Bell's inequality violation with non-negative Wigner functions

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A Bell inequality violation allowed by the two-mode squeezed state, whose Wigner function is nonnegative, is shown to hold only for correlations among dynamical variables that cannot be interpreted via a local hidden variable theory. Explicit calculations and interpretation are given for Bell's suggestion that the EPR (Einstein, Podolsky, and Rosen) state will not allow violation of Bell's inequality, in conjunction with its Wigner representative being nonnegative. It is argued that Bell's theorem disallowing the violation of Bell's inequality within a local hidden-variable theory depends on the dynamical variables having a definite value—assigned by the local hidden variables—even when they cannot be simultaneously measured. The analysis leads us to conclude that Bell's inequality violation is to be associated with endowing these definite values to the dynamical variables, and *not* with their locality attributes.

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I. INTRODUCTION

In his article entitled "EPR (=Einstein, Podolsky and Rosen) correlations and EPW (=Eugene Paul Wigner) Distributions," Bell [1] studied the possibility of underpinning quantum theory with local hidden variables (LHV's) [2] in the case of two spinless particles. He analyzed the correlations arising from measurements of positions of these particles in free space—a situation closer to the original one envisaged by EPR [3]—utilizing the fact that Wigner's distribution [4] simulates a local "classical" model of such correlations in phase space. Bell suggested [1] that the nonnegativity of the Wigner function for certain quantum-mechanical states would preclude Bell's inequality violation (BIQV) with such states when one considers the correlations constructed from a dichotomic variable defined as the sign of the coordinates of the particles.

We first recall a few properties of the Wigner function [5]. One can show that the expectation value of any operator $\hat{A}$ in a state defined by the density matrix $\hat{\rho}$ can be expressed as

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int d\lambda\, W_{\hat{\rho}}(\lambda) W_{\hat{A}}(\lambda), \tag{1.1}$$

where $W_{\hat{\rho}}(\lambda)$ and $W_{\hat{A}}(\lambda)$ are the Wigner representative of the density matrix $\hat{\rho}$ and the quantal operator $\hat{A}$, respectively, defined in Eqs. (2.13) and (2.12) below, and λ designates the appropriate phase space coordinates, i.e., $\lambda = (\mathbf{q},\mathbf{p}) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$, n being the number of degrees of freedom. It should be noted that in Bell's considerations of local hidden variables, the values of the observables obey the so-called Bell's factorization [2,6], which leaves the value of each observable independent of the "setting" of the other. In the expressions for two-particle correlations in terms of the Wigner representatives, when each of the dynamical variables depends on its own phase-space coordinates, this factorization is satisfied automatically. This is our justification for referring to the description in terms of the Wigner function as *local* [1].

We illustrate the above considerations using the two-mode squeezed state (TMSS) $|\zeta\rangle$, defined as

$$|\zeta\rangle = \exp[\zeta(a_1^\dagger a_2^\dagger - a_1 a_2)]|00\rangle \equiv S(\zeta)|00\rangle. \tag{1.2}$$

In a quantum optics problem, $a_1^\dagger$ and $a_2^\dagger$ represent the creation operators for photons in two different channels and $|00\rangle$ is the vacuum associated with these two channels; a_1 and a_2 denote the corresponding annihilation operators (Ref. [7]). Equation (1.2) defines the operator S.

Alternatively, we can think of a two-one-dimensional-particle problem (i.e., a problem with two degrees of freedom) with "dimensionless" position and momentum operators $\hat{q}_1$, $\hat{p}_1$ and $\hat{q}_2$, $\hat{p}_2$, respectively, through which one defines, as usual, the raising and lowering operators

$$a_\alpha^\dagger = \frac{1}{\sqrt{2}}(\hat{q}_\alpha - i\hat{p}_\alpha), \tag{1.3a}$$

$$a_\alpha = \frac{1}{\sqrt{2}}(\hat{q}_\alpha + i\hat{p}_\alpha) \tag{1.3b}$$

(where $\alpha=1,2$). The ground state $|00\rangle$ is, as usual, annihilated by the operator a_α.

In the limit of the squeezing parameter ζ increasing without limit, the state (1.2) approaches the EPR state [3] $|\mathrm{EPR}\rangle = \delta(q_1 - q_2)$, as can be readily seen writing the state (1.2) in the coordinate representation as (we use well known normal ordering formula [8])

$$\langle q_1 q_2|\zeta\rangle = \frac{1}{\cosh\zeta}\sum_{n=0} \tanh^n \zeta \langle q_1 q_2|nn\rangle \underset{\zeta\to\infty}{\rightarrow} \sim \delta(q_1 - q_2). \tag{1.4}$$

Now, the Wigner function, W_ζ, of the TMSS is given by [9]

$$W_\zeta(q_1,q_2,p_1,p_2) = \frac{1}{\pi^2} \exp[-\cosh(2\zeta)(q_1^2+q_2^2+p_1^2+p_2^2) - 2\sinh(2\zeta)(q_1q_2 - p_1p_2)]. \quad (1.5)$$

It is clearly *non-negative* for all q's and p's, and thus may be considered as a distribution in phase space (q_1,q_2,p_1,p_2) associated with the state $|\zeta\rangle$. Thus we may refer to the variables (q_1,q_2,p_1,p_2) as local hidden variables, and correlations weighed with $W_\zeta(q_1,q_2,p_1,p_2)$ should preclude BIQV for dynamical variables for which this may be a legitimate view [10].

