, transmit the higher-dimensional systems states across the network. Unlike many other schemes for quantum computation and communication, these networks do not require qubit couplings to be switched on and off. When restricted to N-qubit spin networks of identical qubit couplings, we show that $2\log_3 N$ is the maximal perfect communication distance for hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits then perfect state transfer can be achieved over arbitrarily long distances in a linear chain. This paper expands and extends the work done by Christandl *et al.*, Phys. Rev. Lett. **92**, 187902 (2004).

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I. INTRODUCTION

An important task in quantum-information processing is the transfer of quantum states from one location (A) to another location (B). In a quantum-communication scenario this is rather explicit, since the goal is the communication between distant parties A and B (e.g., by means of photon transmission). Equally, in the interior of quantum computers good communication between different parts of the system is essential. The need is thus to transfer quantum states and generate entanglement between different regions contained within the system. There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. This can be generally defined as a collection of interacting qubits on a graph, whose dynamics is governed by a suitable Hamiltonian, e.g., the Heisenberg or XY Hamiltonian. One way to accomplish this task is by multiple applications of controlled swap operations along the communication line. Every external manipulation, however, inevitably induces noise in the system. It is therefore desirable to minimize the amount of external control on the system, to the point that they do not require any external manipulation whatsoever.

Quantum communication over short distances through a spin chain, in which adjacent qubits are coupled by equal strength has been studied in detail, and an expression for the fidelity of quantum state transfer has been obtained [1,2]. Similarly, in Ref. [3], near perfect state transfer was achieved for uniform couplings provided a spatially varying magnetic field was introduced. The propagation of quantum information in rings has also been investigated [4].

In our work we focus on the situation in which state transfer is *perfect*, i.e., the fidelity is unity, and in which we can design networks such that this can be achieved over arbitrarily long distances. We will also consider the case in which *no* external control is required during the state transfer, i.e., we consider the case in which we have, after manufacturing the network, no further control over its dynamics. In general this will lead us to think about more complicated networks than the linear chain or chains with preengineered nearest-neighbor interaction strengths. We provide two alternative methods for understanding how perfect state transfer is achieved with preengineered couplings. This paper expands and extends the work done in [5]. Subsequent work has examined the extension of this problem to higher excitation subspaces [6]. The subject of perfect state transfer has been independently studied in the first and second excitation subspaces in [7], where its implementation in an array of quantum dots was considered.

More specifically, we address the problem of arranging N interacting qubits in a network such that it allows for perfect transfer of any quantum state over the longest possible distance. Two qubits are coupled via an XY interaction if an edge connects the two corresponding sites. We show further how one can use these networks to transfer entangled quantum states and to generate entanglement between distant sites in the network. The connection between our approach and the continuous-time quantum random walk is highlighted and, in a particular example, contrasted with the correspond-

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ing result for the classical continuous-time random walk.

The paper is organized as follows. In Sec. II we set the scene by introducing the problem of perfect state transfer in quantum spin systems. In Sec. III we derive a necessary condition for our problem in the case of graphs with mirror symmetry. This is used in Sec. IV to give a limitation of the transfer in chains with uniform couplings. In Sec. V we investigate hypercube geometry as a way to enlarge the previously found limit and compare our result to a classical analog in Sec. VI. The quantum walk on the hypercube is found useful to derive a *modulated* spin chain in Sec. VII that allows perfect transfer over arbitrary distances. We will exhibit a beautiful group-theoretic interpretation of this chain in Sec. VIII. In Secs. X and XI we consider applications for entanglement transfer and the introduction of arbitrary phase gates on the fly.

II. STATE TRANSFER IN QUANTUM SPIN SYSTEMS

In order to set the scene, let us first consider quantum state transfer over a general quantum network. We define a general finite quantum network to be a simple, connected, finite graph $G:=\{V(G),E(G)\}$, where $V(G)$ denotes the finite set of its vertices and $E(G)$ the set of its edges. Two vertices $i,j\in V(G)$ are adjacent if $(i,j)\in E(G)$. To any such graph G one can associate an adjacency matrix $A(G)$ whose elements satisfy

$$A_{ij}(G):=\begin{cases}1 & \text{if } (i,j)\in E(G),\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

A quantum spin system associated with such a graph is defined by attaching a spin-$\frac{1}{2}$ particle to each vertex of the graph. To each vertex $i\in V(G)$ we can therefore associate a Hilbert space $\mathcal{H}_i\simeq\mathbb{C}^2$. The Hilbert space associated with G is then given by

$$\mathcal{H}_G=\bigotimes_{i\in V(G)}\mathcal{H}_i=(\mathbb{C}^2)^{\otimes N}, \tag{2}$$

where $N:=|V(G)|$ denotes the total number of vertices in G.

We define the distance $d(i,j)$ between any two vertices $i,j\in V(G)$ to be the number of edges of the shortest path between i and j, i.e., the graph geodesic between the two vertices.

Consider the dynamics of the system to be governed by the quantum-mechanical Hamiltonian

$$H_G=\frac{1}{2}\sum_{(i,j)\in E(G)}J_{ij}[\sigma_i^x\sigma_j^x+\sigma_i^y\sigma_j^y]. \tag{3}$$

We use the symbols σ_i^x,σ_i^y, and σ_i^z to denote the familiar Pauli matrices acting on the on-site Hilbert space $\mathcal{H}_i$, and J_{ij} is the coupling strength between the ith and jth sites on the graph. Note that $J_{ij}=J_{ji}$ since H_G is Hermitian. The total z component of the spin, given by

$$\sigma_{\text{tot}}^z:=\sum_{i\in V(G)}\sigma_i^z, \tag{4}$$

is conserved, i.e., $[\sigma_{\text{tot}}^z,H_G]=0$. Hence the Hilbert space $\mathcal{H}_G$ decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator σ_{tot}^z.

For the purpose of quantum state transfer, it suffices to restrict our attention to the N-dimensional eigenspace of σ_{tot}^z, which corresponds to the eigenvalue $(2-N)/2$. Let us denote this subspace by $\mathcal{S}_G$. Initial quantum states that are in this subspace will remain there under time evolution. A basis state in $\mathcal{S}_G$ corresponds to a spin configuration in which all the spins except one are down and one spin is up. Such a basis state can hence be denoted by the ket $|j\rangle$, where j is the vertex in G at which the spin is up. Thus $\{|j\rangle\,|\,j\in V(G)\}$ denotes a complete set of orthonormal basis vectors spanning $\mathcal{S}_G$.

When restricted to the subspace $\mathcal{S}_G$, H_G is represented, in the above-mentioned basis, by an $N\times N$ matrix which is *identical* to the adjacency matrix $A(G)$ of the underlying graph G.[1] The time evolution of the system under the action of the Hamiltonian H_G can be interpreted as a *continuous-time quantum walk* on G (first considered by Farhi and Gutmann in 1998 [8]; see also [9]). This is because the latter is defined as the time evolution in an N-dimensional Hilbert space spanned by states $\{|j\rangle\}$, where $j\in V(G)$, with a Hamiltonian given by the adjacency matrix of G.

