Perfect state transfer, effective gates, and entanglement generation in engineered bosonic and fermionic networks

Man-Hong Yung[1,2,*] and Sougato Bose[3,†]
[1]*Physics Department, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3080, USA*
[2]*Physics Department, The Chinese University of Hong Kong, Hong Kong, China*
[3]*Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, England*
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We show how to achieve perfect quantum state transfer and construct effective two-qubit gates between distant sites in engineered bosonic and fermionic networks. The Hamiltonian for the system can be determined by choosing an eigenvalue spectrum satisfying a certain condition, which is shown to be both sufficient and necessary in mirror-symmetrical networks. The natures of the effective two-qubit gates depend on the exchange symmetry for fermions and bosons. For fermionic networks, the gates are entangling (and thus universal for quantum computation). For bosonic networks, though the gates are not entangling, they allow two-way simultaneous communications. Protocols of entanglement generation in both bosonic and fermionic engineered networks are discussed.

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I. INTRODUCTION

Quantum state transfer, two-qubit gates, and entanglement are essential in quantum information theory and quantum computation [1]. Recently, there have been many proposals [2–12] exploiting the *free* evolution of spin networks for accomplishing these tasks. The main idea is to minimize the spatial and dynamical control, which is experimentally challenging, on the interactions between qubits. Imperfect state transfer over homogeneous spin chains has been studied [2,6] for Heisenberg and *XY* Hamiltonians. A measurement-based state transfer scheme [8] has been suggested for dual-spin channels. Perfect state transfer [3], state inversion [4], and graph state generation [5] have been proposed for engineered spin chains in which the couplings between qubits are tunable. Quantum computation using permanently coupled spin chains has been proposed [6,10]. Furthermore, other dynamical properties [11] of spin chains and state transfer schemes [12] have also been studied.

In this paper, we generalize the results in [3,4,6] for engineered networks. In Refs. [3,4] two types of engineered networks which accomplish perfect quantum state transfer have been presented. These networks depend on the known properties of special functions, and hence the choice of the eigenvalue spectrum and the network couplings is limited. One of the aims of this paper is to show how one could "design" such engineered networks without reference to any special functions. One simply has to choose an eigenvalue spectrum from an infinite set of possibilities satisfying a certain condition [cf. Eq. (12)], which is both sufficient and necessary. The network couplings can then be found by solving a structured inverse eigenvalue problem. As a consequence of this approach we note that even a single infinitely deep square well or a single harmonic well enables perfect state transfer from across a distance. Inspired by the recent rapid experimental development in optical lattices (see, e.g., [13] and references therein), our discussion will be presented in terms of fermionic and bosonic networks with the presence or absence of a boson or fermion at a site representing the 0 and 1 states of a qubit. Certain spin networks are classified to be fermionic, as we shall discuss.

Another aim of this paper is to show that effective two-qubit gates over remote qubit pairs can be constructed in those engineered networks. For fermionic ones, including spin chains, the effective gates are entangling and hence universal for quantum computation. For bosonic networks, the gates are not entangling, but they allow two-way communication for different pairs of sites simultaneously without mutual interference. Finally, protocols for entanglement generation and transfer will also be discussed. In contrast to the scheme proposed in [14], these protocols require minimal spatial and temporal control on individual qubits.

II. ENGINEERED NETWORKS

We start with a system consisting of spinless fermions (or bosons) hopping freely in a network of N lattice sites. In fact, the particles need not literally be spinless, but they all need to be polarized in the same spin state, and there should not be any interactions involving spin. The Hamiltonian is therefore of the following form:

$$H = \sum_{\langle i,j\rangle} \omega_{ij}(a_i^\dagger a_j + a_j^\dagger a_i) + \sum_{j=1}^{N} \lambda_j n_j, \tag{1}$$

where $\langle i,j\rangle$ denotes nearest-neighbor coupling, ω_{ij} is the time-independent coupling constant between the site i and site j, and λ_j represents the strength of the external static potential at site j. The annihilation operators a_j obey the standard (anti)commutation relations for bosons (fermions) and $n_j = a_j^\dagger a_j$ is the number operator. This model may be considered as the strong tunneling limit of the Hubbard model

*Electronic address: myung2@uiuc.edu
†Electronic address: sougato.bose@qubit.org

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[15] for fermions and Bose-Hubbard model [16] for bosons. In particular, for one-dimensional fermionic chains, this model can be mapped to spin chains in which spins are coupled through the *XY* Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N-1} \omega_j(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + \frac{1}{2}\sum_{j=1}^{N} \lambda_j(\sigma_j^z + 1) \qquad (2)$$

by the Jordan-Wigner transformation [17]. Therefore, such spin chains will be classified as fermionic, even though the individual spins are distinguishable.