As was mentioned above, Bell suggested [1] that the non-negativity of the Wigner function of the EPR state would preclude BIQV with this state when one considers the correlations of a dichotomic variable defined as the sign of the coordinates of the particles. The correlations considered in that work are those that are involved in the CHSH [11] inequality, i.e., the inequality that is often studied in terms of the Bell operator [12]. (In the present paper, Bell's inequality and BIQV refer to this CHSH inequality.) Bell's original argument that nonnegativity of Wigner's function suffices to preclude BIQV was shown [13] to be inaccurate. Difficulties in handling the normalization of the EPR state considered by Bell were shown to involve a misleading factor.

The TMSS's were studied extensively since the early 1980s in connection with BIQV in general and, in particular, for their connection to the EPR state [14–20]. These studies focused on the polarization as the observable (= dynamical variable). Banaszek and Wodkiewicz [9] noted that *while the Wigner function of the TMSS is non-negative, it allows for BIQV*, when the dynamical variable involved in the correlations is the parity. Their study was extended by Chen *et al.* [21] who showed, by using appropriately defined spinlike variables [which, together with the parity operator, close an SU(2) algebra], that the TMSS, $|\zeta\rangle$, allows the maximal possible [22,23] BIQV for $\zeta \to \infty$, i.e., when it is maximally entangled [24] and, as stated above, it tends to the EPR state. An alternative parametrization (termed configurational) for spinlike operators was given in [25]. This choice of dynamical variables is more convenient for our analysis as it involves the dynamical variables considered by Bell and admits a simple interpretation.

Our study aims at clarifying the relation between the non-negative Wigner function of the TMSS, $|\zeta\rangle$, for all values of ζ, the dynamical variables involved in the CHSH inequality [11,12] and the possibility of BIQV. The latter, by Bell's theorem [2,6], prohibits the underpinning of the theory with a local hidden-variables theory. Note that this attribute (non-negativity) of the Wigner function depends on the variables over which it is defined [26].

The paper is organized as follows. In the next section we describe the properties that should be required of a quantum mechanical problem in order that its translation in terms of Wigner representatives can be legitimately considered as a local hidden-variables theory. We then divide the problem indicated in the last paragraph into three levels. The first level, which the works hitherto addressed, is to consider BIQV with the TMSS, viz., with a state having non-negative Wigner function. In this connection we give in Sec. III a brief review of Chen *et al.* [21] considerations and those of Ref. [25]. We argue that the former approach [21] involves, exclusively, dynamical variables whose Wigner representatives are physically unsuitable for allowing a local hidden-variables theory underpinning (in addition, they do not fulfil the property of boundedness, a mathematical condition that enters the derivation of Bell's inequality). Such dynamical variables that are ineligible for a local hidden-variables theory in phase space (the domain of Wigner's function [26]) are termed *improper* or *dispersive* dynamical variables; the definition of these terms and their justification is included in Sec. II. We then consider the next level of the problem, viz., where in addition to having the non-negative Wigner function of $|\zeta\rangle$, we have a dynamical variable that is proper (or nondispersive), i.e., one that can be accounted for by the local hidden variables that the phase space provides (indeed it is the very one considered by Bell [1]: the sign of the coordinate of the particle). However, we show that its mates, i.e., its rotated (we use here the spin analogy) partner(s) which, with it, must be present in the Bell operator [12], are dispersive (they are also not bounded) and hence, again, no local hidden-variables theory can be sustained here. We also discuss the alternative approach of retaining the original dynamical variables and rotating the wave function and show that in this case it leads to a *non* non-negative Wigner function. In Sec. IV we finally study the last level which is the one considered by Bell. In addition to having the non-negative Wigner function and the proper dynamical variable, its "rotated" mates are now obtained by time evolution with a "free" Hamiltonian. For this case we show that the evolved dynamical variable remains non-dispersive, or alternatively (perhaps less surprising), the "rotated" wave function continues to give rise to a non-negative Wigner function. We thus arrive at the conclusion that Bell's expectation [1] that the EPR state will not allow BIQV is confirmed. However, our approach underscores the importance of the perhaps not sufficiently stressed assumption involved in the derivation of Bell's inequalities [2,11], viz., that the local hidden-variables theory be such that the dynamical variables are defined simultaneously, even when they cannot be measured simultaneously. This point was noted before [27–32]. Indeed, such a requirement is tantamount to having the local hidden variables endowing physical reality (in the EPR sense [3]) to the dynamical variables measurable attributes.

To remain close to the formalism as discussed by Bell [1], we shall throughout refer to changes in the dynamical variables as "evolution." This retains complete generality, since to define the evolution we can choose a Hamiltonian leading to the required change.

II. HIDDEN VARIABLES AND WIGNER'S FUNCTIONS

We consider bounded quantum-mechanical operators $\hat{A}$ associated with dynamical variables for a given physical system, with eigenvalues a_n. By a proper rescaling, we can always have

$$|a_n| \leq 1. \quad (2.1)$$

In a hidden-variables theory we assume that we have variables λ endowed with a probability distribution

$$\rho(\lambda) \geqslant 0, \tag{2.2}$$

such that to every operator $\hat{A}$ we associate, according to some recipe, a function $A(\lambda)$—a "representative" of the dynamical variable in terms of the hidden variable λ—that takes on, as its possible values, the eigenvalues a_n. When this is feasible, we say that we are dealing with a "proper" dynamical variable. Then, if $A(\lambda)$ is the representative of the operator $\hat{A}$, $A^k(\lambda)$ should be the representative of the operator $\hat{A}^k$, where k is an integer. We then speak of a "nondispersive" dynamical variable. As a consequence, the $A(\lambda)$'s are bounded as

