Quantum-state transfer via the ferromagnetic chain in a spatially modulated field

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We show that a perfect quantum-state transmission can be realized through a spin chain possessing the commensurate structure of an energy spectrum, which is matched with the corresponding parity. As an exposition of the mirror inversion symmetry discovered by Albanese *et al.* (e-print quant-ph/0405029), the parity matched commensurability of the energy spectra helps us to present preengineered spin systems for quantum information transmission. Based on these theoretical analyses, we propose a protocol of near-perfect quantum-state transfer by using a ferromagnetic Heisenberg chain with uniform coupling constant, but an external parabolic magnetic field. The numerical results show that the initial Gaussian wave packet in this system with optimal field distribution can be reshaped near perfectly over a longer distance.

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Recently quantum information processing (QIP) protocols have been considered with quantum spin [1–3] (or quasispin [4]) systems. The simple spin chains have been explored as a coherent data bus [5–8]. It provides us with a quantum channel for perfect transmission of quantum states when the spin chain is preengineered [9]. An isotropic antiferromagnetic spin ladder system was proposed as a novel robust kind of quantum data bus [10]. Due to a large spin gap existing in such a perfect medium, the effective Hamiltonian of the two connected spins can be archived as that of a Heisenberg type by adiabatic elimination, which possesses an effective coupling strength inversely proportional to the distance of the two spins and thus the quantum information can be transferred between the two spins separated by a longer distance; i.e., the characteristic time of quantum state transferring linearly depends on the distance. Such a gapped spin system can be used as a perfect quantum channel for perfect quantum state transmission if local measurements of the individual spins can be implemented. In fact, it has been proved by Verstraete *et al.* [2,3] that the ground states of a spin system with energy gap possess an infinite entanglement length opposed to their finite correlation length.

The physical process of quantum-state transmission through a quantum spin system can be understood as a dynamical permutation (or translation) preserving the initial shape of a quantum state of the involved two qubits, which can be realized as a specific evolution of the total quantum spin system from an initial wave function localized on a single site of the lattice to another at long distance. Most recently it was discovered that, if there exists a mirror inversion symmetry (MIS) with respect to its center in the spin chain, such quantum evolution can occur dynamically at certain instants [11]. Such a scheme for quantum-state transmission is much appreciated since no dynamical control is required over individual qubits. In this article we will revisit this elegant conception by explicitly considering the spectrum structure and the corresponding parities of such a MIS system. We discover that the MIS can be implemented in a universal quantum spin system with a commensurate spectral structure matching the corresponding parities. With the help of this discovery, we, in principle, can propose various scenarios for perfect and near-perfect quantum information transmission through the preengineered quantum spin chains. We give an example that seems to be more complicated than those by others.

Furthermore, a scheme, based on our theoretical analysis to realize near-perfect quantum-state transfer, is proposed with the quantum channel by a ferromagnetic Heisenberg chain with uniform coupling constant, but an external parabolic magnetic field. Numerical results show that, for the optimal field distribution, this system can perform a near-perfect transfer for a Gaussian wave packet over a longer distance.

To sketch our central idea, let us first consider a single-particle system with the usual spatial refection symmetry (SRS) in the Hamiltonian H. Let P be the spatial refection operator. The SRS is implied by $[H,P]=0$. Now we prove that, after time π/E_0, any state $\psi(\vec{r})$ can evolve into the reflected state $\pm\psi(-\vec{r})$ if the eigenvalues ε_n match the parities p_n in the following way:

$$\varepsilon_n = N_n E_0, \quad p_n = \pm(-1)^{N_n} \tag{1}$$

for arbitrary positive integer N_n and

$$H\phi_n(\vec{r}) = \varepsilon_n \phi_n(\vec{r}), \quad P\phi_n(\vec{r}) = p_n \phi_n(\vec{r}). \tag{2}$$

Here, $\phi_n(\vec{r})$ is the common eigen wave function of H and P, $\vec{r}$ the position of the particle. We call Eq. (1) the spectrum-parity-matching condition (SPMC).