The spin system on the graph G described above plays the role of a (noiseless) quantum channel. We see below that the continuous-time quantum walk on G can be viewed as a quantum state transfer along the channel.

The process of transmitting a quantum state from A to B proceeds in four steps:

(1) Initialization of the spin system to the state $|\underline{0}\rangle:=|0_A0\cdots00_B\rangle$, which corresponds to the configuration of all spins down. This state is a zero-energy eigenstate of H_G.[2]

(2) Creation of the quantum state $|\psi\rangle_A\in\mathcal{H}_A$ (at vertex A) which is to be transmitted. Let $|\psi\rangle_A=\alpha|0\rangle_A+\beta|1\rangle_A$ with $\alpha,\beta\in\mathbb{C}$ and $|\alpha|^2+|\beta|^2=1$.

(3) Time evolution of the system for an interval of time, say t_0.

(4) Recovery of the state at vertex B, the latter being given by the reduced density matrix ρ_B acting on $\mathcal{H}_B$.

The state of the entire spin system after step 2 is given by

$$\begin{aligned}|\Psi(t=0)\rangle&=|\psi_A00\cdots00_B\rangle=\alpha|0_A00\cdots00_B\rangle\\&\quad+\beta|1_A00\cdots00_B\rangle=\alpha|\underline{0}\rangle+\beta|1\rangle.\end{aligned} \tag{5}$$

It evolves in time t to

$$|\Psi(t)\rangle=\alpha|\underline{0}\rangle+\sum_{j=1}^{N}\beta_j(t)|j\rangle \tag{6}$$

with complex coefficients $\alpha,\beta_j(t)$, where $1=|\alpha|^2+\Sigma_{j=1}^N|\beta_j(t)|^2$. The initial conditions are given by $\beta_A(0)=\beta$ and $\beta_j(0)=0$ for all $j\neq A$. The coefficient α does not change in time, as $|\underline{0}\rangle$ is the zero-energy eigenstate of H_G. Hence, it does not even acquire a phase factor during the evolution of the state.

[1] In Ref. [16] it was shown, starting from G, how to construct graphs whose adjacency matrix governs the time evolution in higher-excitation subspaces.

[2] Experimentally, this initialization can, for example, be achieved by application of a magnetic field to align the spins.

The output state at B after a time t is given by the reduced density matrix

$$\rho_B(t) \coloneqq \mathrm{Tr}_{\mathcal{H}_{G\setminus\{B\}}}|\Psi(t)\rangle\langle\Psi(t)| = \begin{pmatrix} 1-|\beta_B(t)|^2 & \alpha\beta_B^*(t) \\ \alpha^*\beta_B(t) & |\beta_B(t)|^2 \end{pmatrix}. \tag{7}$$

A measure of the overlap between the input state $\rho_A \coloneqq |\psi\rangle\langle\psi|$ and the output state is the fidelity

$$F(\rho_A,\rho_B(t)) \coloneqq \mathrm{Tr}\sqrt{\rho_A^{1/2}\rho_B(t)\rho_A^{1/2}} = \sqrt{\langle\psi|\rho_B(t)|\psi\rangle} = \sqrt{|\alpha|^2(1-2|\beta_B|^2+\beta_B\beta^*+\beta_B^*\beta)+|\beta_B|^2} \tag{8}$$

where it is understood that β_B depends on t.

Since the $|0\rangle_A$ component of the state $|\psi\rangle_A$ is invariant under the evolution, it suffices to focus on the evolution of the $|1\rangle_A$ component of the state, i.e., to the choice $\alpha=0$ and $\beta=1$ in Eq. (5). It is therefore convenient to consider the *transfer fidelity*

$$f_{AB}(t) \coloneqq \beta_B(t) \equiv \langle B|e^{-iH_Gt}|A\rangle, \tag{9}$$

where $|A\rangle \equiv |1\rangle = |1_A0\cdots00_B\rangle$ and $|B\rangle \equiv |N\rangle = |0_A0\cdots01_B\rangle$ and we have taken $\hbar=1$.

Here we focus only on *perfect state transfer*. This means that we consider the condition

$$|f_{AB}(t_0)| = 1 \quad \text{for some } 0 < t_0 < \infty, \tag{10}$$

which we interpret to be the signature of perfect communication (or perfect state transfer) between A and B in time t_0. The effect of the modulus in (10) is that the state at B, after transmission, will no longer be $|\psi\rangle$, but will be of the form

$$\alpha|0\rangle + e^{i\phi}\beta|1\rangle. \tag{11}$$

The phase factor $e^{i\phi}$ is not a problem because ϕ is independent of α and β and will thus be a known quantity for the graph, which we can correct for with an appropriate phase gate.

The perfect communication distance $d(A,B)$ is given by the distance on the graph, for which perfect state transfer is possible. For a fixed number of qubits N, our aim is to find quantum networks which maximize $d(A,B)$. We achieve this in two different ways. (1) By fixing the nearest-neighbor couplings to be identical but considering more complicated graphs (see Secs. IV and V). (2) By considering linear chains but allowing the nearest-neighbor couplings to be different (Secs. VII and VIII).

Note that if there is perfect communication between A and B in a time t_0, then perfect communication also occurs for all times t satisfying

$$t = (2n+1)t_0, \quad \text{where } n \in \mathbb{Z}, \tag{12}$$

provided the graph is mirror symmetric (see Sec. III).

III. CONDITIONS FOR PERFECT STATE TRANSFER IN SYSTEMS WITH MIRROR SYMMETRY

In the rest of the paper, we will examine different graphs for the purposes of perfect state transfer. These graphs will have mirror symmetry. By mirror symmetry, we mean that the graph is identical from the points of view of A and B. So, a linear chain with A and B at opposite ends is an example of such a system. The obvious question is how can we tell if a proposed graph will permit perfect state transfer? A necessary condition, as we will show, is that the ratios of differences of the eigenvalues of the Hamiltonian H_G must be rational provided the graph is mirror symmetric.

With a system capable of perfect state transfer, initialized in the state $|A\rangle$, at time t_0 we have the state

$$e^{-iH_Gt_0}|A\rangle = e^{i\phi}|B\rangle \tag{13}$$

but by the definition of a symmetric system, A and B are entirely equivalent, and thus after another period of time t_0, we have the state

$$e^{-iH_G2t_0}|A\rangle = e^{-iH_Gt_0}e^{i\phi}|B\rangle = e^{i2\phi}|A\rangle \tag{14}$$

and thus the system is periodic, up to a phase 2ϕ, with period $2t_0$. Thus we conclude that *a mirror symmetric system must be periodic if it is to allow perfect state transfer*. This may be written most simply as

$$|\langle A|e^{-iH_G2t_0}|A\rangle| = 1 \tag{15}$$

for some time $0 < t_0 < \infty$.