$$|A(\lambda)| \leqslant 1. \tag{2.3}$$

In a two-particle problem, if the dynamical variable $\hat{A}$ is associated with particle 1 and $\hat{B}$ with particle 2, the requirement that $A(\lambda)$ be independent of the setting **b** of the instrument that measures particle 2 and $B(\lambda)$ be independent of the setting **a** of the instrument that measures particle 1 makes the theory local [2]. For this two-particle problem we now introduce two other dynamical variables, $\hat{A}'$ and $\hat{B}'$, associated with particles 1 and 2, respectively, and not commuting, in general, with $\hat{A}$ and $\hat{B}$, respectively. To these new dynamical variables we associate the functions $A'(\lambda)$ and $B'(\lambda)$, respectively. Notice that the functions $A(\lambda)$ and $A'(\lambda)$ for particle 1 [and similarly $B'(\lambda)$ and $B'(\lambda)$ for particle 2] *assign a definite value to the two dynamical variables, whether they can be measured simultaneously or not*. Then one can prove the CHSH inequality

$$|\langle \mathcal{B}(\lambda) \rangle| \equiv \left| \int \mathcal{B}(\lambda)\rho(\lambda)d\lambda \right| \leqslant 2, \tag{2.4}$$

where $\mathcal{B}$ is given by

$$\mathcal{B} = A(\lambda)B(\lambda) + A(\lambda)B'(\lambda) + A'(\lambda)B(\lambda) - A'(\lambda)B'(\lambda). \tag{2.5}$$

We designate the above inequality BIQ. In other words, dealing with proper dynamical variables (PDV) implies Eq. (2.3) which, in turn, implies BIQ:

$$(\text{PDV}) \Rightarrow (2.3) \Rightarrow (\text{BIQ}), \tag{2.6}$$

so that

$$(\text{PDV}) \Rightarrow (\text{BIQ}). \tag{2.7}$$

Conversely, in a hidden-variables model in which Eq. (2.2) is fulfilled, a violation of BIQ (to be called BIQV) implies that Eq. (2.3) is not fulfilled, and hence that we are not dealing with PDV's, i.e.,

$$(\text{BIQV}) \Rightarrow \overline{(2.3)} \Rightarrow (\overline{\text{PDV}}), \tag{2.8}$$

so that

$$(\text{BIQV}) \Rightarrow (\overline{\text{PDV}}). \tag{2.9}$$

(The bar on a proposition indicates its negation.) We mentioned these conditions with some care because of the various applications that we shall be concerned with in the following sections.

Let us mention that when we deal with dichotomic variables, i.e., with operators having only two eigenvalues (± 1), one can prove that the QM expectation value for any two-particle state $|\Psi\rangle$ of the Bell operator [12]

$$\mathcal{B} = \hat{A}\hat{B} + \hat{A}\hat{B}' + \hat{A}'\hat{B} - \hat{A}'\hat{B}' \tag{2.10}$$

satisfies the Cirel'son inequality [22]

$$|\langle \Psi | \hat{\mathcal{B}} | \Psi \rangle| \leqslant 2\sqrt{2}. \tag{2.11}$$

We now discuss a specific way of implementing the above local hidden-variables program in terms of the theory of Wigner's functions. We define the Wigner representative $W_{\hat{Q}}(q,p)$ of the quantal operator $\hat{Q}$ (for one degree of freedom) as [33]

$$W_{\hat{Q}}(q,p) = \int e^{-ip\cdot y} \left\langle q + \frac{1}{2}y \right| \hat{Q} \left| q - \frac{1}{2}y \right\rangle dy, \tag{2.12}$$

while the Wigner function for the density operator is defined with an extra factor of $1/2\pi$ for each degree of freedom, i.e., for one degree of freedom:

$$W_{\hat{\rho}}(q,p) = \frac{1}{2\pi} \int e^{-ip\cdot y} \left\langle q + \frac{1}{2}y \right| \hat{\rho} \left| q - \frac{1}{2}y \right\rangle dy. \tag{2.13}$$

Then one can prove that the expectation value of an operator $\hat{A}$ with the density matrix $\hat{\rho}$ is [33]

$$\text{Tr}(\hat{\rho}\hat{A}) = \int W_{\hat{\rho}}(q,p) W_{\hat{A}}(q,p) dq dp. \tag{2.14}$$

One can easily see that $W_{\hat{Q}}(q,p)$ of Eq. (2.12) can also be expressed as

$$W_{\hat{Q}}(q,p) = \text{Tr}[\hat{Q}\hat{\Omega}(q,p)], \tag{2.15a}$$

$$\hat{\Omega}(q,p) = \int \left| q - \frac{1}{2}y \right\rangle e^{-ip\cdot y} \left\langle q + \frac{1}{2}y \right| dy, \tag{2.15b}$$

an expression that will be useful later.

It can be shown [34] that the only wave function whose Wigner representative is non-negative is a Gaussian: in this case, the associated Wigner function is apparently interpretable as a probability density in phase space [see Eq. (2.2)]. The TMSS of Eq. (1.2) is an example where this interpretation is indeed feasible. If, in addition, the Wigner representatives of the dynamical variables under study are of the proper, or nondispersive, nature required above, we have a candidate for a local hidden-variables theory, where the local hidden variables are represented by the canonical variables q and p. It seems clear from the outset that it will be rather exceptional for a dynamical variable to fall into this category. It is the purpose of the discussion that follows in the present section to identify a class of operators $\hat{A}$ that do correspond to proper dynamical variables. Although the analysis is certainly not exhaustive, it serves the purpose of indicating a number of sufficient conditions leading to proper dynamical variables. For simplicity, the analysis will be restricted to systems with only one degree of freedom.