The proof of the above rigorous conclusion is a simple, but heuristic exercise in basic quantum mechanics. In fact, for the spatial refection operator, $P\psi(\vec{r})=\pm\psi(-\vec{r})$. For an arbitrarily given state at $t=0$, $\psi(\vec{r},t)|_{t=0}=\psi(\vec{r})$, this state evolves to

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$$\psi(\vec{r},t) = e^{-iHt}\psi(\vec{r}) = \sum_n C_n e^{-iN_nE_0t}\phi_n(\vec{r}) \tag{3}$$

at time t, where $C_n = \langle\phi_n|\psi\rangle$. Then, at time $t=\pi/E_0$, we have

$$\begin{aligned}\psi(\vec{r},\pi/E_0) &= \sum_n C_n(-1)^{N_n}\phi_n(\vec{r})\\ &= \pm\sum_n C_n p_n \phi_n(\vec{r})\\ &= \pm P\psi(\vec{r})\\ &= \pm\psi(-\vec{r}).\end{aligned} \tag{4}$$

This is just the central result [11] discovered for a quantum spin system that the evolution operator can become a parity operators $\pm P$ at some instant $t=(2n+1)\pi/E_0$—that is,

$$\exp[-iH\pi/E_0] = \pm P. \tag{5}$$

From the above arguments we have a consequence that if the eigenvalues $\varepsilon_n = N_nE_0$ of a one-dimensional (1D) Hamiltonian H with spatial refection symmetry are odd-number spaced—i.e., $N_n - N_{n-1}$ are always odd—any initial state $\psi(x)$ can evolve into $\pm\psi(-x)$ at time $t=\pi/E_0$. In fact, for such a 1D system, the discrete states alternate between even and odd parity. Consider that the eigenvalues $\varepsilon_n = N_nE_0$ are odd-number spaced. The next-nearest level must be even-number spaced, and then the SPMC is satisfied. Obviously, the 1D SPMC is more realizable for the construction of the model Hamiltonian to perform perfect state transfer.

Now, we can directly generalize the above analysis to many-particle systems. For the quantum spin chain, one can identify the above SRS as the MIS with respect to the center of the quantum spin chain. As in the discussion in Ref. [11], we write MIS operation

$$P\Psi(s_1,s_2,\ldots,s_{N-1},s_N) = \Psi(s_N,s_{N-1},\ldots,s_2,s_1) \tag{6}$$

for the wave function $\Psi(s_1,s_2,\ldots,s_{N-1},s_N)$ of spin chain. Here, $s_n=0,1$ denotes the spin values of the nth qubit. According to this representation of SRS and our discovered SPMC, many spin systems can be preengineered for perfect quantum-state transfer. For instance, two-site spin-$\frac{1}{2}$ Heisenberg system the simplest example which meets the SPMC. Recently, Christandl *et al.* [9] proposed an N-site XY chain with elaborately designed modulated coupling constants between two nearest-neighbor sites, which ensures a perfect state transfer. It is easy to find that this model corresponds the SPMC for the simplest case $N_n=n$.

In the following, we propose a class of different models for perfect state transfer, whose spectrum structures obey our SPMC exactly. Consider an N-site spin-$\frac{1}{2}$ XY chain with the Hamiltonian

$$H = 2\sum_{i=1}^{N-1} J_i[S_i^xS_{i+1}^x + S_i^yS_{i+1}^y], \tag{7}$$

where S_i^x, S_i^y, and S_i^z are Pauli matrices for the ith site and J_i the coupling strength for near-neihbor interaction. For the open boundary condition, this model is equivalent to the spinless fermion model. The equivalent Hamiltonian can be written as

$$H = \sum_{i=1}^{N-1} J_i^{[k]} a_i^\dagger a_{i+1} + \text{H.c.}, \tag{8}$$

where $a_i^\dagger, a_i$ are the fermion operators. This describes a simple hopping process in the lattice.