Let us examine the general state of a periodic system with period $2t_0$. We can write

$$|\psi(2t_0)\rangle = \sum_j a_j e^{-i2E_jt_0}|j\rangle = e^{i2\phi}\sum_j a_j|j\rangle \tag{16}$$

for eigenstates $|j\rangle$ of H_G with corresponding eigenvalues E_j. Hence for all of the stationary states $|i\rangle$, we have the condition

$$2E_it_0 - 2\phi = 2k_i\pi \tag{17}$$

where the k_i's are integers. Eliminating ϕ between two of these, we get that

$$(E_i - E_j)2t_0 = 2\pi(k_i - k_j) \tag{18}$$

and eliminating the t_0 between any two of these $(E_{i'} \neq E_{j'})$ gives

$$\frac{E_i - E_j}{E_{i'} - E_{j'}} = \frac{k_i - k_j}{k_{i'} - k_{j'}} \in \mathbb{Q}, \tag{19}$$

where $\mathbb{Q}$ denotes the set of rational numbers. As the k_i's are integers, this implies that the ratio is rational. Hence, a symmetric system capable of perfect state transfer must be periodic, which is equivalent to the requirement that the ratios of the differences of the eigenvalues are rational.

IV. LIMITATIONS FOR PERFECT COMMUNICATION OF A UNIFORMLY COUPLED CHAIN

It is desirable to maximize the distance over which communication is possible for a fixed number of qubits. The optimal arrangement, in this case, is just a linear chain of N qubits, where A and B are the qubits at opposite ends of the chain.

Let us start with the *XY* chain of qubits, with uniform couplings $J_{i,i+1}=1$ for all $1\leqslant i\leqslant N-1$. The Hamiltonian reads

$$H=\frac{1}{2}\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y. \tag{20}$$

In this case, one can compute $f_{AB}(t)$ explicitly by diagonalizing the Hamiltonian or the corresponding adjacency matrix. The eigenstates and the corresponding eigenvalues are given by

$$|\tilde{k}\rangle=\sqrt{\frac{2}{N+1}}\sum_{n=1}^{N}\sin\left(\frac{\pi kn}{N+1}\right)|n\rangle, \tag{21}$$

$$E_k=-2\cos\frac{k\pi}{N+1}, \tag{22}$$

with $k=1,\ldots,N$. Thus

$$f_{AB}(t)=\frac{2}{N+1}\sum_{k=1}^{N}\sin\left(\frac{\pi k}{N+1}\right)\sin\left(\frac{\pi kN}{N+1}\right)e^{-iE_kt}. \tag{23}$$

Perfect state transfer from one end of the chain to another is possible for $N=2$ and 3, where we find that $f_{AB}(t)=-i\sin(t)$ and $f_{AB}(t)=-[\sin(t/\sqrt{2})]^2$, respectively.

We have shown that perfect state transfer is possible for chains containing two or three qubits. We will now prove that it is not possible to get perfect state transfer for $N\geqslant 4$.

A chain is symmetric about its center. Hence the condition Eq. (19) for perfect state transfer applies, i.e.,

$$\frac{E_m-E_n}{E_{m'}-E_{n'}}\in\mathbb{Q} \tag{24}$$

where the E_m's are eigenvalues of the unmodulated chain, as given in (22). We will explicitly show that there is a set of eigenvalues for which this expression does not hold for all $N\geqslant 4$.

We are free to choose any values for the indices (provided $E_{m'}\neq E_{n'}$), so let us choose that $m=2$, $n=N-1$, $m'=1$, and $n'=N$. Hence we see, using Eq. (22), that we require

$$\frac{\cos[2\pi/(N+1)]}{\cos[\pi/(N+1)]}\in\mathbb{Q} \tag{25}$$

to hold for perfect state transfer. To find the values of N for which this holds, we make use of algebraic numbers. An *algebraic number* x is a complex number that satisfies an equation of the form

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0, \tag{26}$$

with integral coefficients a_i. Every algebraic number α satisfies a unique polynomial equation of least degree. The degree of this polynomial is called the *degree* of α.

If α satisfies a monic polynomial (i.e., a polynomial with $a_0=1$) then it is called an *algebraic integer* of degree n. Note that an algebraic integer of degree n is also an algebraic number of degree n. Rational numbers are algebraic numbers with degree 1, and numbers with degree $\geqslant 2$ are irrational.

Lehmer proved (for example, see [10]) that if $N>1$ and $\gcd(k,N+1)=1$, then $\cos[\pi k/(N+1)]$ is an algebraic integer of degree $\phi(2(N+1))/2$, where ϕ is the Euler phi function.

For $n\geqslant 3$ it can be shown [11] that

$$\phi(n)\geqslant\frac{n}{e^{\gamma}\log\log n+3/\log\log n} \tag{27}$$

holds, with $\gamma\approx 0.5772$ Euler's constant. Using this bound, and by inspection of values not covered by the bound, we see that $\phi(2(N+1)/2)\geqslant 3$ for $N\geqslant 6$.

We need to prove that if $\cos\theta$ is an algebraic number of degree $\geqslant 3$, the quotient

$$\frac{\cos 2\theta}{\cos\theta}$$

is irrational, where $\theta=\pi/(N+1)$.

Assume that this expression is rational, i.e.,

$$\frac{\cos 2\theta}{\cos\theta}=\frac{p}{q},\quad p,q\in\mathbb{Z}. \tag{28}$$

Using the trigonometric identity

$$\cos 2\theta=2\cos^2\theta-1 \tag{29}$$

we can write

$$(\cos\theta)^2-\frac{p}{2q}\cos\theta-\frac{1}{2}=0 \tag{30}$$

which has rational coefficients. According to the definition, $\cos\theta$ is therefore algebraic with degree $\leqslant 2$. Given that, from Eq. (27), $\cos\theta$ is an algebraic number of degree $\geqslant 3$, then we have a contradiction and therefore $(\cos 2\theta)/\cos\theta$ must be irrational.

Hence we see that for $N\geqslant 6$ perfect state transfer is impossible because $\deg(N)\geqslant 3$. This simply leaves $N=4$ and $N=5$ unproved, which can be done by straightforward evaluation. Thus for $N\geqslant 4$

$$\frac{\cos 2\theta}{\cos\theta}\notin\mathbb{Q}.$$

Hence, perfect state transfer is impossible for unmodulated chains of length $N\geqslant 4$.

V. PERFECT STATE TRANSFER OVER GREATER DISTANCES

Perfect state transfer over arbitrary distances is impossible for a simple unmodulated spin chain. Clearly it is desirable to find a graph that allows state transfer over larger distances, and to that end we examine the d-fold Cartesian product of the two-link (three-vertex) chain G. We denote this by G^d.

In general the Cartesian product of two graphs $G:=\{V(G),E(G)\}$ and $H:=\{V(H),E(H)\}$ is a graph $G\times H$ whose vertex set is $V(G)\times V(H)$ and two of its vertices (g,h) and (g',h') are adjacent if and only if one of the following holds: (i) $g=g'$ and $\{h,h'\}\in E(H)$; (ii) $h=h'$ and $\{g,g'\}\in E(G)$.

Let $A=(1,1,1,\ldots,1)$ and $B=(3,3,3,\ldots,3)$ denote the antipodal points of G^d.