Consider a function $f(x)$, where $-\infty \leqslant x \leqslant \infty$, bounded as $|f(x)| \leqslant 1$.

(1) We define the operator $\hat{A}_1 = f(\hat{q})$ through its spectral representation as

$$\hat{A}_1 = f(\hat{q}) = \int_{-\infty}^{\infty} |q'\rangle f(q')\langle q'|dq'. \qquad (2.16)$$

The eigenvalues of this operator are $f(x)$, so that its spectrum lies in the interval $[-1,1]$. For instance, (a) $f(x) = \tanh x$ gives a continuous spectrum in the interval $[-1,1]$; (b) $f(x) = \operatorname{sgn} x$ (where the sgn function takes on the value 1 for $x > 0$ and -1 for $x < 0$) has a discrete spectrum, consisting of the two values 1 and -1.

One can easily show that the Wigner representative of the operator $f(\hat{q})$ of Eq. (2.16) is

$$W_{f(\hat{q})}(q',p') = f(q'), \qquad (2.17)$$

a function which takes on, as its values, precisely the eigenvalues of the operator $f(\hat{q})$. According to our nomenclature, we are thus dealing with a proper dynamical variable. In these examples we see the nondispersive property explicitly, since

$$W_{[f(\hat{q})]^k}(q',p') = [W_{f(\hat{q})}(q',p')]^k. \qquad (2.18)$$

(2) Similar considerations apply to the operator $\hat{A}_2 = f(\hat{p})$.

(3) Another case, which is very relevant for our future considerations, is that of the operator

$$\hat{A}_3 = f(\hat{\bar{q}}), \qquad (2.19)$$

where

$$\hat{\bar{q}} = a\hat{q} + b\hat{p}, \qquad (2.20)$$

(a and b being numerical constants) is a linear combination of the position and momentum operators $\hat{q}$ and $\hat{p}$. If we add, to Eq. (2.20), the following one:

$$\hat{\bar{p}} = c\hat{q} + d\hat{p}, \qquad (2.21)$$

c and d being numerical constants satisfying the condition

$$ad - bc = 1, \qquad (2.22)$$

then the pair of equations (2.20) and (2.21) can be considered as a transformation from the canonical position and momentum operators $\hat{q}$ and $\hat{p}$ to the new ones $\hat{\bar{q}}$ and $\hat{\bar{p}}$. Thanks to the condition (2.22), the commutator $[\hat{q},\hat{p}] = [\hat{\bar{q}},\hat{\bar{p}}] = i$ is preserved and the transformation is canonical: it is the quantum-mechanical counterpart [35] of the classical linear canonical transformation obtained from Eqs. (2.20) and (2.21) by removing the "hats" and considering the q, p, $\bar{q}$ and $\bar{p}$ as c-number canonical variables; in the classical problem it is the Poisson bracket that is preserved by the transformation.

The operators $\hat{\bar{q}}, \hat{q}$ have the same spectrum, and so do the operators $\hat{\bar{p}}, \hat{p}$; we can thus relate the two members of each pair through the unitary transformation

$$\hat{\bar{q}} = V^{\dagger}\hat{q}V, \qquad (2.23a)$$

$$\hat{\bar{p}} = V^{\dagger}\hat{p}V. \qquad (2.23b)$$

The eigenstates of $\hat{\bar{q}}$ and $\hat{\bar{p}}$, to be designated by $|q')$ and $|p')$, respectively, i.e.,

$$\hat{\bar{q}}|q') = q'|q'), \qquad (2.24a)$$

$$\hat{\bar{p}}|p') = p'|p'), \qquad (2.24b)$$

are related to the eigenstates $|q'\rangle$, $|p'\rangle$ of $\hat{q}$ and $\hat{p}$, respectively, as

$$|q') = V^{\dagger}|q'\rangle, \qquad (2.25a)$$

$$|p') = V^{\dagger}|p'\rangle. \qquad (2.25b)$$

In terms of the eigenstates $|q')$ of $\hat{\bar{q}}$, Eq. (2.24a), we can write the spectral representation of the operator $\hat{A}_3$ of Eq. (2.19) as

$$\hat{A}_3 = f(\hat{\bar{q}}) = \int_{-\infty}^{\infty} |q')f(q')(q'|dq'. \qquad (2.26)$$

Using Eqs. (2.25a) and (2.16), we can write further

$$\hat{A}_3 = f(\hat{\bar{q}}) = V^{\dagger}\int_{-\infty}^{\infty} |q'\rangle f(q')\langle q'|Vdq' \qquad (2.27a)$$

$$= V^{\dagger}f(\hat{q})V. \qquad (2.27b)$$

From Eq. (2.26) we read off the eigenvalues of the operator $\hat{A}_3 = f(\hat{\bar{q}})$ as $f(x)$, just as for $\hat{A}_1 = f(\hat{q})$: in point of fact, a unitary transformation [Eq. (2.27b)] does not change the spectrum.