Since the Hamiltonian H commutes with the total number operator $n_{\text{tot}}=\Sigma_{j=1}^N n_j$ or the total z-spin operator $S_{\text{tot}}^z=\Sigma_{j=1}^N \sigma_j^z$, the Hilbert space can be decomposed into subspaces consisting of the eigenstates of n_{tot} or S_{tot}^z. Furthermore, as the particles are noninteracting, the eigenstates in the n-particle subspace are the antisymmetrized (symmetrized) products of the single-particle eigenstates for fermions (bosons).

A. Quantum state transfer

Quantum state transfer over a network is similar to the quantum random walk problem, where a variety of networks are equivalent to one-dimensional chains [3,18]. Therefore, we will now focus on a chain of N sites. For $j=1,2,\ldots,N$, let $|\boldsymbol{j}\rangle$ be the state where a single fermion (or boson) is at the site j but is in the empty state $|0\rangle$ for all other sites and $|\mathbf{0}\rangle$ be the vacuum state where all sites are empty. For spin chains, $|\mathbf{0}\rangle$ corresponds to the state where all the spins are in the spin-down state $|\downarrow\rangle$ and $|\boldsymbol{j}\rangle$ corresponds to a spin-up state $|\uparrow\rangle$ for the jth spin and spin-down for all other spins. The Hamiltonian in this single-particle subspace can be written in a tridiagonal form, which is real and symmetric:

$$H_N = \begin{pmatrix} \lambda_1 & \omega_1 & 0 & \cdots & 0 \\ \omega_1 & \lambda_2 & \omega_2 & \cdots & 0 \\ 0 & \omega_2 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \omega_{N-1} \\ 0 & 0 & 0 & \omega_{N-1} & \lambda_N \end{pmatrix}. \qquad (3)$$

The quantum state transfer protocol involves two steps: initialization and evolution. First, a quantum state $\alpha|0\rangle + \beta|1\rangle$ to be sent is encoded at site x. The initial state of the network is described by $|\boldsymbol{\varphi}_x\rangle=\alpha|\mathbf{0}\rangle+\beta|\boldsymbol{x}\rangle$. Then, the network couplings ω_j and λ_j are switched on and the whole system is allowed to evolve under $U(t)=\exp(-iHt)$ for a fixed time interval $t=\tau$. The final state becomes

$$U(\tau)|\boldsymbol{\varphi}_x\rangle = \alpha|\mathbf{0}\rangle + \beta\sum_{j=1}^{N} f_{j,x}^N(\tau)|\boldsymbol{j}\rangle, \qquad (4)$$

where $f_{j,x}^N(\tau)=\langle\boldsymbol{j}|e^{-iH\tau}|\boldsymbol{x}\rangle$. Any site y is in a mixed state if $|f_{y,x}^N(\tau)|<1$, which also implies that the state transfer from site x to y is imperfect. Our goal here is to find a set of ω_j and λ_j to realize perfect state transfer.

In [4], it is shown that when the couplings are chosen such that $H_N=S_x$, where S_x is the x component of the spin operator of a spin $S=(N-1)/2$, or alternatively $H_N=\mathbf{L}\cdot\mathbf{S}$, subject to the constraint that the z component of the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$ being zero, then a mirror inversion of eigenstates with respect to the center of a linear chain can be implemented. This implies that a quantum state at site x can be transferred perfectly to its *mirror-conjugate* site $\overline{x}=N-x+1$,

$$U(\tau)|\boldsymbol{\varphi}_x\rangle = \alpha|\mathbf{0}\rangle + e^{-i\phi_N}\beta|\overline{\boldsymbol{x}}\rangle, \qquad (5)$$

where in general $\phi_N\neq 0$ and a single-qubit operation on the site $\overline{x}$ is required to remove it, in order to reconstruct the original state there.