According to our SPMC, we can present different models [labeled by different positive integer $k(\in\{0,1,2,\ldots\})$] through preengineering of the coupling strength as

$$J_i = J_i^{[k]} = \sqrt{i(N-i)} \tag{9}$$

for even i and

$$J_i = J_i^{[k]} = \sqrt{(i+2k)(N-i+2k)}$$

for odd i. By a straightforward calculation, one can find the k-dependent spectrum

$$\varepsilon_n = -N + 2(n-k) - 1 \tag{10}$$

for $n=1,2,\ldots,N/2$ and

$$\varepsilon_n = -N + 2(n+k) - 1 \tag{11}$$

for $n=N/2+1,\ldots,N$. The corresponding k-dependent eigenstates are

$$|\phi_n\rangle = \sum_{i=1}^{N} c_{ni}|i\rangle = \sum_{i=1}^{N} c_{ni}a_i^\dagger|0\rangle, \tag{12}$$

where the coefficients can be determined by

$$c_{n2} = \frac{\varepsilon_n c_{n1}}{\sqrt{(1+2k)(N-1+2k)}},$$

$$\cdots$$

$$0 = \sqrt{(i+2k+1)(N-i+2k-1)}c_{ni+2} - \varepsilon_n c_{ni+1} + \sqrt{i(N-i)}c_{ni} \quad (i\text{ is even}),$$

$$0 = \sqrt{(i+2k)(N-i+2k)}c_{ni} - \varepsilon_n c_{ni+1} + \sqrt{(i+1)(N-i-1)}c_{ni+2} \quad (i\text{ is odd}),$$

$$\cdots$$

$$c_{nN} = \frac{\sqrt{(1+2k)(N-1+2k)}c_{nN-1}}{\varepsilon_n}. \tag{13}$$

It is obvious that the model proposed in Ref. [9] is just the special case of our general model in $k=0$. For arbitrary k, we can easily check that it meets the our SPMC by a straightforward calculation. Thus we can conclude that these spin systems with a set $S^{[k]}$ of preengineered couplings $J_i^{[k]}$ can serve the perfect quantum channels and allow us to transfer the quantum information of spin qubits.

In the above arguments we show the possibility to implement the perfect state transfer of any quantum state over arbitrarily long distances in a quantum spin chain. The crucial point to this end is that one need to locally engineer couplings between the spins in the specific way. Now we need consider the possibility to engineer couplings for the practical quantum information processing. As usual, we expect to transfer quantum information over a much longer distance. For this purpose the spin chain must be longer and thus contain too many degrees of freedom. Since the dimension of the Hilbert space of the many-body system grows exponentially with the size of the system, there must be enormous parameters to be exactly engineered. In this sense it is almost impossible to engineer a real spin system so that it possesses energy levels to exactly satisfy the SPMC. To overcome the difficulties, there is a naive way that one chooses some special states to be transported, which is a coherent superposition of a commensurate part of the whole set of eigenstates. For example, we consider a truncated Gaussian wave packet for an anharmonic oscillator with lower eigenstates to be harmonic. It is obvious that such a system allows some special states to transfer with high fidelity. We can implement such an approximate harmonic system in a natural spin chain without the preengineering of couplings. Our strategy is to apply a modulated external field.

Let us consider the Hamiltonian of a $(2N+1)$-site spin-$\frac{1}{2}$ ferromagnetic Heisenberg chain

$$H = -J\sum_{i=1}^{2N} \vec{S}_i \cdot \vec{S}_{i+1} + \sum_{i=1}^{2N+1} B(i) S_i^z, \qquad (14)$$

with uniform coupling strength $-J<0$, but in the parabolic magnetic field

$$B(i) = 2B_0(i-N-1)^2, \qquad (15)$$

where B_0 is a constant. In single-excitation invariant subspace with the fixed z component of total spin $S^z=(2N-1)/2$, this model is equivalent to the spinless fermion hopping model with the Hamiltonian

$$H = -\frac{J}{2}\sum_{i=1}^{2N}(a_i^\dagger a_{i+1} + \text{H.c.}) + \frac{1}{2}\sum_{i=1}^{2N+1} B(i) a_i^\dagger a_i, \qquad (16)$$

where we have neglected a constant in the Hamiltonian for simplicity. For the single-particle case with the site basis