The next step is analyze the properties of the Wigner representative of the operator $\hat{A}_3 = f(\hat{\bar{q}})$. We first make a more general statement: from Eqs. (2.15) one can show that the Wigner representatives of two operators $\hat{A}$ and $V^{\dagger}\hat{A}V$, V being the unitary operator discussed above, are related by

$$W_{V^{\dagger}\hat{A}V}(q',p') = W_{\hat{A}}(aq' + bp', cq' + dp'). \qquad (2.28)$$

In other words, if the operator $\hat{A}$ undergoes the unitary transformation $\hat{A} \Rightarrow V^{\dagger}\hat{A}V$, the Wigner representative is affected precisely by the classical linear canonical transformation of which Eqs. (2.20) and (2.21) are the quantum-mechanical counterpart. Now, if we apply this result to the operators $f(\hat{q})$ and $V^{\dagger}f(\hat{q})V = f(a\hat{q} + b\hat{p})$ of Eq. (2.27b), we find

$$W_{f(a\hat{q}+b\hat{p})}(q',p') = W_{f(\hat{q})}(aq' + bp', cq' + dp'), \qquad (2.29)$$

and, using Eq. (2.17) for the right-hand side, we finally obtain

$$W_{f(a\hat{q}+b\hat{p})}(q',p') = f(aq' + bp'), \tag{2.30}$$

which clearly reduces to Eq. (2.17) when $a=1$ and $b=0$.

Right after Eq. (2.27b) we identified the spectrum of $f(a\hat{q}+b\hat{p})$ as $f(x)$. Now, Eq. (2.30) tells us that the Wigner representative of this operator takes on, as its values, exactly the eigenvalues of the quantum-mechanical operator: we are thus dealing with a proper dynamical variable. As a result, *we have found a class of observables*, i.e., $f(a\hat{q}+b\hat{p})$ which, together with their Wigner representative, i.e., $f(aq'+bp')$, *are termed proper dynamical variables*.

As an application, suppose that we have a two-particle problem, with the Wigner function associated with the wave function being non-negative. Suppose also that we choose, as the operators $\hat{A}$, $\hat{A}'$ to be associated with particle 1, any two (in general noncommuting) of the proper ($\Rightarrow$ nondispersive [36]) dynamical variables discussed above, like $\hat{A}_1$, $\hat{A}_2$, or $\hat{A}_3$, and similarly for the operators $\hat{B}$, $\hat{B}'$ to be associated with particle 2. Then the CHSH inequality (2.4) must be fulfilled, according to the discussion given right before that equation. In the presentation carried out in Sec. IV below, $\hat{A}$ is taken as $\text{sgn}(\hat{q})$, i.e., as $\hat{A}_1$ above, Eq. (2.16), case (b); $\hat{A}'$ is taken as $\hat{A}_3$ above, Eq. (2.19), again with $f(x)=\text{sgn}(x)$, for two options for the coefficients a and b. Similar choices are made for $\hat{B}$ and $\hat{B}'$. For these cases, the validity of the CHSH inequality (2.4) is verified explicitly in Sec. IV.

In contrast, it is easy to give examples of dynamical variables that do not fulfill the above property of having a Wigner function taking, as its values, the eigenvalues of the quantum operator. For instance, for the observable

$$\hat{A} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2), \tag{2.31}$$

the quantum-mechanical spectrum is $n+1/2$ ($n=0,1,2,\ldots$). [This spectrum is not bounded in the sense of Eq. (2.1); it just serves as an example to illustrate the point.] In contrast, its Wigner representative is

$$W_{(1/2)(\hat{p}^2+\hat{q}^2)}(q',p') = \frac{1}{2}[(p')^2 + (q')^2], \tag{2.32}$$

which takes on *any* value in $[0,\infty]$: the dynamical variable (2.31), together with its Wigner representative (2.32), is thus improper. Some of the observables considered in Sec. III below will, indeed, fail to be proper.

As one further application of Eq. (2.28), consider the variation of $\text{Tr}(\hat{\rho}\hat{A})$, Eq. (2.14), when the operator $\hat{A}$ is subjected to the unitary transformation $\hat{A}\Rightarrow V^{\dagger}\hat{A}V$; obviously, the same answer is obtained if, instead, $\hat{\rho}$ is transformed as $\hat{\rho}\Rightarrow V\hat{\rho}V^{\dagger}$: transforming the operator will be called the Heisenberg picture, whereas transforming the state will be designated as the Schrödinger picture. We can calculate the change of the Wigner representative of $\hat{\rho}$ from Eq. (2.28), valid for any Hermitean operator, replacing $\hat{A}$ by $\hat{\rho}$ and V by its inverse, with the result

$$W_{V\hat{\rho}V^{\dagger}}(q',p') = W_{\hat{\rho}}(dq' - bp', -cq' + ap'), \tag{2.33}$$

which will be useful later.

The relevance of the Heisenberg vs the Schrödinger picture in the present context lies in the fact that we can calculate the four terms occurring in the expectation value of the Bell operator (2.10) appearing in Eq. (2.11) in either of the two pictures. In fact we can write

$$\langle\psi|\hat{A}\hat{B}|\psi\rangle = \int W_{\psi}W_A W_B dpdq, \tag{2.34a}$$

$$\langle\psi|\hat{A}\hat{B}'|\psi\rangle = \int W_{\psi}W_A W_{B'} dpdq = \langle\psi_2'|\hat{A}\hat{B}|\psi_2'\rangle = \int W_{\psi_2'}W_A W_B dpdq, \tag{2.34b}$$

$$\langle\psi|\hat{A}'\hat{B}|\psi\rangle = \int W_{\psi}W_{A'} W_B dpdq = \langle\psi_1'|\hat{A}\hat{B}|\psi_1'\rangle = \int W_{\psi_1'}W_A W_B dpdq, \tag{2.34c}$$

$$\langle\psi|\hat{A}'\hat{B}'|\psi\rangle = \int W_{\psi}W_{A'} W_{B'} dpdq = \langle\psi_{12}'|\hat{A}\hat{B}|\psi_{12}'\rangle = \int W_{\psi_{12}'}W_A W_B dpdq, \tag{2.34d}$$

where $dpdq=dq_1dq_2dp_1dp_2$. If the operator $\hat{A}'(1)$ associated with particle 1 is obtained from $\hat{A}(1)$ via the unitary transformation

$$\hat{A}'(1) = V^{\dagger}(1)\hat{A}(1)V(1), \tag{2.35}$$

then $|\psi_1'\rangle$ in Eq. (2.34c) denotes

$$|\psi_1'\rangle = V(1)|\psi\rangle, \tag{2.36}$$

with a similar notation for the other states. An important issue that will arise naturally in the following sections is precisely that of the properties of the Wigner function associated with the transformed operators and states, and the relevance of those properties for the fulfillment of CHSH inequality. The interplay between the Heisenberg and Schrödinger pictures that we just outlined will thus be very relevant in what follows.

