Relativistic spectral comparison theorem in two dimensions

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We demonstrate a spectral comparison theory for the two-dimensional Dirac equation which states that for a given mass, if two time-independent attractive potentials are different such that $V_a < V_b$, the corresponding energy spectrum satisfies the condition $E_a < E_b$. As an illustrative example, the comparison relation is applied to calculate the energy spectrum of the two-dimensional Dirac equation with a Coulomb plus linear potential.

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I. INTRODUCTION

There have been consistent investigations since the Dirac equation was found. Some new results have been reported in recent years: for example, Levision's theorem for the Dirac equation of two [1], three [2], and N dimensions [3], the spectral comparison theorem [4], the supercriticality and transmission resonances in one [5] and three dimensions [6], and the solutions of the Dirac equation with shape-invariant potential in two [7], three [8], and N dimensions [9].

The spectral comparison theorem in the nonrelativistic quantum mechanics proved for the discrete part of the energy spectrum can be formulated as [10]

$$V_a < V_b \Rightarrow E_a < E_b, \tag{1}$$

where V is an attractive potential and E is the discrete spectrum. The theorem of Eq. (1) is valid provided that the wave functions have no nodes—that is to say, for the wave functions in the bottom of angular momentum subspace of the three-dimensional Dirac equation [4]. In lower-dimensional field theory and condensed matter physics, the two-dimensional systems seem to some new features. However, a study of the discrete spectral comparison theorem for the two-dimensional Dirac equation is still lacking to our knowledge, which is the purpose of this paper.

This paper is organized as follows. Section II is devoted to the introduction of the two-dimensional Dirac equation. The derivation of the spectral comparison theorem is given in Sec. III. We apply the new two-dimensional relativistic comparison theorem to calculate the spectrum of a Coulomb plus linear potential in Sec. IV.

II. DIRAC EQUATION OF 2+1 DIMENSIONS

The Dirac equation of two dimensions is given by

$$\sum_{\mu=0}^{2} i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi = m\psi, \tag{2}$$

where m is the mass of the particle and

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = i\sigma_1. \tag{3}$$

In this paper, we only discuss the zero component of A_μ, which is nonvanishing and cylindrically symmetric; i.e., we consider the special case

$$A_1 = A_2 = 0, \quad eA_0 = V(r). \tag{4}$$

The wave function is written as

$$\psi(t,\vec{r}) = (2\pi)^{-1/2}\exp[-iEt]r^{-1/2}\begin{pmatrix} f(r)e^{i(j-1/2)\phi} \\ g(r)e^{i(j+1/2)\phi} \end{pmatrix}, \tag{5}$$

where j denotes the total angular momentum, $j = \pm 1/2, \pm 3/2, \ldots$. The radial components f and g become the following set of first-order coupled differential equations

$$\frac{d}{dr}g(r) + \frac{j}{r}g(r) = [E - V(r) - m]f(r), \tag{6}$$

$$-\frac{d}{dr}f(r) + \frac{j}{r}f(r) = [E - V(r) + m]g(r). \tag{7}$$

Recently, the bound-state solutions and scattering amplitudes for the Coulomb potential were derived [7]. It is demonstrated that the boundary conditions for radial wave functions are given by

$$f(0) = g(0) = 0, \tag{8}$$

$$f(\infty) = g(\infty) = 0, \tag{9}$$

which are very useful in following calculations.

III. SPECTRAL COMPARISON THEOREM

Considering two different attractive potentials V_a and V_b, which satisfy the condition $V_a < V_b$, and writing the corresponding pairs of radial wave functions as $\{f_a, g_a\}$ and $\{F_b, G_b\}$, we can obtain following coupled radial equations

$$\frac{d}{dr}g_a(r) + \frac{j}{r}g_a(r) = [E_a - V_a(r) - m]f_a(r), \tag{10}$$

$$-\frac{d}{dr}f_a(r) + \frac{j}{r}f_a(r) = [E_a - V_a(r) + m]g_a(r), \tag{11}$$

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$$\frac{d}{dr}G_b(r) + \frac{j}{r}G_b(r) = [E_b - V_b(r) - m]F_b(r), \quad (12)$$

$$-\frac{d}{dr}F_b(r) + \frac{j}{r}F_b(r) = [E_b - V_b(r) + m]G_b(r). \quad (13)$$

In terms of the prescriptions of F_b in Eq. (10) and g_a in Eq. (13), we can obtain

$$\frac{d}{dr}[g_a(r)F_b(r)] = [-m - V_a(r) + E_a]f_a(r)F_b(r) + [-m + V_b(r) - E_b]g_b(r)F_b(r). \quad (14)$$

