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Generation of entanglement in regular systems

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We study dynamical generation of entanglement in bipartite quantum systems, characterized by purity (or linear entropy), and caused by the coupling between the two subsystems. The explicit semiclassical theory of purity decay is derived for integrable classical dynamics of the uncoupled system and for localized (general Gaussian wave packet) initial states. Purity decays as an algebraic function of (time) × (strength of perturbation), independently of the Planck's constant.

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Detailed understanding of *entanglement*, being one of the most distinct features of the quantum world, is an issue of high importance, particularly in view of recent efforts to build quantum devices that will manipulate (pure states of) individual quantum systems. The loss of control over entanglement, i.e., decoherence, in such a device is one of the major obstacles that we have to overcome.

In the present paper we are going to study dynamical generation of entanglement in bi-partite systems. Initially separable pure states will get entangled due to the coupling between two subsystems. Here we consider systems where the uncoupled part of the Hamiltonian in both subsystems generates regular (integrable) dynamics in the classical limit. The motivation to study entanglement generation in systems with regular uncoupled dynamics comes from the fact that such systems are quite common both in experiments and as theoretical models. For instance, if the uncoupled system consists of a number of uncoupled one degree of freedom (DOF) systems then it is integrable. Such is the case in various proposals for quantum computation, e.g., ion traps. Further, the experimentally realizable Jaynes-Cummings model, where decoherence for cat states [1] has actually been experimentally measured, is also an integrable system. Still further, a standard model of decoherence [2] consists of an infinite number of harmonic oscillators. If the bath consists of a finite number of harmonic oscillators this falls under the domain of our theory. Recently [3] it has been pointed out that the decoherence for truly macroscopic superposition is so fast that the usual master equation approach is not valid anymore. On this very short "instantaneous" time scale any system will effectively behave as a regular one (i.e., correlations do not decay yet). There have been several related studies of purity decay: Ref. [4] numerically compared purity decay in classically regular and chaotic regimes, and further Refs. [5–7] used time-dependent perturbation theory in order to explain the semiclassical behavior of purity decay. For a random-matrix approach to purity, see Ref. [8]. Most recently, Jacquod [9] suggested some universal forms of asymptotic purity decay based on the semiclassical expansion in terms of classical orbits along the lines of [10].

Time evolution of the system will be governed by the Hamiltonian

$$H = H_0 + \delta V, \quad H_0 = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B, \tag{1}$$

where H_0 is the uncoupled part of the Hamiltonian and V is the coupling between the two subsystems responsible for the generation of entanglement. The strength of this coupling is given by a dimensionless parameter δ. We will use subscripts A and B to denote the two subsystems. The state of the whole system at time t is simply $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, with a unitary propagator $U(t) = \exp(-iHt/\hbar)$. Let us define the time-averaged coupling

$$\bar{V} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, V(t), \tag{2}$$

where $V(t)$ is the coupling operator in the interaction picture, $V(t) = U_0^\dagger(t) V U_0(t), U_0(t) = \exp(-iH_0 t/\hbar)$, i.e., propagated with the uncoupled part of the Hamiltonian. We shall assume a situation, typical for a regular H_0, where $\bar{V}$ is a nontrivial operator, different from zero or a multiple of the identity [11]. We wish to stress that the coupling V will typically break the integrability of H_0.

The entanglement between the two subsystems, for a pure state $|\psi(t)\rangle$, is characterized by a purity

$$I(t) = \mathrm{tr}_A[\rho_A^2(t)], \quad \rho_A(t) = \mathrm{tr}_B[\rho(t)] \tag{3}$$

where $\rho(t) \coloneqq |\psi(t)\rangle\langle\psi(t)|$. If and only if the purity $I(t)$ is less than 1, the two subsystems are entangled; otherwise they are in a separable (product) state. Our initial state will always be a product one, $|\psi(0)\rangle = |\psi_A(0)\rangle \otimes |\psi_B(0)\rangle$; hence $I(0)=1$. The initial states $|\psi_{A,B}(0)\rangle$ will be Gaussian wave packets. The time dependence of the purity $I(t)$ will then tell us how fast the two subsystems get entangled due to the coupling V.