After having given a panoramic view of the hidden-variables problem and Wigner's function we now turn to a study of the three levels outlined in the Introduction.

III. THE EPR-EPW PROBLEM

As outlined in the Introduction, we consider the so-called EPR-EPW problem [1,13] in successive levels. The first level is as follows: Given a state, $|\zeta\rangle$ in our case, whose Wigner representative function is non-negative, does such a state allow BIQV?

The answer to this was shown [9,21] to be in the affirmative. The dynamical variable considered was the parity, S_z ($\hat{N}$ being the number operator),

$$S_z \equiv \sum_{n=0}^{\infty} [|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|] = -(-1)^{\hat{N}}. \quad (3.1)$$

In Ref. [21], "rotated" parity operators were introduced:

$$S_x = \sum_{n=0}^{\infty} [|2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1|], \quad (3.2)$$

$$S_y = i\sum_{n=0}^{\infty} [|2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|]. \quad (3.3)$$

These operators close an $su(2)$ algebra and are viewed as components of a 3-dimensional vector operator. We may thus consider a "rotation" in parity space by, e.g.,

$$S'_x(\vartheta) = e^{(i\vartheta/2)S_z} S_x e^{(-i\vartheta/2)S_z} = S_x \cos\vartheta - S_y \sin\vartheta = \mathbf{S}\cdot\mathbf{n}, \quad (3.4)$$

with $\mathbf{n}$ a unit vector which, in this case, is in the "$x-y$" plane of the parity space. It will be convenient for us later to refer to the above as the "time evolution" of S_x under the "Hamiltonian" S_z in Eq. (3.4): in this way we refer to the "rotation" angle, ϑ, as the time, t. Sticking to the geometric notation, the Bell operator [12] of Eq. (2.10) is

$$\mathcal{B} = \mathbf{S}^1\cdot\mathbf{n}\mathbf{S}^2\cdot\mathbf{m} + \mathbf{S}^1\cdot\mathbf{n}'\mathbf{S}^2\cdot\mathbf{m} + \mathbf{S}^1\cdot\mathbf{n}\mathbf{S}^2\cdot\mathbf{m}' - \mathbf{S}^1\cdot\mathbf{n}'\mathbf{S}^2\cdot\mathbf{m}', \quad (3.5)$$

and the Bell inequality we study is

$$|\langle\mathcal{B}\rangle| \leqslant 2. \quad (3.6)$$

Varying $\mathbf{n},\mathbf{n}'$ and $\mathbf{m},\mathbf{m}'$ to maximize $|\langle\hat{\mathcal{B}}\rangle|$ for the state $|\zeta\rangle$ we get [25]

$$|\langle\zeta|\mathcal{B}|\zeta\rangle| = 2\sqrt{1+F^2(\zeta)}, \quad (3.7)$$

$$F(\zeta) = \langle\zeta|S_x^1 S_x^2|\zeta\rangle = \tanh 2\zeta. \quad (3.8)$$

Thus the state $|\zeta\rangle$ allows BIQV, even though the Wigner function of the corresponding density operator may be viewed as a probability density of local hidden variables (the phase space coordinates). However, as was stressed in the Introduction, this does not violate Bell's theorem which prohibits BIQV for a local hidden-variables theory. Thus the correlations appearing in the Bell operator have the structure [5]

$$\langle\zeta|S_z^1 S_z^2|\zeta\rangle = \int_{-\infty}^{\infty} dp_1 dq_1 dp_2 dq_2 W_\zeta(p_1,q_1,p_2,q_2) \times W_{S_z^1}(p_1,q_1) W_{S_z^2}(p_2,q_2). \quad (3.9)$$

Here, the factorization of the Wigner function of the two channels is automatic. As explained in detail in Sec. II, for the right-hand side of Eq. (3.9) to be interpretable as a local hidden-variables theory, aside from a nonnegative Wigner function for the state, W_ζ, we require that the Wigner representatives of the dynamical variables, the S_z's in this case, give the observable values of these dynamical variables, viz., the eigenvalues of the quantal parity operator (for the phase point: q,p). As already indicated, we refer to a dynamical variable with this property as a *proper* or *non*dispersive dynamical variable [36]. This is not the case for any of the parity operators, S_i ($i=x,y,z$); in fact, e.g., we can easily verify that

$$W_{S_z}(q,p) = -\pi\delta(\alpha) = -\pi\delta(q)\delta(p), \quad \alpha = q+ip. \quad (3.10)$$

This clearly is not an eigenvalue of the parity operator (which is ± 1). Thus in this case this dynamical variable is *im*proper or dispersive [36]. Therefore, we are not dealing here with a local hidden-variables theory. [In addition, Eq. (3.10) makes clear the assertion made in the Introduction that the Wigner representative of $\hat{S}_z$ violates the property of boundedness.]

We have thus completed the discussion of the first level of the EPR-EPW problem: nothing new was gained but we considered examples that will serve us below.