In the next section, we will introduce a systematic way to find the sets of ω_j and λ_j for state inversion and hence perfect state transfer even if we did not use any of the above examples. Instead of solving the eigenvalue problem, we will first *choose* a desired eigenvalue spectrum [cf. Eq. (12)] for H_N and the solutions for ω_j and λ_j can be found from the spectrum and the symmetrical properties of H_N. It is therefore an inverse eigenvalue problem.

B. Symmetrical properties of H_N

State inversion by free evolution crucially depends on (a) the reflection symmetry and (b) the eigenvalue spectrum. By reflection symmetry, we mean for $j=1,2,\ldots,\lfloor N/2\rfloor$,

$$\lambda_j = \lambda_{\overline{j}} \quad \text{and} \quad \omega_j = \omega_{N-j} \neq 0. \qquad (6)$$

Thus, H_N has double symmetries (also known as persymmetric), along both the main diagonal and the second diagonal. We shall now show that if the above symmetries are present in H_N, then one only needs the eigenvalue spectrum to satisfy a certain condition [cf. Eq. (12)] in order to achieve state inversion. This condition will later be shown not only sufficient but also necessary for state transfer in mirror-symmetric networks. As a consequence of the above symmetries, the eigenvectors $|\boldsymbol{e_k}\rangle=\Sigma_{j=1}^N a_j^k|\boldsymbol{j}\rangle$ have definite parities—i.e., being either even or odd with respect to the mirror-conjugate operation $j\rightarrow\overline{j}$. In fact, the eigenvectors can be determined [cf. Eq. (10)] explicitly. However, we need to know which one would change sign when inverted. This can be determined by the interlacing property described below.

Let $P_N(E)=\Pi_{k=0}^{N-1}(E-E_k)$ be the characteristic polynomials of H_N and denote the jth leading principal minor (i.e., the characteristic polynomial obtained by the first j rows and columns of a matrix) of the matrix $EI-H_N$ by $P_j(E)$, where I is the $N\times N$ identity matrix and E is a real number. With $P_0\equiv 1$ and $P_1\equiv E-\lambda_1$, the sequence of $P_j(E)$ is a Sturm sequence [19] and for $j=1,2,\ldots,N$, it satisfies a recurrence relation

$$P_j(E) = (E-\lambda_j)P_{j-1}(E) - \omega_{j-1}^2 P_{j-2}(E). \qquad (7)$$

The Sturm sequence has an important property: the roots E_k^j of P_j interlace those of P_{j-1}—i.e.,

$$E_{j-1}^j < E_{j-2}^{j-1} < E_{j-2}^j < \cdots < E_1^j < E_0^{j-1} < E_0^j. \qquad (8)$$

This implies that

$$\mathrm{sgn}[P_{N-1}(E_k)] = (-1)\mathrm{sgn}[P_{N-1}(E_{k-1})], \tag{9}$$

where $E_k \equiv E_k^N$ and $P_N(E_k)=0$. We shall show immediately that this interlacing property of the Sturm sequence determines the parity of the eigenvectors.

1. Parity

It is known that the coefficients a_j^k, $j=2,3,\ldots,N$, of the eigenvectors are given [19] by

$$a_j^k = \frac{P_{j-1}(E_k)}{\omega_1\omega_2\cdots\omega_{j-1}}a_1^k, \tag{10}$$

with a_1^k determined by the normalization condition $\Sigma_{j=1}^N |a_j^k|^2 = 1$. We note that the parity of the eigenvectors can be determined by checking the relative sign of *any* pair of mirror-conjugate coefficients. For convenience, we consider $\mathrm{sgn}[a_N^k/a_1^k] = \mathrm{sgn}[P_{N-1}(E_k)/\omega_1\omega_2\cdots\omega_{N-1}] = (-1)^\nu \mathrm{sgn}[P_{N-1}(E_k)]$, where $(-1)^\nu \equiv \mathrm{sgn}[\omega_1\omega_2\cdots\omega_{N-1}]$. From Eqs. (9) and (10), if the eigenvectors are ordered in decreasing eigenvalues—i.e., $E_0 > E_1 > \cdots > E_{N-1}$—the parities of them change *alternatively*. Since $P_{N-1}(E_0) > 0$, the parity of the highest-energy eigenstate $|\boldsymbol{e}_0\rangle$ is only determined by $(-1)^\nu$. It is even (i.e., $\nu=0$), if all $\omega_j > 0$. As the parity changes alternatively, once the parity of $|\boldsymbol{e}_0\rangle$ is known, the parities of all other eigenvectors can be inferred immediately. These can be summarized as