Let us proceed with the calculation of the purity decay $I(t)$. We should observe that propagating the state backward in time with a separable uncoupled dynamics $U_0(t)$ does not change the value of the purity, so $\rho(t)$ in Eq. (3) can be replaced by

$$\rho^M(t) = M(t)\rho(0)M^\dagger(t), \quad M(t) = U_0^\dagger(t)U(t), \tag{4}$$

where $M(t)$ is the echo operator used in the theory of fidelity decay [12,13]. The matrix $\rho^M(t)$ represents the evolution of our pure state in the interaction picture. As just explained above, the purity (3) is equal to

$$I(t) = \mathrm{tr}_A[\{\rho_A^M(t)\}^2], \quad \rho_A^M(t) = \mathrm{tr}_B[\rho^M(t)]. \tag{5}$$

An advantage of the representation (5) over (3) is the fact that the echo operator $M(t)$ is, unlike the forward evolution $U(t)$, close to an identity for small δ so one may use perturbative or asymptotic expansions in δ. We follow the approach of Ref. [12] and use the Baker-Campbell-Hausdorff formula $e^{\delta V}e^{\delta W} = \exp\{\delta(V+W) + \frac{1}{2}\delta^2[V,W] + \cdots\}$ for continuous products (see, e.g., [14]) to simplify the expression for the echo operator $M(t)$. The lowest-order term in the exponential is $(\delta/\hbar)\int dt\, V(t)$. For times larger than some classical averaging time t_{av}, in which $\overline{V}$ [Eq. (2)] converges, this term can be rewritten as $(\delta/\hbar)\overline{V}t$. The second-order term in δ can be shown to grow with time as $\delta^2 t\bar{\bar{V}}/\hbar$, where $\bar{\bar{V}}$ [12] has an $\hbar$-independent classical limit. By induction higher orders can be estimated to grow as $\sim \delta^r t^{r-1}/\hbar$. Therefore, provided only $\delta \ll \delta_c$, where $\delta_c \approx \overline{V}_{\mathrm{eff}}/\bar{\bar{V}}_{\mathrm{eff}}$ is $\hbar$ independent, higher orders in δ can be neglected and we end up with a very simple expression for the echo operator,

$$M(t) = e^{-i\delta t\overline{V}/\hbar}. \tag{6}$$

So the echo operator can be interpreted as the propagator with an effective Hamiltonian $\delta\overline{V}$. We proceed with a semiclassical evaluation of the purity, a procedure completely analogous to a similar calculation for the fidelity [15]. We use the notation in which small latin letters denote classical limiting observables (e.g., Weyl symbols) of the corresponding operators denoted by capital latin letters. For example, let $\mathbf{j} = (\mathbf{j}_A, \mathbf{j}_B)$ denote a $(d = d_A + d_B)$-dimensional vector of classical canonical actions of the completely integrable uncoupled classical Hamiltonian $h_0 = h_A + h_B$. d_A and d_B are the numbers of DOF's of the subsystems A and B, respectively. In quantum mechanics, one has a vector of mutually commuting action operators $\mathbf{J}$, with a common set of eigenvectors, denoted by a multi-index $\mathbf{n} \in \mathbb{Z}^d$ of quantum numbers, $\mathbf{J}|\mathbf{n}\rangle = \hbar(\mathbf{n} + \boldsymbol{\alpha})|\mathbf{n}\rangle \approx \hbar\mathbf{n}|\mathbf{n}\rangle$ where $\boldsymbol{\alpha}$ are the Maslov indices. Here and below "$\approx$" means equal in the leading order in $\hbar$. The purity (5) can now be written as a sum over a d-dimensional lattice of quantum numbers, using the fact that since $\overline{V}$ commutes with H_0 it is diagonal in the basis $|\mathbf{n}\rangle$, and in the leading semiclassical order (in $\hbar$) we can replace the summation by an integral over the classical action space. Further, we replace the operator $\overline{V}$ by its classical limit $\overline{v}(\mathbf{j})$, which is a conserved quantity so it is a function of d classical actions $\mathbf{j}$ only. Let us denote by $p(\mathbf{j}) = p_A(\mathbf{j}_A)p_B(\mathbf{j}_B)$ the classical limit of the initial density $\langle\mathbf{n}|\rho(0)|\mathbf{n}\rangle$. For our initial product state of two wave packets each of the two densities is a Gaussian,