$$\{|n\rangle = |0,0,\ldots,\overset{n'\text{th}}{1},0\ldots\rangle | n=1,2,\ldots\},$$

the matrix presentation of the Hamiltonian (16) is

$$H_{JJ} = \sum_n \left\{ -\frac{1}{2}E_J(|n+1\rangle\langle n| + |n\rangle\langle n+1|) + 4E_c(n-n_g)^2|n\rangle\langle n| \right\}, \qquad (17)$$

which is just the same as that of the Hamiltonian of Josephson junction in the Cooper-pair number basis in Ref. [12] for $E_J=J, E_c=2B_0$. Analytical analysis and numerical results can show that the lower-energy spectrum is indeed quasiharmonic in the case $E_J \gg E_c$ [13]. Although the eigenstates of the Hamiltonian (14) do not satisfy the SPMC precisely, especially for the high-energy range, there must exist some Gaussian wave packet states expanded by the lower eigenstates, which can be transferred near perfectly.

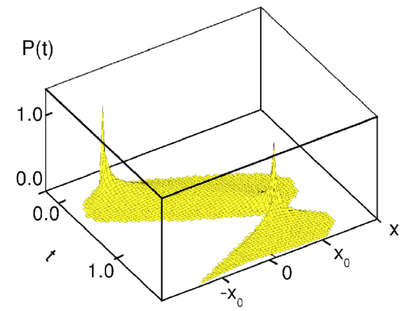

FIG. 1. Schematic illustration of the time evolution of a Gaussian wave packet. $P(t)$ is the probability of the wave packet, where the unit of t is $\alpha^2\pi$. It shows that the near-perfect state transfer over a long distance is possible in the quasiharmonic system.

We consider a Gaussian wave packet at $t=0$, $x=N_A$ as the initial state:

$$|\psi(N_A,0)\rangle = C\sum_{i=1}^{2N+1} e^{-\alpha^2(i-N_A-1)^2/2}|i\rangle, \qquad (18)$$

where $|i\rangle$ denotes the state with $2N$ spins in the down state and only the ith spin in the up state and C is the renormalization factor. The coefficient α^2 is determined by the width of the Gaussian wave packet Δ:

$$\alpha^2 = \frac{4\ln 2}{\Delta^2}. \qquad (19)$$

The state $|\psi(0)\rangle$ evolves to

$$|\psi(t)\rangle = e^{-iHt}|\psi(N_A,0)\rangle \qquad (20)$$

at time t and the fidelity of state $|\psi(0)\rangle$ transferring to the position N_B is defined as

$$F(t) = |\langle\psi(N_B,0)|e^{-iHt}|\psi(N_A,0)\rangle|. \qquad (21)$$

In Fig. 1 the evolution of the state $|\psi(0)\rangle$ is illustrated schematically. From the investigation of Ref. [13], we know that for small $N_A=-N_B=-x_0$, where N_B is the mirror counterpart of N_A, but in the large-Δ limit, if we take $B_0=8(\ln 2/\Delta^2)^2$, $F(t)$ has the form

$$F(t) = \exp\left[-\frac{1}{2}\alpha^2 N_A^2\left(1+\cos\frac{2t}{\alpha^2}\right)\right], \qquad (22)$$

which is a periodic function of t with maxima 1. This is in agreement with our above analysis. However, in quantum communication, what concerns us is the behavior of $F(t)$ in the case of the transfer distance $L \gg \Delta$, where $L=2|N_A| = 2|N_B|$. For this purpose the numerical method is performed for the case $L=500$, $\Delta=2,4,6$, and $B_0=8(\ln 2/\Delta^2)^2\lambda$. The factor λ determines the maximum fidelity, and then the optimal field distribution can be obtained numerically. In Figs. 2(a)–2(c) the functions $F(t)$ are plotted for different values of λ. It shows that for the given wave packets with $\Delta=2$, 4, and 6, there exists a range of λ during which the fidelities $F(t)$