The second level of the EPR-EPW problem is when, in addition to having a nonnegative Wigner function for the state, *we have a dynamical variable whose Wigner representative is the eigenvalue of the dynamical variable*—i.e., it is a *proper* or nondispersive dynamical variable. Would this situation allow BIQV? Would it conform to Bell's theorem? Recently [10,25] an alternative configuration was discussed for the parity operators. In this alternative configuration the operators are given in the q representation. Denoting the operators in this configuration by Π_i ($i=x,y,z$), we have

$$\Pi_z \equiv -\int_0^{\infty} dq[|\mathcal{E}\rangle\langle\mathcal{E}| - |O\rangle\langle O|] = S_z. \quad (3.11)$$

Here

$$|\mathcal{E}\rangle = \frac{1}{\sqrt{2}}[|q\rangle + |-q\rangle], \quad |O\rangle = \frac{1}{\sqrt{2}}[|q\rangle - |-q\rangle], \quad (3.12)$$

so that

$$\Pi_z = -\int_{-\infty}^{\infty} dq[|q\rangle\langle -q|]. \quad (3.13)$$

The equality $\langle n|\Pi_z|n'\rangle = \langle n|S_z|n'\rangle$ is easily verifiable. The natural vectorial operators that close an $su(2)$ algebra with Π_z are

$$\Pi_x = \int_0^{\infty} dq[|\mathcal{E}\rangle\langle O| + |O\rangle\langle\mathcal{E}|], \quad (3.14)$$

$$\Pi_y = i\int_0^\infty dq[|\mathcal{E}\rangle\langle O| - |O\rangle\langle\mathcal{E}|]. \tag{3.15}$$

We note that Π_x is diagonal in q, i.e.,

$$\Pi_x = \int_0^\infty dq[|q\rangle\langle q| - |-q\rangle\langle -q|] = \text{sgn}(\hat{q}) \tag{3.16}$$

is the spectral representation of Π_x. Its Wigner function is

$$W_{\Pi_x}(q,p) = \text{sgn}\, q, \tag{3.17}$$

i.e., it gives the eigenvalues (± 1) of the operator and hence is a *proper* (nondispersive) dynamical variable, just as in the discussion of Eq. (2.16), case (b), of Sec. II. In this case, with Π_i, much like in the previous case (with the S_i, $i=x,y,z$) it is easy to get BIQV by selecting the appropriate orientational parameters. For convenience, while retaining complete generality, we consider the choice of the orientational parameters by choosing the times (for both channels) of the evolution of $\Pi_x^1(t_1)$, $\Pi_x^2(t_2)$ under the Hamiltonian $H=\Pi_z$. (We note that Bell [1] considered the same case with $\zeta\to\infty$, i.e., the EPR state, but with the free Hamiltonian, $H=p^2/2$.)

Direct calculations show that by appropriate choices of the times (t_1, t_1' and t_2, t_2') we obtain

$$\langle\hat{\mathcal{B}}\rangle = 2\sqrt{2\overline{F}}(2\zeta), \quad \overline{F} = \frac{2}{\pi}\arctan(\sinh 2\zeta). \tag{3.18}$$

Thus we see that in this case, where seemingly the quantal description may be given a local hidden-variables underpinning, we get a BIQV which, we are told, is an impossibility. However, the present Bell operator involves not only the "proper" dynamical variable, Π_x, but also Π_y which evolves via our Hamiltonian, $H=\Pi_z$. The latter, i.e., Π_y, is *not* a proper dynamical variable. In fact, its Wigner representative is given by

$$W_{\Pi_y}(q,p) = -\delta(q)\text{P}\frac{1}{p}, \tag{3.19}$$

where P stands for the "principal value." Thus, once again, no local hidden-variables underpinning for the correlation involved in Eq. (3.18) is possible, after all. (The boundedness condition for the Wigner representatives is violated as well.)

We may attempt to consider the problem in a "Schrödinger-like manner" by applying the time evolution operator to the state $|\zeta\rangle$; this, however, leads to a new state, $|\zeta'\rangle$, whose Wigner function is no longer non-negative over all phase space. This can be proven most readily by considering an alternative expression for the state $|\zeta\rangle$ obtained in [25], i.e.,

$$|\zeta\rangle = \int_o^\infty\int_0^\infty dqdq'[(g_+ + g_-)|\mathcal{E}\mathcal{E}'\rangle + (g_+ - g_-)|OO'\rangle], \tag{3.20}$$

where

$$g_\pm(q,q,;\zeta) = \langle qq'|S(\pm\zeta)|00\rangle = \frac{1}{\sqrt{\pi}}\exp\left\{-\frac{1}{2}[q^2 + q'^2 \mp 2qq'\tanh(2\zeta)] \times\cosh(2\zeta)\right\}. \tag{3.21}$$

Using this expression for $|\zeta\rangle$, we have directly

$$e^{-i\gamma\Pi_z}|\zeta\rangle = |\zeta'\rangle = \cos\gamma|\zeta\rangle + \sin\gamma|-\zeta\rangle, \tag{3.22}$$

and the Wigner function of $|\zeta'\rangle$ is no longer non-negative [34].

IV. BILINEAR HAMILTONIANS

Level 3 of our EPR-EPW problem is closer to Bell's original program [1]; it involves the study of cases wherein (1) the states have non-negative Wigner representatives which, at some limit, reduce to the EPR state—our $|\zeta\rangle$ is such a state. (2) There is a dynamical variable (=observable) that is nondispersive (=proper), i.e., such that the Wigner representative of its quantal version gives its eigenvalues in terms of our local hidden variables: p, q—our Π_x is such a dynamical variable. We inquire for possible BIQV when this dynamical variable evolves via Hamiltonians which leave the Wigner representative of the state under study non-negative. Alternatively, we inquire for BIQV when our dynamical variables evolve by Hamiltonians which allow the initially proper dynamical variable to remain so. In the next paragraphs we study the relationship between these two alternatives.