We prove that for *any dimension d*

$$|f_{AB}(t)| = 1 \quad \text{for } t = t_0 = \frac{\pi}{\sqrt{2}}. \tag{31}$$

Hence, t_0 is the time for perfect communication between the vertices A and B of G^d.

Let $\{\lambda_i(G), 1 \leqslant i \leqslant |V(G)|\}$ and $\{\lambda_j(H), 1 \leqslant j \leqslant |V(H)|\}$ denote the set of eigenvalues of the graphs G and H, respectively. The eigenvalues of the adjacency matrix of their Cartesian product $G \times H$ are precisely the numbers $\lambda_i(G) + \lambda_j(H)$, with $1 \leqslant i \leqslant |V(G)|$ and $1 \leqslant j \leqslant |V(H)|$, where each number is obtained as many times as its multiplicity as an eigenvalue of the adjacency matrix $A(G \times H)$. This is because

$$A(G \times H) = A(G) \otimes \mathbb{1}_{V(H)} + \mathbb{1}_{V(G)} \otimes A(H), \tag{32}$$

where $\mathbb{1}_{V(H)}$ is the $|V(H)| \times |V(H)|$ identity matrix (see, e.g., [12]).

The eigenvalues of the adjacency matrix of G^d are given by

$$\{j\sqrt{2} | j \in \{0, \pm 1, \pm 2, \ldots, \pm d\}\} \tag{33}$$

and therefore the ratios of differences of the eigenvalues are all rational.

As already observed, the Hamiltonian of a system coupled via nearest-neighbor *XY* interactions is identical to the adjacency matrix. This will hold equally for the Cartesian product of G. Hence,

$$H = A(G^d) = \sum_{j=0}^{d-1} \mathbb{1}^{\otimes j} \otimes A(G) \otimes \mathbb{1}^{\otimes d-j-1}, \tag{34}$$

$$e^{-iHt} = (e^{-iA(G)t})^{\otimes d}. \tag{35}$$

Thus, if we select a time $t = \pi/\sqrt{2}$, then we get perfect state transfer along each dimension. Each term in the tensor product of (35) applies to a different element of the basis (for example, each acts on a different 1 in the definition of A, or a different 3 in B). We therefore achieve perfect state transfer between A and B [as well as between any qubit and its mirror, such as $(1,1,1,2,3) \rightarrow (3,3,3,2,1)$]. The fidelity of the state transfer is simply the dth power of the fidelity for the original chain (23).

Perfect transfer of a single-qubit state can also be achieved between the antipodes of a one-link hypercube in any arbitrary dimension d, in a constant time $t_0 = \pi/2$. This is because perfect transfer occurs across a chain of two qubits in this time.

We can also extend this to the one-link hypercube which is coupled via the Heisenberg interaction. This is because, in the case of a two-qubit chain, the Hamiltonian in the single-excitation subspace is represented by a matrix with identical diagonal elements, and hence is the same as the Hamiltonian of an *XY* model up to a constant energy shift, which just adds a global phase factor.

Note that perfect transfer is possible across a ring of four spin-1/2 particles. The topology of this is exactly the same as a two-fold Cartesian product of a one-link chain, hence it is a special case of the hypercube we have been discussing (whether it is coupled with the Heisenberg or *XY* coupling).

VI. CLASSICAL CONTINUOUS-TIME RANDOM WALK ON THE HYPERCUBE

In the previous section we showed that for hypercubes generated from both the one-link (two-vertex) and the two-link (three-vertex) chain, the perfect state transfer time t_0 is independent of the dimension d. In this section we will investigate the behavior of the mean hitting time of the classical continuous-time symmetric random walk on G^d, which we denote by t_{cl}, and compare it to t_0. We will focus our attention on the two-link hypercube, since in the quantum case it provides us with a greater communication distance $d(A,B)$ than does the one-link hypercube. Unlike t_0, we show that t_{cl} grows exponentially with the dimension d [Eq. (52)]. We also note that the case of the one-link hypercube has previously been studied [13].

A two-link hypercube G^d is generated by taking the d-fold Cartesian product of the graph $G := \{V(G), E(G)\}$ where $V(G) = \{1,2,3\}$ and $E(G) = \{\{1,2\},\{2,3\}\}$. Hence, the state space for the classical continuous-time random walk on G^d is $\{1,2,3\}^d$. Transitions are allowed from a vertex $\underline{x} \in G^d$ to $\underline{x} \pm \underline{e}_i$ where $\underline{e}_i$ is the ith unit vector and $\underline{x}$ is a d-dimensional vector with components x_i, i.e., transitions are allowed to all the nearest neighbors with equal probability.

If T is the random variable defined as the hitting time of B, for a random walk starting at A, then

$$t_{cl} := \mathbb{E}(T) \tag{36}$$

where $\mathbb{E}(T)$ denotes the expectation value of T. The random variable T can be written as

$$T = \sum_{i=1}^{N} X_i, \tag{37}$$

where N is a random variable which gives the number of jumps that the random walker undergoes in going from A to B, and the X_i's are the holding times between successive jumps. We have

$$\mathbb{E}(T) = E\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\mathbb{E}(X_1) = \mathbb{E}(N), \tag{38}$$

where we have made use of the fact that the X_i's are independent and identically distributed with mean $\mathbb{E}(X_i) = 1$. Note that $\mathbb{E}(N)$ is the mean hitting time of the corresponding jump chain, which is a discrete-time Markov chain. Hence, to estimate the mean hitting time t_{cl} of the continuous-time random walk, it suffices to consider the discrete-time random walk given by the jump chain of the original walk.

All the information that we need is contained within the $3^d \times 3^d$ transition matrix P. An element $P_{A,B}$ is the probability of transition from A to B; hence after N steps the probability of hitting B (irrespective of whether it has previously hit) is

$$\underbrace{(P \times P \times \ldots \times P)}_{N}{}_{1,3^d},$$

which we shall denote as $P^N_{1,3^d}$, and hence

$$\mathbb{E}(N) = \sum_{n=2d}^{\infty} n P^n_{1,3^d} \prod_{m=2d-1}^{n-1} (1 - P^m_{1,3^d}). \tag{39}$$

Since we have a two-link hypercube, it will always take at least $2d$ steps to get from one corner to the opposite one. In fact, because we have a two-link hypercube, it will always take an even number of steps to hit the opposite corner. Thus, $P^m_{1,3^d}=0$ for odd m and $m<2d$.

So, all we are interested in is the element $P^{2n}_{1,3^d}$, which we expect to tend toward a constant value for large n, i.e., $(P^2P^{2n})_{1,3^d} \approx P^{2n}_{1,3^d}$ as $n\to\infty$. Thus, we are confronted with the problem of finding the first element in the eigenvector of P^2 which has eigenvalue 1 (see, for example, [14]).

Take, as an example, the special case of $d=2$ where the first row of P^{2n} is given by

$$\left(a_n\ 0\ \frac{1}{6}\ 0\ \frac{1}{3}\ 0\ \frac{1}{6}\ 0\ b_n\right).$$

The 0's occur because in an even number of steps, you cannot get to a point that is odd. For $n=1$, $a_n=\frac{1}{3}$ (the probability of returning to the start node) and $b_n=0$ (the probability of getting to the exit node). The sum of all the elements in the row must be 1, and hence $a_n+b_n=\frac{1}{3}$ for all n. We find a recursion relation for b_n, $\frac{1}{9}+\frac{1}{3}b_n=b_{n+1}$, and as $n\to\infty$, we find that $b_n\to\frac{1}{6}$.