On the other hand, according to the prescriptions of G_b in Eq. (11) andf_a in Eq. (12), we can also obtain

$$\frac{d}{dr}[f_a(r)G_b(r)] = [-m - V_b(r) + E_b]f_b(r)F_b(r) + [-m + V_a(r) - E_a]g_a(r)G_b(r). \quad (15)$$

Finally, we can find from Eqs. (14) and (15)

$$[(E_a - E_b) - (V_a - V_b)][f_a(r)F_b(r) + g_a(r)G_b(r)] = -\frac{d}{dr}[f_a(r)G_b(r) - g_a(r)F_b(r)]. \quad (16)$$

Integrating Eq. (16) and using the boundary conditions given by Eqs. (8) and (9), we can obtain

$$\int_0^\infty (E_a - E_b)[f_a(r)F_b(r) + g_a(r)G_b(r)]dr = \int_0^\infty (V_a - V_b)[f_a(r)F_b(r) + g_a(r)G_b(r)]dr. \quad (17)$$

If the wave functions have no nodes, the factors at each side of Eq. (17) have the same sign, and hence, the spectral comparison theorem given by Eq. (1) for the two-dimensional Dirac equation is proved. It should be emphasized that the potentials and eigenvalues are both real. If potential parameters stray into a region such that the corresponding energy spectrum turns complex, then Eq. (17) would no longer lead to the result given by Eq. (1), since the complex levels cannot be well ordered.

IV. APPLICATION TO THE COULOMB PLUS LINEAR POTENTIAL

In order to investigate the application of the above comparison theorem, we need two ordered potentials satisfying the condition

$$V^{(t)}(r) \geqslant V(r). \quad (18)$$

$V(r)$ is chosen as the Coulomb plus linear potential which has been discussed by Mehta and Patil [11] and is given by

$$V(r) = -\frac{Z}{r} + \lambda r, \quad (19)$$

where λ is a positive constant. For the comparison potential $V^{(t)}(r)$ we can generate not only one but a set of "tangential" potentials by using the method of "potential envelopes" [12].

The envelope method requires a soluble base potential taken as the pure hydrogenic potential $-\varsigma/r = \varsigma h(r)$. The two-dimensional Dirac equation with this potential has a discrete spectrum given exactly by [7]

$$D = m\left[1 + \frac{\varsigma^2}{(\sqrt{j^2 - \varsigma^2} + n)^2}\right]^{-1/2}, \quad (20)$$

where $n = 1, 2, \ldots$.

By using the transformation $V(r) = \eta(h(r))$ of the pure Coulomb potential, we can obtain

$$\eta(h(r)) = Zh - \frac{\lambda}{h}. \quad (21)$$

It is straightforward to find out that $\eta'(h) > 0$ and $\eta''(h) < 0$; namely, $\eta(h)$ is monotonously increasing and concave. As a result, the tangent line of $\eta(h)$ is a shifted-Coulomb potential of the form (for detailed discussions see Ref. [13])

$$V^{(t)}(r) = A(t) + B(t)h(r) = [\eta(h(t) - h(t)\eta'(h(t))] + \eta'(h(t))h(r), \quad (22)$$

where $r = t$ is the point of contact with V. The potential inequality given by Eq. (18) is valid. The energy spectrum corresponding to the shifted-Coulomb potential given by Eq. (22) can be obtained exactly and is given with the help of known pure hydrogenic energy spectrum D of Eq. (20) [12]:

$$\varepsilon^{(t)} = A(t) + D(B(t)) \geqslant D. \quad (23)$$

The energy spectrum inequality in Eq. (23) follows the potential inequality given by Eq. (18). This indicates that our energy comparison theorem in the two-dimensional Dirac equation is correct. However, some remarks should be noticed. If the spin wave functions are node free, the above conclusion is valid. For the Coulomb-like potential, the number of nodes near $r = 0$ is the same for f and g only if $j = -1/2$ [14]. Therefore we must restrict our discussion to the energy spectrum at the bottoms of the angular momentum subspaces—namely, to those with $j = -1/2$ and $n = 1$.

V. CONCLUSION

The spectral comparison theorem for the two-dimensional Dirac equation has been established and is seen to be a very useful tool for predicting the spectral spectrum without actually solving the eigenvalue problems. However the theorem is valid for an energy spectrum limited at the bottom of the angular momentum subspaces. This limitation may be overcome by further study.

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