$$p_a(\mathbf{j}_a) = C\exp\{-(\mathbf{j}_a - \mathbf{j}_a^*)\Lambda_a(\mathbf{j}_a - \mathbf{j}_a^*)/\hbar\}, \tag{7}$$

where the subscript a takes values A or B, depending on the subsystem, $\mathbf{j}_a^*$ is the position of the initial packet, Λ_a is a positive squeezing matrix, and $C = (\hbar/\pi)^{d_a/2}\sqrt{\det\Lambda_a}$ is a normalization constant. The purity can now be written as an integral,

$$I(t) \approx \hbar^{-2d}\int d\mathbf{j}\, d\tilde{\mathbf{j}}\, \exp\left(-i\frac{\delta t}{\hbar}\Phi\right)p(\mathbf{j})p(\tilde{\mathbf{j}}),$$

$$\Phi = \overline{v}(\mathbf{j}_A,\mathbf{j}_B) - \overline{v}(\tilde{\mathbf{j}}_A,\mathbf{j}_B) + \overline{v}(\tilde{\mathbf{j}}_A,\tilde{\mathbf{j}}_B) - \overline{v}(\mathbf{j}_A,\tilde{\mathbf{j}}_B). \tag{8}$$

Note that $I(t)$ is written simply as a double average over the classical action space of the phase factor, weighted with initial densities. For a comparison between the purity and the corresponding classical analog see Ref. [16]. Next we expand the phase Φ around the position $\mathbf{j}^* = (\mathbf{j}_A^*, \mathbf{j}_B^*)$ of the initial packet. The constant and the linear terms cancel exactly and the lowest-order nonvanishing term is quadratic,

$$\Phi \approx (\mathbf{j}_A - \tilde{\mathbf{j}}_A)\cdot\overline{v}''_{AB}(\mathbf{j}^*)(\mathbf{j}_B - \tilde{\mathbf{j}}_B) + \cdots, \tag{9}$$

where $\overline{v}''_{AB}$ is a $d_A \times d_B$ matrix of mixed second derivatives of $\overline{v}$ evaluated at the position of the initial packet,

$$(\overline{v}''_{AB})_{kl} = \frac{\partial^2\overline{v}}{\partial(\mathbf{j}_A)_k\,\partial(\mathbf{j}_B)_l}. \tag{10}$$

Using this expansion in the integral for purity we see that the resulting $2d$-dimensional integral is Gaussian and can therefore be expressed in terms of a determinant of a $2d \times 2d$ matrix. Using special properties of the resulting matrix the determinant can be reduced [17] to a determinant of a $d_A \times d_A$ matrix, with the final result

$$I(t) = \frac{1}{\sqrt{\det[\mathbb{1} + (\delta t)^2 u]}}, \quad u = \Lambda_A^{-1}\overline{v}''_{AB}\Lambda_B^{-1}\overline{v}''_{BA}, \tag{11}$$

where u is a $d_A \times d_A$ matrix involving $\overline{v}''_{AB}$ and its transpose $\overline{v}''_{BA}$. Note that the matrix u is a classical quantity (independent of $\hbar$) that depends only on the observable $\overline{v}$ and on the position of the initial packet. This explicit formula for purity decay is the main result of the present paper.[1]

Before discussing its consequences let us recall its range of validity. The restrictions are rather weak: $\overline{v}$ must be nonvanishing (typical for regular systems) and smooth on the scale of the initial packet proportional to $\sqrt{\hbar}$, time must be larger than the averaging time $t > t_{\mathrm{av}}$ and the coupling must be small $\delta < \delta_c$. Note that δ_c does not depend on $\hbar$. In addition, the phase Φ should increment by a small amount for neighboring quantum numbers, which translates into the condition $\delta t\|\overline{v}''_{AB}\| < 1/\hbar$.