Let us return to the general case. The location of a given node of the hypercube can be represented by $|x_1,x_2,\ldots,x_d\rangle$ where $x_i\in\{1,2,3\}$, i.e., x_i specifies, for the ith dimension, which of the three nodes we are positioned on. All of the properties of a given node depend only on r and d, where r is a count of the number of $x_i=2$. For example, the transition rate from one node at r to an adjacent node is

$$P_{\text{nearest neighbor}} = \frac{1}{d+r}. \tag{40}$$

This quantity can be understood because the transition probability is the same for all connected nodes, and all $x_i=1$ or 3 are only connected to $x_i=2$ in the ith dimension, whereas $x_i=2$ has two links.

Given that all properties of a node depend only on r and d, which must also be true for the eigenvector a. Hence, we can denote the elements of a by a_r. The element of the eigenvector that we are interested in corresponds to the position $|1,1,\ldots,1\rangle$, i.e., $r=0$, so we want to find the element a_0. We will now find the elements of a, the eigenvector of P^2.

In two steps, there are only five ways to get to a specific lattice point at a distance $r=2k$.

(1) Start at that point, make one step away, and then make the same step in reverse. This happens with a probability

$$p_{\text{return}} = \frac{1}{d+2k}\left(\frac{d-2k}{d+2k+1} + \frac{4k}{d+2k-1}\right). \tag{41}$$

(2) Start at a distance $2k-2$. There are $4\binom{2k}{2}$ equivalent jumps, which each occur with probability

$$\frac{2}{(d+2k-2)(d+2k-1)}. \tag{42}$$

(3) Start at a distance $2k+2$. There are $\binom{d-2k}{2}$ of these jumps, each occurring with probability

$$\frac{2}{(d+2k+2)(d+2k+1)}. \tag{43}$$

(4) Start at a distance $2k$ and go around two edges of a square (e.g., $|1,2\rangle\to|1,1\rangle\to|2,1\rangle$). There are $4k(d-2k)$ points from which this type of move will get us to the specific node we are interested in, and this transition happens with a probability

$$\frac{1}{d+2k}\left(\frac{1}{d+2k+1} + \frac{1}{d+2k-1}\right). \tag{44}$$

(5) Start at a distance $2k$ and travel the length of a chain (e.g., $|1,2\rangle\to|2,2\rangle\to|3,2\rangle$). There are $d-2k$ such jumps, each occurring with probability

$$\frac{1}{(d+2k)(d+2k+1)}. \tag{45}$$

Knowing these, it is possible to write down the elements $P^2\cdot a$, so we can solve for the eigenvector:

$$\begin{aligned} &4\binom{2k}{2}\frac{2}{(d+2k-2)(d+2k-1)}a_{2k-2} \\ &\quad + \binom{d-2k}{2}\frac{2}{(d+2k+2)(d+2k+1)}a_{2k+2} \\ &\quad + \frac{1}{d+2k}\left(\frac{(4k+2)(d-2k)}{d+2k+1} + \frac{4k(d-2k+1)}{d+2k-1}\right)a_{2k} = a_{2k}. \end{aligned} \tag{46}$$

Starting with the special case of $k=0$, we see that

$$a_0 = \frac{d}{d+2}a_2 \tag{47}$$

and the general solution

$$a_{2k} = \frac{d+2k}{d+2k+2}a_{2k+2} \tag{48}$$

can then be proved by induction. We thus have the required elements, and just need to normalize them, remembering that there are $2^{d-2k}\binom{d-2k}{2}$ identical elements a_{2k},

$$\sum_{k=0}^{[d/2]} 2^{d-2k}\binom{d}{2k}a_{2k} = 1. \tag{49}$$

This gives that

$$a_{2k} = \frac{d+2k}{2d3^{d-1}} \tag{50}$$

and, in particular,

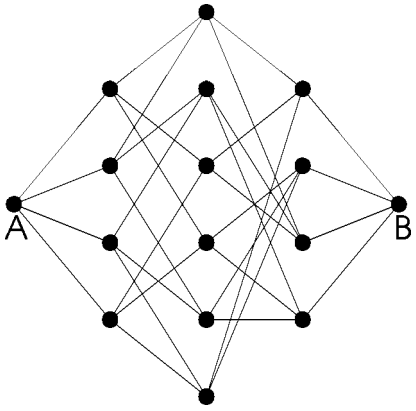

FIG. 1. An example of a five-column graph that allows perfect state transfer between either end.

$$a_0 = \frac{3}{2}3^{-d}. \tag{51}$$

We know that, as $n \to \infty$, $P_{1,3^d}^{2n} \to a_0$. Taking this value for all n, we can evaluate Eq. (39) to find that the mean hitting time is given by

$$t_{cl} = \mathbb{E}(T) = \sum_{n=d}^{\infty} 2na_0 \prod_{m=d}^{n-1} (1 - a_0) = 2d - 2 + \frac{2}{a_0} \approx \frac{4}{3}3^d. \tag{52}$$

The quantum analog of this mean hitting time is given by the time for perfect state transfer between the antipodal points A and B. We proved in the previous section that this time is a constant $t_0 = \pi/\sqrt{2}$. On comparing this with Eq. (52), we conclude that the graph G^d provides an example of a graph for which the quantum case leads to an exponential separation.

VII. PROJECTING A HYPERCUBE ONTO A SPIN CHAIN

Encouraged by the ability of the hypercube to allow perfect state transfer (Sec. V), we examine the one-link hypercube from a different angle. Such a graph falls into a general category of graphs G that have the property that the vertices can be arranged in columns so that there are no edges between the vertices within any column, and edges only join vertices in adjacent columns. Further, each vertex in column i must have the same number of incoming (from column $i-1$) and outgoing (to column $i+1$) edges as all other vertices in that column. See Fig. 1 for an example.

Representing the one-link d-dimensional hypercube in such a form, we allow the graph G to consist of N_C columns. The size of each column (the column occupation) is given by $b_i := |G_i| = \binom{N_C-1}{i-1}$ and the vertices in each G_i are labeled G_{ij}, $j = \{1, \ldots, b_i\}$. The ith column is $i-1$ edges away from a corner (say A) of the hypercube.