$$|\overline{\boldsymbol{e}_k}\rangle = (-1)^{k+\nu}|\boldsymbol{e}_k\rangle, \tag{11}$$

for $k=0,1,\ldots,N-1$, where $|\overline{\boldsymbol{e}_k}\rangle \equiv \Sigma_{j=1}^N a_j^k |\overline{\boldsymbol{j}}\rangle$.

C. Mirror inversion

Next, we require, for some time interval τ, the eigenvalue spectrum of H_N to satisfy the relation

$$e^{-iE_k\tau} = (-1)^{\pm k} e^{-i\phi_N}, \tag{12}$$

where ϕ_N is independent of k and the $\pm$ sign has to be taken consistently for all k. For simplicity, we assume all $\omega_j > 0$. Consider $U(\tau)|\boldsymbol{x}\rangle = \Sigma_{k=0}^{N-1} e^{-iE_k\tau}|\boldsymbol{e}_k\rangle\langle\boldsymbol{e}_k|\boldsymbol{x}\rangle$. When $|\boldsymbol{e}_k\rangle$ is replaced with $(-1)^{-k}|\overline{\boldsymbol{e}_k}\rangle$, together with the relation in Eq. (12), the completeness relation $I = \Sigma_{k=0}^{N-1}|\overline{\boldsymbol{e}_k}\rangle\langle\overline{\boldsymbol{e}_k}|$, and the double-inversion relation $\langle\boldsymbol{e}_k|\boldsymbol{j}\rangle = \langle\overline{\boldsymbol{e}_k}|\overline{\boldsymbol{j}}\rangle$, one can show that

$$U(\tau)|\boldsymbol{x}\rangle = e^{-i\phi_N}|\overline{\boldsymbol{x}}\rangle. \tag{13}$$

From Eq. (5), consequently, quantum states can be transported from any site x to its mirror-conjugate site $\overline{x}$ after a fixed period τ. Once a spectrum is determined, the search for the solutions of ω_j and λ_j becomes an inverse eigenvalue problem. There are some efficient algorithms available in the literature for accomplishing the task—for example [20] and references therein.

Here we also note that the condition in Eq. (12) is not only sufficient but also *necessary* for perfect state transfer in mirror-symmetric networks. To prove that it is necessary, we set for some time τ, $1 = |\langle\overline{\boldsymbol{x}}|U(\tau)|\boldsymbol{x}\rangle| = |\Sigma_k \langle\boldsymbol{e}_k|\boldsymbol{x}\rangle|^2 e^{i\varphi_k}| \leq \Sigma_k |\langle\boldsymbol{e}_k|\boldsymbol{x}\rangle|^2 = 1$, where $e^{i\varphi_k} \equiv e^{-iE_k t}(-1)^k$ and we have used the normalization condition in the last step. As the above equality must hold, $e^{i\varphi_k}$ should be a constant phase (independent of k), and hence the condition in Eq. (12) follows.

1. Example

Two types of spectrums, $E_k = -k$ and $E_k = k(k+q)$ for some rational number q and $k=0,1,2,\ldots,N$, suggested in [4] can easily be shown to satisfy Eq. (12). However, these spectra are related to some known examples of special functions. To illustrate the generality of the method, we consider a 4×4 tridiagonal matrix with the spectrum $E_0=1$, $E_1=2$, $E_2=3$, and $E_3 = 2(1+m)$, for any integer $m \geqslant 1$. The condition (12) is satisfied with $\tau=\pi$ and $\phi_N=0$. One of the solutions for the Hamiltonian [of the form of Eq. (3)] is found to be

$$\begin{pmatrix} a & c & 0 & 0 \\ c & b & d & 0 \\ 0 & d & b & c \\ 0 & 0 & c & a \end{pmatrix}, \tag{14}$$

with $a = 2 + 1/(2m)$, $b = m + 2 - 1/(2m)$, $c = \sqrt{1 - 1/(4m^2)}$, and $d = m$. The generality of generating engineered chains for perfect state transfer is thus clear. On the other hand, it is interesting to note that in the limit $m \gg 1$, one may want to put $a \approx 2$, $b \approx m$ $c \approx 1$, and $d = m$. However, since $a \sim O(1)$, although $m \gg 2$, changing b from $m+2$ to m would cause a large error. This is also confirmed numerically. Therefore, in such a limit, the requirement of precision is very high. In this sense, energy spectra that yield more uniform coupling are more desirable from the engineering point of view.