The most prominent feature of the formula (11) for the purity decay for the initial product wave packets is its $\hbar$ independence. In the linear response calculation this $\hbar$ independence has already been theoretically predicted [7] as well as numerically confirmed [6]. Here we have a full expression to all orders. We also see that the scaling of the decay time t_d on which $I(t)$ decays is $t_d \sim 1/\delta$. This means that the purity will decay on a very long time scale and so the wave packets are universal pointer states [18], i.e., the most robust states. For small δt we can expand the determinant and we get initial quadratic decay $I(t) = 1 - \frac{1}{2}(\delta t)^2\mathrm{tr}[u] + \cdots$. For large times we use the fact that det $(\mathbb{1} + zu)$ is a polynomial in z of order $r = \mathrm{rank}(u)$, so we have asymptotic power law decay $I(t) \simeq \mathrm{const} \times (\delta t)^{-r}$. Note that the rank of u is bounded by

[1]The very same expression holds also for a generalization of purity to echo dynamics, the so-called echo purity (or purity fidelity), first used in [13].

the minimal of the subspace dimensions, i.e., $1 \leq r \leq \min\{d_A, d_B\}$, since the definition (3) is symmetric with respect to interchanging the roles of the subspaces A and B. Let us give two simple examples. (i) For $d_A=1$ and for *any* d_B we will always have asymptotic power law decay with $r=1$. If a single DOF of the subsystem A is coupled with all DOF's of the subsystem B, e.g., $\overline{v}=j_A \otimes (j_{B1}+j_{B2}+\cdots)$, then $|\overline{v}''|^2 \propto d_B$ and we have $I(t) \simeq 1/(\delta t \sqrt{d_B})$. (ii) Let us consider a multidimensional system where the matrix u is of rank 1 so it can be written as a direct product of two vectors, $u=\mathbf{x} \otimes \mathbf{y}$. The determinant occurring in $I(t)$ is then simply det $[\mathbb{1}+(\delta t)^2 u]=1+(\delta t)^2 \mathbf{x}\cdot\mathbf{y}$. Such is the case for instance if we have a coupling of the same strength between all pairs of DOF's. The dot product is in this case $\mathbf{x}\cdot\mathbf{y} \propto d_A d_B$ and we have $I(t) \simeq 1/(\delta t \sqrt{d_A d_B})$, i.e., the power of the algebraic decay is independent of both d_A and d_B.

In Ref. [9] the author predicted a universal decay of $I(t)$ as t^{-d_A} for short times, and t^{-2d_A} for long times (assuming $d_A=d_B$). The crossover time between two regimes is predicted to be independent of δ and depends only on the size of the initial packet. We note that the short-time prediction of Ref. [9] is consistent with our result. On the other hand, there is a discrepancy of the results for asymptotically long times. Perhaps this inconsistency is due to inapplicability of the semiclassical orbit expansion [9] for asymptotically long times.

We continue with a numerical demonstration of the theoretical prediction for purity decay (11). For the first example we take a (1+1)-DOF system, $d_A=d_B=1$, of two anharmonic oscillators with the uncoupled Hamiltonian

$$H_0 = \gamma_A(\hbar a_A^\dagger a_A - \Delta)^2 + \gamma_B(\hbar a_B^\dagger a_B - \Delta)^2, \tag{12}$$

where $a^\dagger$ and a are standard boson raising and lowering operators. For the coupling we take

$$V = \hbar^2 (a_A^\dagger + a_A)^2 (a_B^\dagger + a_B)^2. \tag{13}$$

The corresponding classical Hamiltonian h reads

$$h = \gamma_A(j_A - \Delta)^2 + \gamma_B(j_B - \Delta)^2 + 16\delta j_A j_B \sin^2\theta_A \sin^2\theta_B \tag{14}$$