The only edges are between vertices of adjacent columns. From each column there must be a set of edges going forward to the next column, and another set going back to the previous one. These are denoted in the following manner:

$$\mathcal{P}_i^{\text{for}} := \{(G_{ij}, k) : j \in \{1, \ldots, b_i\}, k \in \{1, \ldots, r_i\}\}, \tag{53}$$

$$\mathcal{P}_i^{\text{back}} := \{(G_{ij}, k) : j \in \{1, \ldots, b_i\}, k \in \{1, \ldots, s_i\}\}, \tag{54}$$

where r_i and s_i denote the number of forward and backward edges, respectively, for the ith column. Clearly, if all the edges are to have ends, $|\mathcal{P}_i^{\text{for}}| = |\mathcal{P}_{i+1}^{\text{back}}|$. Since there is only a single qubit in the first column ($b_1 = 1$), each vertex in the second column has only a single edge going backward ($s_2 = 1$). With this constraint, and that s_i and r_i must be integers for all $1 \leqslant i \leqslant N_C$, we require that

$$b_i r_i = b_{i+1} s_{i+1}, \tag{55}$$

$$\frac{r_i}{s_{i+1}} = \frac{N_C - i}{i}. \tag{56}$$

The solution that we will choose for this is $r_i = N_C - i$, $s_i = i - 1$, which certainly satisfies all conditions. Thus we have a graph such that for every pair of numbers (i, j), G_{ij} is connected with $N_C - i$ vertices in G_{i+1} and each vertex in G_{i+1} is connected with $i-1$ vertices in G_i.

Let us define the vectors that span the *column space* $\mathcal{H}_C$,

$$|\text{col } i\rangle := \frac{1}{\sqrt{b_i}} \sum_{j=1}^{b_i} |G_{ij}\rangle. \tag{57}$$

Childs *et al.* [9] note that the evolution with the adjacency matrix H_G of G for this general class of networks (not just the hypercube), starting in G_{11}, always remains in the column space $\mathcal{H}_C$ because every vertex in column i is connected to the same number of vertices in column $i+1$ and every vertex in column $i+1$ is connected to the same number of vertices in column i.

Thus, we can restrict our attention to the column space $\mathcal{H}_C$ for the purpose of perfect state transfer from G_{11} to $G_{N_C 1}$. The matrix elements of the adjacency matrix of G, restricted to this subspace are given by

$$J_i := \langle \text{col } i | H_G | \text{col } i+1 \rangle = \sqrt{i(N_C - i)}, \tag{58}$$

$$J = \begin{pmatrix} 0 & J_1 & 0 & 0 & \ldots & 0 \\ J_1 & 0 & J_2 & 0 & \ldots & 0 \\ 0 & J_2 & 0 & J_3 & \ldots & 0 \\ 0 & 0 & J_3 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & J_{N_C-1} \\ 0 & 0 & 0 & 0 & J_{N_C-1} & 0 \end{pmatrix}. \tag{59}$$

This can be seen as follows:

$$\begin{aligned} \langle \text{col } i | H_G | \text{col } i+1 \rangle &= \frac{1}{\sqrt{b_i b_{i+1}}} \sum_{j=1}^{b_i} \sum_{j'=1}^{b_{i+1}} \langle G_{i,j} | H_G | G_{i+1,j'} \rangle \\ &= \frac{1}{\sqrt{b_i b_{i+1}}} b_i (N_C - i) = \sqrt{i(N_C - i)}. \end{aligned} \tag{60}$$

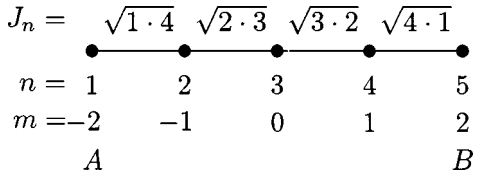

FIG. 2. Couplings J_n that admit perfect state transfer from A to B in a five-qubit chain. Eigenvalues m of the equivalent spin-2 particle are also shown. This is the projection of Fig. 1 onto a chain.

Hence, the above graph exhibits the same behavior as the XY chain with "engineered" coupling strengths J_i:

$$H = \frac{1}{2}\sum_{i=1}^{N_C-1} J_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y). \tag{61}$$

Such a chain must allow perfect state transfer over any length N_C (where $|A\rangle \equiv |\text{col } 1\rangle$, $|B\rangle \equiv |\text{col } N_C\rangle$) because the hypercube does. In the next section we prove that this is the case using a more physically motivated derivation.

The number of vertices in the graph G is given by $|G| = 2^{N_C-1}$; hence it has *communication distance* of $\log_2|G|$. The two-link hypercube in contrast has communication distance $2\log_3|G|$. One should note, however, that the degree of each vertex is bounded linearly.

Some examples of this graph are provided here for different numbers of columns. For $N_C=2$, two-qubit chain ($d=1$ one-link hypercube); for $N_C=3$, square ($d=2$ one-link hypercube); for $N_C=5$, for example, Fig. 1 which reduces to an engineered chain, as shown in Fig. 2.

For the purpose of perfect state transfer, we have stated that the d-dimensional, one-link hypercube is equivalent to the graph G. The equivalence is obvious for the case of $d=1$ and 2. The general proof arises by considering how the Cartesian product of a graph is taken when you extend the product from $(d-1)$ to d dimensions.

Assume the numbers of vertices in two adjacent columns are n_i and n_{i+1} in the $(d-1)$-dimensional hypercube. In the $(i+1)$th column of the d-dimensional hypercube, there must still be the n_{i+1} vertices, plus each of the vertices in the previous column have one more edge (from taking the Cartesian product). Hence the total number of vertices is $n_i + n_{i+1}$. Assuming that the $(d-1)$-dimensional hypercube has column occupations given by a binomial distribution, this specifies that the d-dimensional hypercube does as well. Since we know that these column occupations hold for $d=2$, then by induction this must hold for any d.

What this does not prove is that the edges between vertices are correct. This is because they are not necessarily correct. While a hypercube must have a specific set of edges, the construction of the graph G did not specify which vertices had to be connected to which other ones, we just made sure we got the correct number of forward and backward edges. In that sense, the general graph G is a "scrambled" hypercube. No matter what this scrambling is, G still reduces to the same chain.

VIII. STATE TRANSFER OVER ARBITRARY DISTANCES

Suppose we have N_C qubits in a chain, with only one qubit in state $|\uparrow\rangle \equiv |1\rangle$ and all others in state $|\downarrow\rangle \equiv |0\rangle$. We previously labeled these as $|j\rangle$, denoting that the single excitation is on the jth qubit. One may associate a fictitious spin $\frac{1}{2}(N_C-1)$ particle with this chain and relabel the basis vectors as $|m\rangle$, where $m=-\frac{1}{2}(N_C-1)+j-1$, as illustrated in Fig. 2.

The input vertex $|A\rangle$ can be labeled as $|j=1\rangle$ or $|m=-\frac{1}{2}(N_C-1)\rangle$ and the output vertex $|B\rangle$ as $|j=N_C\rangle$ or $|m=+\frac{1}{2}(N_C-1)\rangle$. Now, consider the Hamiltonian

$$H = \lambda J_x = \frac{1}{2}\lambda\sum_{i=1}^{N_C-1} J_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) \tag{62}$$

which has the same matrix form as Eq. (59), with a scaling constant λ.