2. Continuous systems

An interesting consequence of Eq. (12) can also be found in infinite-dimensional systems. The eigenvalue spectra allowed by Eq. (12) correspond to some canonical systems such as harmonic well or infinite square well. In those cases, the necessary criterion for state inversion—namely, the parity of the eigenstates—is automatically satisfied. For example, the energy spectrum of an infinite square well is quadratic $E_k \propto k^2$, for $k=1,2,3,\ldots$ and the eigenfunctions $\psi_k(x)$ have a definite parity $\psi_k(x) = (-1)^{k-1}\psi_k(x)$. One can show that (also mentioned in [21]) any single-particle wave function $\Psi(x,t)$ at x will be transported (up to a $-$ sign) to $-x$, $\Psi(x,t) = -\Psi(-x, t+\tau)$, for a period of $\tau = 2\pi\hbar/E_1$, where E_1 is the ground state energy. This property has been discussed recently in the literature on fractional wave- function revivals [22], but its relevance to quantum communication and its connection to the above general theory linking eigenvalue spectrum to perfect state transfer has not been appreciated. For example, one can think of the following strategy for communicating perfectly through those continuous systems. We can encode the information of a qudit (not necessarily qubit) to the spin degree of freedom of a boson or fermion in a state $\Psi(x,t=0)$ which is initially localized around x. At $t=\tau$, the particle will arrive $\overline{x}$ and the information can be extracted.

D. Effective two-qubit gates

Two identical fermions (bosons) at sites x and y is described by the antisymmetrized (symmetrized) product state

$|\boldsymbol{xy}\rangle=(1/\sqrt{2})(|\boldsymbol{x}\rangle|\boldsymbol{y}\rangle\pm|\boldsymbol{y}\rangle|\boldsymbol{x}\rangle)$, with $-$ $(+)$ sign for fermions (bosons). Similarly, the two-particle eigenstates $|\boldsymbol{e}_{kl}\rangle=(1/\sqrt{2})(|\boldsymbol{e}_k\rangle|\boldsymbol{e}_l\rangle\pm|\boldsymbol{e}_l\rangle|\boldsymbol{e}_k\rangle)$ are also antisymmetrized or symmetrized accordingly. For many fermion excitations, the states are more convenient to be represented by a Slater determinant. Besides, in the mapping of spin states to fermionic states, we adopt the convention [4,5] that the site indices are arranged in ascending order. Thus, in a spin chain having spin-up states at $\boldsymbol{x}$ and $\boldsymbol{y}$ but spin-down for all other spins, the spin state corresponds to the fermionic state $|\boldsymbol{xy}\rangle$ if $x<y$ and $|\boldsymbol{yx}\rangle$ if $y<x$.

By effective gate, we mean the configuration of all the sites after the network evolution is the same as before, except that the state of the pair of qubits at x and $\overline{x}$ is changed according to a logic gate U_x. The simplest way is to choose all other sites to be empty (or all other spins being the spin-down state for spin chains). Using similar tricks as before, one can show that

$$U(\tau)|\boldsymbol{xy}\rangle=e^{-2i\phi_N}|\overline{\boldsymbol{xy}}\rangle. \tag{15}$$