where θ_a are the canonical angles. The initial wave packet on both subsystems is a boson coherent state

$$|\psi_A(0)\rangle = |\psi_B(0)\rangle = |\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle, \tag{15}$$

where $|0\rangle$ is the ground state. The parameter α is chosen as $\alpha=\sqrt{j^*/\hbar}$ with $j^*=0.1$. The squeezing parameter for the coherent states (15) is $\Lambda_{A,B}=1/(2j^*)$. Other parameters of the Hamiltonian are $\gamma_A=1, \gamma_B=0.6456$. The offset $\Delta=1.2$ was chosen in order to have nonzero classical frequency $\partial h/\partial j$ at the position of the initial packet. This is needed in order for $\overline{v}$ to be well defined. The time-averaged coupling is calculated easily, $\overline{v}=4j_A j_B$. The matrix u is now just a number, $u=(8j^*)^2$. Theoretical prediction for the purity decay is thus

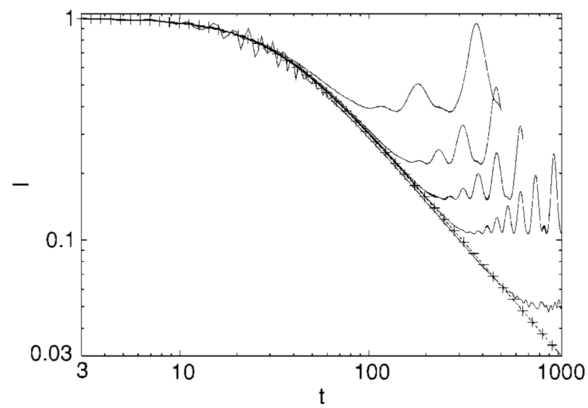

FIG. 1. Purity decay for (1+1)-DOF system (12) and (13) for $\delta=0.04$ and different $1/\hbar=10$, 25, 50, 100, 500, from top to bottom. Dashed line with pluses is the theoretical formula (16).

$$I(t) = \frac{1}{\sqrt{1+(8j^*\delta t)^2}}. \tag{16}$$

The results of numerical simulation together with the theory are shown in Figs. 1 and 2. In Fig. 1 we see that the decay is indeed ħ independent, apart from a finite-size fluctuating plateau after a long-time. The size of this plateau is of the order $I(t\to\infty) \sim 1/N_{\text{eff}}$, where $N_{\text{eff}} \sim \sqrt{8j^*/\hbar}$ is the effective Hilbert space dimension, i.e., the number of action eigenstates overlapping with the initial coherent state (15). Strong revivals for large $\hbar$ are a consequence of the small number of available states N_{eff} and the low dimensionality. Revivals are expected to be less pronounced for larger dimensionalities d_A, d_B, similarly as for the fidelity [15]. For large times one can clearly observe asymptotic t^{-1} decay of the purity. In Fig. 2 we fix $\hbar$ and change the coupling strength δ instead. Apart from oscillations we see a good agreement with the theory also for large δ. Oscillations for times $t<10$ are a consequence of the fact that the time averaging of V [Eq. (2)] converges only after some averaging time t_{av} which is of order ~ 10 in our case.

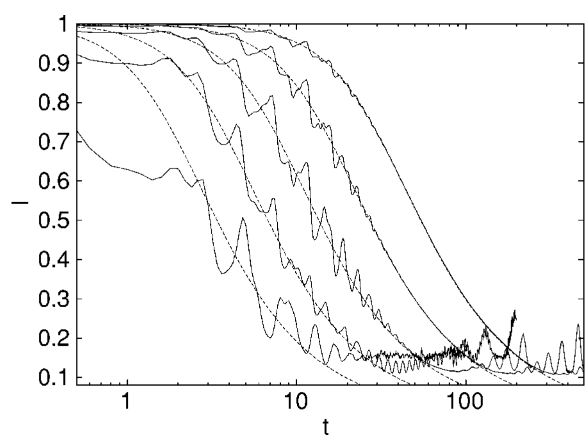

FIG. 2. Purity decay for (1+1)-DOF system (12) and (13) for $\hbar=1/100$ and different $\delta=0.64$, 0.32, 0.16, 0.08, 0.04, from left to right. Dashed lines give the theoretical prediction (16).