This corresponds to the flipped spin hopping between the vertices j and $j+1$ with a probability amplitude of J_j. Now, let us choose H to be proportional to the angular momentum operator J_x or J_y for some spin $J=\frac{1}{2}(N_C-1)$ particle. In this case the matrix elements J_j are $\frac{1}{2}\sqrt{j(N_C-j)}$ (these are the same as the elements derived in the previous section up to a numerical factor). The evolution of the excitation in the chain is governed by the operator

$$U(t) = \exp(-i\lambda t\, J_x), \tag{63}$$

which represents a rotation of the fictitious spin $J=\frac{1}{2}(N_C-1)$ particle. The matrix elements $\langle j'|U|j\rangle$ are well known. Thus working out [15] or looking up an appropriate representation of SU2 gives

$$f_{AB}(t) = \langle B|U(t)|A\rangle = \left[-i\sin\left(\frac{\lambda t}{2}\right)\right]^{N_C-1}. \tag{64}$$

Thus we get perfect transfer of the state from $|1\rangle$ to $|N_C\rangle$ in a constant time $t_0=\pi/\lambda$. We can select N_C-1 to be divisible by 4 and this eliminates the phase shift caused by the factor of $-i$.

Note that the case of $N_C=2$ is just the same as an unmodulated spin chain of the same length, so the calculation done previously (23) is expected to give the same result. This it does, provided we remember that in the current situation the coupling strength is $\lambda/2$, whereas it was simply set to 1 in the original situation.

Is there any other interqubit interaction in the chain that gives Hamiltonian (59) when restricted to the single-excitation subspace? The first choice is the XY model with modulated interactions, another one is the Heisenberg model. If we try the Heisenberg model of the form

$$\frac{1}{2}\sum_{j=1}^{N_C-1} J_j\sigma_j\cdot\sigma_{j+1}, \tag{65}$$

we obtain

$$\begin{pmatrix} D_1 & J_1 & 0 & 0 & \cdots & 0 \\ J_1 & D_2 & J_2 & 0 & \cdots & 0 \\ 0 & J_2 & D_3 & J_3 & \cdots & 0 \\ 0 & 0 & J_3 & D_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & J_{N_C-1} \\ 0 & 0 & 0 & 0 & J_{N_C-1} & D_{N_C} \end{pmatrix} \tag{66}$$

where $D_j=\frac{1}{2}(\Sigma_{k=1}J_k)-J_{j-1}-J_j$. In order to get rid of the diagonal elements in the matrix above we can apply a magnetic field in the z direction, i.e., we add an extra term to Eq. (65),

$$\frac{1}{2}\sum_{j=1}^{N_C-1} J_j\sigma_j\cdot\sigma_{j+1}+\sum_{j=1}^{N_C} B_j\sigma_j^z, \tag{67}$$

with $B_j=\frac{1}{2}(J_{j-1}+J_j)-[1/2(N_C-2)]\Sigma_{k=1}^{N_C-1}J_k$.

All this means that we can distribute a quantum state over any distance with fidelity equal to 1 as long as we engineer the interqubit interactions, e.g., the interqubit distances in the chain, and apply a suitable spatially varying magnetic field.

IX. SCALING RELATIONS AND ENERGY CONSIDERATIONS

In the previous section we showed that a spin chain with engineered interactions can be used to transfer a quantum state in fixed time t_0. To compare the computational complexity of the proposed spin chain, it is customary to consider what happens to the energy of the system as the number of spins in the chain increases. One physical assumption that we might make, for example, is that the maximum coupling strength is a fixed size. This maximum occurs at the middle of the chain and is

$$\lambda J_{\lfloor N/2\rfloor}\sim\lambda N.$$

Hence, to keep this coupling a constant strength, λ must scale with $1/N$ and $t_0=\pi/\lambda$ must scale with N.

A second concern is what might happen if we tried to extract our state at a time $t_0-\delta t$. The fidelity of the state transfer is easily approximated from Eq. (64) so for small δt we get

$$f_{AB}(t_0-\delta t)\approx 1-\frac{\pi^2(N_C-1)}{8}\left(\frac{\delta t}{t_0}\right)^2.$$

Finally, we could ask the question about what happens in the presence of manufacturing errors. In particular, we shall consider what happens if the errors affect only the eigenvalues of the system. This is not the entire story for the spin chain, because we should also consider what happens to the eigenvectors (and, in particular, how well they maintain their symmetry about the center of the chain since all the eigenvectors are either symmetric or antisymmetric). However, in the case of a double application of the chain (which corresponds to nothing happening to the stored state), we learned in Sec. III that it is only the eigenvalues that matter.

Let us assume that we have made some manufacturing errors when producing our spin chain, i.e., we have some errors that are time independent. The ideal energies of the eigenstates are E_i and the actual energies are E_i':

$$f_{AA}=\langle A|e^{-iH2t_0}|A\rangle,$$

$$|A\rangle=\sum_i a_i|i\rangle=\sum_i a_i e^{-2it_0E_i}|i\rangle,$$

$$f_{AA}=\sum_i |a_i|^2 e^{-2it_0(E_i'-E_i)}.$$

We can estimate the worst case for the fidelity of the identity transformation, by taking the worst error to be $E_i'-E_i=\delta$ and by assuming that $t_0\delta\ll 1$. The error is then

$$\epsilon=|1-f_{AA}|,$$

$$\epsilon\approx 2t_0\delta,$$

i.e., it scales linearly with N.

X. USING THE CHAIN FOR ENTANGLEMENT TRANSFER

The idea of the rotation of the large spin particle and subsequent calculation can also tell us more about the system. For example, in the same time that we get perfect state transfer from qubit 1 to N_C, we also get perfect state transfer from qubit j to N_C+1-j. Under the action of the J_x rotation, these transfers all have the same phase. This means that the chain can be used to move an entangled state from one end of the chain to another. We can start with the Bell state $1/\sqrt{2}(|01\rangle+|10\rangle)$ on the first two qubits:

$$\frac{1}{\sqrt{2}}(|1\rangle+|2\rangle). \tag{68}$$

In time $t_0=\pi/\lambda$ this will evolve to the state

$$\frac{1}{\sqrt{2}}(|N_C\rangle+|N_C-1\rangle), \tag{69}$$

having thus transferred the Bell state to the other end of the chain. Note that we cannot use the state $1/\sqrt{2}(|00\rangle+|11\rangle)$ because this contains a term with two spins in it, and we have restricted ourselves to the subspace of only a single spin. We point out, however, that the results of [6] show that we will also get perfect state transfer in higher-excitation subspaces and thus, in principle, such a state could be transferred.