If $y=\overline{x}$, then there is an extra factor (-1) for fermionic states but not bosonic states, after exchanging the site indices. We define a new basis with $|00\rangle\equiv|0\rangle$, $|10\rangle\equiv|\boldsymbol{x}\rangle$, $|01\rangle\equiv|\overline{\boldsymbol{x}}\rangle$, and $|11\rangle\equiv|\boldsymbol{x}\overline{\boldsymbol{x}}\rangle$. In this basis, an effective two-qubit gate U_x can be constructed readily for any conjugate pair of sites x and $\overline{x}$ (see Fig. 1):

$$U_x=\begin{pmatrix}1&0&0&0\\0&0&e^{-i\phi_N}&0\\0&e^{-i\phi_N}&0&0\\0&0&0&(-1)^{\eta}e^{-2i\phi_N}\end{pmatrix}, \tag{16}$$

where $\eta=1(0)$ for fermions (bosons). The effective gate for a three-spin chain in [6] is a special case of U_x here. It is known [24] that any two-qubit gate that can create entanglement between two qubits is universal for quantum computation, when assisted by one-qubit operations. Here we assume such one-qubit gates are available and we will show that U_x can create entanglement for fermionic chains (and spin chains) and hence is universal for quantum computation. For bosonic chains, however, U_x is not entangling but it allows two-way communication—i.e., transfer states from both ends simultaneously.

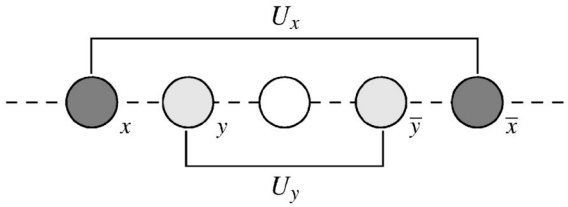

FIG. 1. In engineered fermionic and bosonic chains, when the spectrum of the single-particle Hamiltonian in Eq. (3) satisfies the relation (12), effective gates on mirror-conjugate pairs of sites, such as $(x,\overline{x})$ and $(y,\overline{y})$, can be constructed by free network evolution.

E. Entanglement generation and communication

The entanglement generation protocols in [6] can now be generalized. These protocols require minimal spatial and temporal control of the individual qubits and are also advantageous in that, after extracting the entangled states at sites x and $\overline{x}$, the whole procedures can be repeated by replacing the extracted state with the corresponding initial states. The configuration of the intermediate sites or spins will not be changed after each cycle (except the middle site in protocol 1, which can act as a trigger of the evolution). Moreover, these protocols can be deployed for studying the dynamics of entanglement flow [23].

1. Entanglement generation protocol 1

For a linear bosonic or fermionic chain with odd number of sites, the entanglement generation problem can be mapped to the state transfer problem. In [6], only one specific type of mapping is discussed—namely, the one proportional to S_x. Here, with the enlarged set of choices for the coupling constants, we can generalize the mapping by including the possibility of nonzero diagonal coupling terms. For the sake of comparison with protocol 2, we outline briefly the mapping below.

Suppose the coupling constants still satisfy the symmetry relations in Eq. (6), we consider a basis consisting of maximally entangled states $|\widetilde{j}\rangle\equiv(1/\sqrt{2})(|\boldsymbol{j}\rangle+|\overline{\boldsymbol{j}}\rangle)$ for $j=1,2,\ldots,n-1$, and a state $|\widetilde{n}\rangle\equiv|\boldsymbol{n}\rangle$, where $n=\frac{1}{2}(N+1)$ is the position of the middle site. The Hamiltonian in Eq. (1) acts in this basis as

$$\begin{pmatrix}\lambda_1&\omega_1&0&\cdots&0\\\omega_1&\lambda_2&\omega_2&\cdots&0\\0&\omega_2&\lambda_3&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\sqrt{2}\omega_{n-1}\\0&0&0&\sqrt{2}\omega_{n-1}&\lambda_n\end{pmatrix}, \tag{17}$$

which is also a real and symmetric tridiagonal matrix as H_N (but the size is about half of it). Suppose the initial state is $|\widetilde{n}\rangle$—i.e., a single boson or fermion at the middle site but empty for all other sites. The task of entanglement generation for the remote pair of sites located at 1 and $\overline{1}$ is the same as to rotate from the unentangled state $|\widetilde{n}\rangle$ to the entangled state $|\widetilde{1}\rangle$. This is equivalent to the state transfer problem we have discussed and can be solved in exactly the same way.

On the other hand, for linear chains with an even number of sites, a similar protocol [6] can be used for transferring entanglement from (now redefined) $|\widetilde{n}\rangle\equiv(1/\sqrt{2})(|\boldsymbol{n}\rangle+|\overline{\boldsymbol{n}}\rangle)$, where $n=N/2$, to the remote pair of sites $|\widetilde{1}\rangle$. However, this requires the local pair of sites in the middle to be maximally entangled initially and is therefore an entanglement transfer protocol.