As for the second numerical example we take a (2+2)-DOF system ($d_{A,B}=2$) which is the simplest case where we can find different powers of the asymptotic decay, depending on the topology of the coupling. The uncoupled Hamiltonian now reads

$$H_0 = \gamma_1(\hbar a_1^\dagger a_1 - \Delta)(\hbar a_2^\dagger a_2 - \Delta) + \gamma_2(\hbar a_3^\dagger a_3 - \Delta)(\hbar a_4^\dagger a_4 - \Delta). \tag{17}$$

The subscripts 1 and 2 describe two DOF's of the subsystem A, while 3 and 4 compose the subsystem B. The parameters are $\gamma_1=1, \gamma_2=0.64$, and $\Delta=1.2$. The initial state is a product state of four boson coherent states $|\psi_{1,2,3,4}(0)\rangle=|\alpha\rangle$, all with the same $\alpha=\sqrt{j^*/\hbar}$. For the coupling we consider two cases which will give different powers of the asymptotic decay.

Case I. $V=V_{13}+V_{24}$, where the two coupling terms are of the same form as for the (1+1)-DOF system (13) and the indices denote between which two degrees of freedom the coupling acts. The matrix u as well as the relevant determinant is easily calculated resulting in a simple expression for the purity (11),

$$I(t) = \frac{1}{1+(8j^*\delta t)^2}. \tag{18}$$

We see that we have a quadratic asymptotic decay, $I(t) \simeq 1/(\delta t)^2$.

Case II. All-to-all coupling, $V=V_{13}+V_{14}+V_{23}+V_{24}$, results in a rank-1 ($r=1$) matrix u giving the purity decay (11),

$$I(t) = \frac{1}{\sqrt{1+(16j^*\delta t)^2}}. \tag{19}$$

Results of the numerical simulation for both cases are shown in Fig. 3. The coupling strength and the location of the initial packets are $\delta=0.04, j^*=0.1$ for case I, and $\delta=0.02, j^*=0.2$ for case II. From Fig. 3 we see that one indeed has asymptotic t^{-1} or t^{-2} decay, depending on the topology of the coupling.

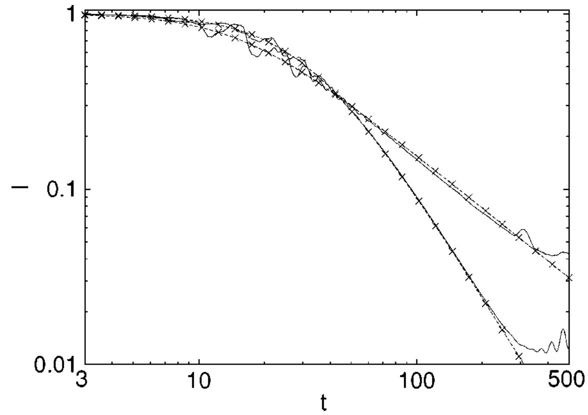

FIG. 3. Purity decay for a (2+2)-DOF system (17) and two different couplings showing different asymptotic power law decay. Full line is the numerics while the two dashed lines with crosses are theoretical predictions, the one with a smaller slope for (19) and the other for (18).

In conclusion, we have derived purity decay for initially localized wave packets in bipartite systems with a nonvanishing (nontrivial) time-averaged coupling operator. This situation naturally occurs in systems where an uncoupled part of the Hamiltonian represents regular dynamics. Purity decays in time inversely proportionally to the coupling strength and is independent of Planck's constant. The decay is algebraic with the asymptotic power law exponent ranging between 1 and the minimal dimension of the subsystems depending on the topology of the coupling.

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