The chain can also be used to distribute an entangled pair between two distant parties. If we create a Bell state

$$\frac{1}{\sqrt{2}}(|0\rangle_{NI}|0\rangle_C+|1\rangle_{NI}|1\rangle_C) \tag{70}$$

between a noninteracting qubit (NI) and the first qubit on the chain (C), then the overall Hamiltonian will be of the form

$$H'=\mathbb{1}\otimes H. \tag{71}$$

Note that the state $|i\rangle_C$ is exactly the same as the state $|i\rangle$ that we were talking about before with the engineered chain, but

we have to be careful not to confuse those states with the states of the noninteracting qubit. The state (70) then evolves as

$$\frac{1}{\sqrt{2}}(|0\rangle_{NI}e^{-iHt}|0\rangle_C + |1\rangle_{NI}e^{-iHt}|1\rangle_C) \tag{72}$$

so after the same t_0, the entangled pair will be the noninteracting qubit and the N_Cth qubit on the chain.

$$\frac{1}{\sqrt{2}}(|0\rangle_{NI}|0\rangle_C + e^{i\phi}|1\rangle_{NI}|N_C\rangle_C). \tag{73}$$

This prescription is sufficient to transfer the entanglement of any general two-qubit density matrix from being between the noninteracting qubit and the first qubit at $t=0$ to being between the noninteracting qubit and the N_Cth qubit on the chain. This can be understood by seeing how the most general density matrix evolves. What we require is that

$$\mathrm{Tr}_{\mathcal{H}_G\setminus\{A\}}[\rho(0)] = \mathrm{Tr}_{\mathcal{H}_G\setminus\{B\}}[\rho(t_0)]. \tag{74}$$

Such a density matrix can be written as

$$\rho(0) = \sum_{(i,j,i',j')\in\{0,1\}} \alpha_{iji'j'}|ij\rangle\langle i'j'|, \tag{75}$$

$$\rho(t) = \sum_{(i,j,i',j')\in\{0,1\}} \alpha_{iji'j'}e^{-iH't}|ij\rangle\langle i'j'|e^{iH't}. \tag{76}$$

So if a single component of this density matrix evolves, giving perfect transfer, so will all the components and therefore so will the density matrix as a whole. This component evolves as

$$e^{-iH't}|i\rangle_{NI}|j\rangle_C\langle i'|_{NI}\langle j'|_C e^{iH't} \equiv |i\rangle_{NI}(e^{-iHt}|j\rangle_C)\langle i'|_{NI}(\langle j'|_C e^{iHt}). \tag{77}$$

After time t_0, if j or j' were 1, then they will have changed to N_C, and if they were 0, they remain as 0. Tracing out the effect of all the spins except for the noninteracting one and the N_Cth qubit will return precisely the same two-qubit density matrix as was initially set up. This then allows the density matrix to be split over the length of the chain.

If we want to transmit the complete density matrix, we just use two of our engineered chains (C_1 and C_2) in parallel ($N_{C_1}=N_{C_2}$). The new Hamiltonian can be written as

$$H'' = H \otimes H \tag{78}$$

and an exactly analogous argument now applies so that if we create the desired state [which could be the Bell state $1/\sqrt{2}(|00\rangle+|11\rangle)$, for example] across the first qubits of C_1 and C_2, then after time t_0, the state has been perfectly transmitted to being on the N_Cth qubits of the two chains. For an example, see Fig. 3.

This scheme will work for both the engineered spin chain and the hypercubes (since the density matrix can be created between the corners of two hypercubes). It is, however, unnecessary in the case of the spin chain, since the work of [6] shows that state transfer occurs in all excitation subspaces, not just the single-excitation subspace, as assumed here. The extension of this to multiqubit entangled states does still yield an advantage since these multiqubit states can be transmitted across the same chain (one qubit at a time) while only having access to the qubits at either end of the chain.

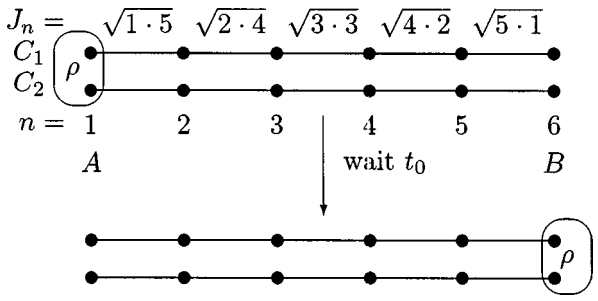

FIG. 3. Scheme for transferring an arbitrary two-qubit density matrix ρ, using two engineered spin chains (C_1 and C_2). This example has a chain length of $N_C=6$.

XI. J_y AND ARBITRARY PHASE GATES

As previously noted, the J_x rotation introduces a phase shift, depending on the length of the chain. There are several ways in which this can be avoided. The simplest is just to select the correct length of chain. In the case of the engineered chain (and also the one-link hypercube), if (N_C-1) is divisible by 4, then there is no phase shift (since $i^4=1$). Similarly with the two-link hypercube, if the dimension of the hypercube is even, there is no phase shift.

Another choice is to use the J_y rotation [which does not give the factor of $-i$ in Eq. (64)]:

$$J_y = H = \frac{1}{2}\sum_{j=1}^{N_C-1} J_j(\sigma_j^y\sigma_{j+1}^x - \sigma_j^x\sigma_{j+1}^y)$$
$$= i\begin{pmatrix} 0 & -J_1 & 0 & 0 & \cdots & 0 \\ J_1 & 0 & -J_2 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & -J_3 & \cdots & 0 \\ 0 & 0 & J_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -J_{N_C-1} \\ 0 & 0 & 0 & 0 & J_{N_C-1} & 0 \end{pmatrix}. \tag{79}$$

Using this in conjunction with the J_x rotation, it is possible, along with the transfer of a state through our spin chain network, to apply an arbitrary phase gate to it during transmission, simply by choosing the correct linear combination of J_x and J_y. Assume that we have picked N_C such that J_x gives a phase shift of i. A combination of

$$\gamma J_x \pm \sqrt{1-\gamma^2}J_y \tag{80}$$

will thus yield a phase shift $e^{i\phi}$ where

$$\tan(\phi) = \frac{\pm\gamma}{\sqrt{1-\gamma^2}}, \tag{81}$$

meaning that the initial state $|\psi\rangle$ will have evolved to the state

$$\alpha|0\rangle + e^{i\phi}\beta|N_C\rangle. \tag{82}$$

The final alternative for negating the phase shift, or applying an arbitrary phase gate during transmission, would be to apply a uniform global magnetic field in the z direction. Applying a field strength B shifts the energy of the single-spin excitation by $B(N_C-2)/2$ and the ground state energy is shifted by $BN_C/2$. Assuming transmission of the state occurs in a time t_0, then B can be selected to give the desired phase shift ϕ by

$$B = \frac{\phi - (\pi/2)(N_C - 1)}{t_0}. \tag{83}$$

XII. SUMMARY

We have shown that perfect state transfer is possible across a network of qubits, allowing only control over the initial design of the network, and no dynamical control.

When the couplings between adjacent qubits are constrained to be equal, we showed that examples of such networks are the one- and two-link d-dimensional hypercubes. Perfect state transfer for three- or more-link hypercube geometries is shown to be impossible. The transfer time is independent of the dimension of the hypercube and for comparative purposes, we calculated the expected hitting time in the classical continuous-time random walk, which increases exponentially with the dimension.

We have also proposed a spin chain of N qubits with nonuniform couplings that allows both state and entanglement transfer. This chain can be interpreted in two ways: first, as a projection of an $(N-1)$-dimensional one-link hypercube and second, as a rotation in the x direction of a fictitious spin $(N-1)/2$ particle.

Finally, we have shown how to effect entanglement transfer and how to introduce phases on the transferred quantum states on the fly.

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