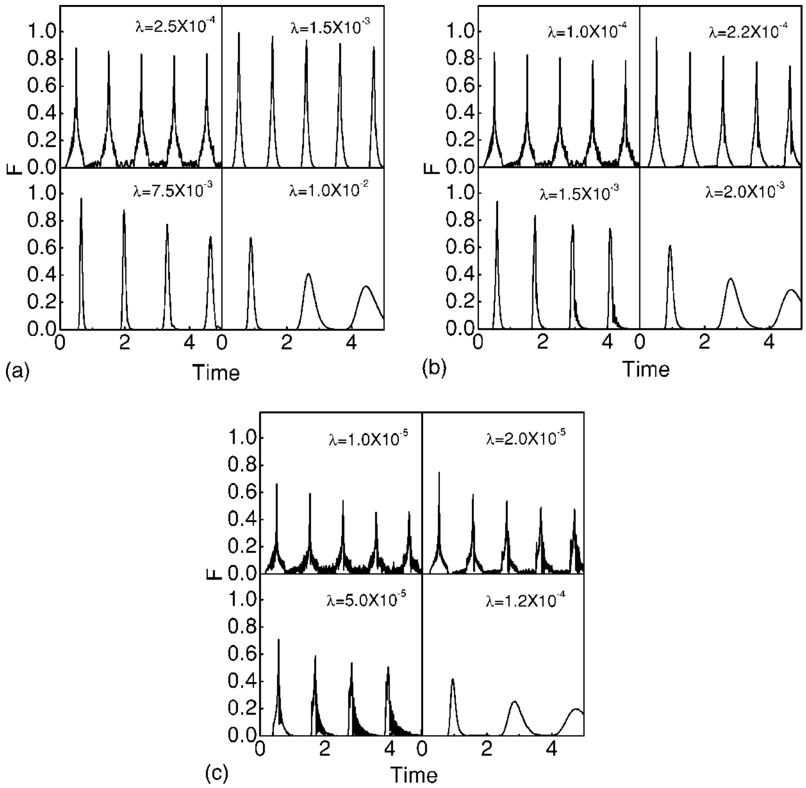

FIG. 2. The fidelities $F(t)$ of the transmission of the Gaussian wave packets with $\Delta=2$, 4, and 6 over the distance $L=500$ are plotted for different values of λ in (a), (b), and (c). The unit of time is $\alpha^2\pi$. It shows that there exist optimal values of λ to get high fidelities up to 0.748, 0.958, and 0.992, respectively.

are up to 0.748, 0.958, and 0.992, respectively. For finite distance, the maximum fidelity decreases as the width of Gaussian wave packet increases. On the other hand, the strength of the external field also determines the value of the optimal fidelity for a given wave packet. Numerical results indicate that it is possible to realize near-perfect quantum-state transfer over a longer distance in a practical ferromagnetic spin chain system. It also shows when $\lambda \rightarrow 0$—i.e., when zero (or uniform) external field is applied—the fidelity decreases rapidly. It is due the cosinusoidal single-excitation dispersion relation of the spin chain.

In summary, we have shown that a perfect quantum transmission can be realized through a universal quantum channel provided by a quantum spin system with spectrum structure, in which each eigenenergy is commensurate and matches with the corresponding parity. According to this SPMC for the mirror inversion symmetry [11], we can implement the perfect quantum information transmission with several preengineered quantum spin chains. For more practical purposes, we prove that an approximately commensurate spin system can also realize near-perfect quantum-state transfer in a ferromagnetic Heisenberg chain with uniform coupling constant, but in an external field. A numerical method has performed to study the fidelity for the system in a parabolic magnetic field. The external field plays a crucial role in the scheme. It induces a lower quasiharmonic spectrum, which can drive a Gaussian wave packet from the initial position to its mirror counterpart. The fidelity depends on the initial position (or distance L), the width of the wave packet Δ, and the magnetic field distribution $B(i)$ via the factor λ. Thus for given L and Δ, proper choice of the factor λ can achieve the optimal fidelity. Finally, we conclude that it is possible to implement near-perfect Gaussian wave packet transmission over a longer distance in many-body systems.

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