The two protocols above require the initialization to be made in the middle of the chains. After the free evolution, the entangled states are then extracted at the ends of the chains. In situations where we are allowed to have access only to the pair of sites we want to entangle, protocol 2, as we shall see next, will be more useful. However, protocol 1

works for both fermionic and bosonic chains but protocol 2 is applicable for fermionic chains only.

2. Entanglement generation protocol 2

We now show that any pair of mirror-conjugate sites x and $\overline{x}$ can be maximally entangled with the application of U_x and the state initialization at x and $\overline{x}$ only. For simplicity, all other sites are set to be empty (or spin-down in applying to spins chains). First of all, for any normalized pure state of two qubits, $a|00\rangle+b|01\rangle+c|01\rangle+d|11\rangle$, where $|a|^2+|b|^2+|c|^2+|d|^2=1$, the concurrence $\mathcal{C}=2|ad-bc|$ is a measure of entanglement [25]. The two sites are unentangled when $\mathcal{C}=0$ and maximally entangled when $\mathcal{C}=1$. Suppose the two sites are initially in a product state—i.e., $ad=bc$—and all other sites being empty. With the application of U_x, the concurrence becomes $2|ad-(-1)^{\eta}bc|=2(1-(-1)^{\eta})|ad|$. Consequently, for fermionic chains (with $\eta=1$), the sites x and $\overline{x}$ can be maximally entangled from any initial product state with $ad=bc$ and $|ad|=\frac{1}{4}$. For example, if the initial state is $|+\rangle\,|+\rangle$ where $|+\rangle=(1/\sqrt{2})(|0\rangle+|1\rangle)$, then the entanglement of the final state can be made explicit by expressing it in the Schmidt form $(1/\sqrt{2})(|0\rangle|\phi_+\rangle+e^{-i\phi_N}|1\rangle|\phi_-\rangle)$, where $|\phi_\pm\rangle=(1/\sqrt{2})(|0\rangle\pm e^{-i\phi_N}|1\rangle)$ and $\langle\phi_+|\phi_-\rangle=0$.

Last, we note that the protocol for generating a class of multipartite entangled states, called graph states, suggested in [5], can also be extended for the more general Hamiltonian in Eq. (1) with various spectra.

3. Two-way communication

For bosonic chains, one can show that the entanglement of any pure state between sites x and $\overline{x}$ is invariant after the application of U_x. In fact, the net effect of the free evolution of the network, with any initial configurations, for a period of τ is an inversion of quantum states about the center of the chain, apart from an extra induced phase $e^{-i\phi_N}$. Nonetheless, this implies the possibility of simultaneous transfer of quantum state from site x to site $\overline{x}$ and vice versa. Let us define the protocol more clearly. Suppose Alice and Bob are sending their states at x and $\overline{x}$, respectively. We consider the initial state is in a product state, which can be written in general as $(a_0|0\rangle_x+a_1|1\rangle_x)(b_0|0\rangle_{\overline{x}}+b_1|1\rangle_{\overline{x}})$, with all other sites being empty. Applying U_x yields $(b_0|0\rangle_x+e^{-i\phi_N}b_1|1\rangle_x)(a_0|0\rangle_{\overline{x}}+e^{-i\phi_N}a_1|1\rangle_{\overline{x}})$. Therefore, both states can be sent simultaneously. Interestingly, different parities can use the same channel, but on different conjugate pair of sites, at the same time without mutual interference.

III. CONCLUSION

We have demonstrated how to perform quantum state transfer and construct effective two-qubit gates in engineered networks in which the coupling constants are determined by the eigenvalue spectrum satisfying a certain condition. This condition is shown to be both sufficient and necessary in mirror-symmetrical networks. The possibility of perfect communication between distant sites of a single harmonic trap or an infinitely deep square well has been discussed. The effective gates for fermionic networks, including spin chains, are entangling and hence can be used for universal quantum computation. Two entanglement generation schemes are proposed. The first one works for both fermionic and bosonic chains but the second one is for the fermionic chains only. Nonetheless, the bosonic chain allows two-way communication for different pair of sites simultaneously without mutual interference.

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