The only non-negative Wigner functions are gaussians [34]. Since Gaussians remain gaussians under linear transformations, it follows that single-particle Hamiltonians that leave the Wigner function non-negative are bilinear ones. We will consider now two such Hamiltonians:

$$H_0(i) = \frac{1}{2}(\hat{p}_i^2 + \omega_i^2\hat{q}_i^2), \tag{4.1a}$$

$$H_f(i) = \frac{1}{2}\hat{p}_i^2, \tag{4.1b}$$

where the subscript $i=1,2$ denotes the particle. For simplicity we shall consider, in H_0, the frequency $\omega_i=1$ for both particles. The second Hamiltonian is the one considered by Bell [1,13].

We consider the harmonic oscillator Hamiltonian H_0 first. Evolution of the state $|\zeta\rangle$, Eq. (1.2), under H_0, during a time t_1 for particle 1 and t_2 for particle 2, gives

$$|\zeta(t_1,t_2)\rangle = |\zeta'\rangle = \exp^{-\zeta(a_1^\dagger a_2^\dagger e^{-i\theta} - a_1a_2e^{i\theta})}|00\rangle, \tag{4.2}$$

where $\theta=t_1+t_2$. The corresponding Wigner function can be obtained either directly from the state (4.2), or from Eq. (1.5), applying Eq. (2.33) with $a=\cos t_i$, $b=\sin t_i$, $c=-\sin t_i$ and $d=\cos t_i$, with the result

$$W_{\zeta(\theta)} = \frac{1}{\pi^2} \exp\{- \cosh(2\zeta)(q_1^2 + q_2^2 + p_1^2 + p_2^2) - 2 \sinh(2\zeta) \times[(q_1q_2 - p_1p_2)\cos \theta - (q_1p_2 + q_2p_1)\sin \theta]\}. \quad (4.3)$$

Direct evaluation of

$$E(t_1,t_2) = \int_{-\infty}^{\infty} dq_1dq_2dp_1dp_2W_{\zeta(\theta)}(q_1,q_2,p_1,p_2)\Pi_x^1\Pi_x^2 \quad (4.4)$$

[here, $\Pi_x^i = \mathrm{sgn}(\hat{q}^i)$, Eq. (3.16)] gives (see Appendix A)

$$E(t_1,t_2) = 1 - \frac{2}{\pi}\chi, \quad \cos \chi = \tanh(2\zeta)\cos \theta. \quad (4.5)$$

We have used the notation of Ref. [1]

$$E(t_1,t_2) = P_{++}(\theta) + P_{--}(\theta) - P_{-+}(\theta) - P_{+-}(\theta). \quad (4.6)$$

The first subscript in P_{++}, P_{+-}, etc., refers to the eigenvalue (i.e., ±1) of Π_x^1 for the first particle and the second subscript refers to that of the second particle Π_x^2: i.e., P_{++} is the integral of $W_{\zeta(\theta)}(q_1,q_2,p_1,p_2)$ [see Eq. (4.4)] over the region $q_1 > 0$, $q_2 > 0$, etc. The alternative view, i.e., allowing Π_x^i to evolve in time, while keeping W_ζ fixed, is readily done (see Appendix B) by noting that $\Pi_x^i(t_i) = \mathrm{sgn}(\hat{q}_i \cos t_i + \hat{p}_i \sin t_i)$ and computing the resulting integral for $E(t_1,t_2)$

$$E(t_1,t_2) = \int_{-\infty}^{\infty} dq_1dq_2dp_1dp_2W_{\zeta}(q_1,q_2,p_1,p_2)\Pi_x^1(t_1)\Pi_x^2(t_2) \quad (4.7)$$

upon the change of variables: $\overline{q_i} = q_i \cos t_i + p_i \sin t_i$ and $\overline{p_i} = -q_i \sin t_i + p_i \cos t_i$. We obviously obtain the same answer at the end. Perhaps more elegantly, one can find the Wigner representative of the time evolution of Π_x^i by applying the general result (2.30) of Sec. II, with $a = \cos t_i$, $b = \sin t_i$, $c = -\sin t_i$, and $d = \cos t_i$.

It is easily shown (cf. Ref. [1]) that, in case the time dependence occurs only in the combination $t_1 + t_2$ [which is the case in the present situation (Eq. (4.2)], the CHSH inequality [11] implies the following inequality:

$$3P_{+-}(\theta) - P_{+-}(3\theta) \geqslant 0. \quad (4.8)$$

In the $\zeta \to \infty$ limit, i.e., when the state $|\zeta\rangle$ is maximally entangled and approaches the EPR state, $\tanh(2\zeta) \to 1$. In this limit $\chi \to \cos^{-1}(\cos \theta) = \theta$ (cf. Appendix A) and $P_{+-}(\theta) = (1/2\pi)\theta$; thus the inequality is saturated. It can be shown that for finite ζ the inequality is always satisfied. Bell suggested that correlations of observables of the type of $\Pi_x^{1,2}$ [cf. Eq. (4.7)] for the EPR state, evolving under the free Hamiltonian, would not allow for BIQV; we observe that this indeed occurs for the harmonic oscillator Hamiltonian used here.

However, his reasoning perhaps was somewhat misleading: the reason is that *it is not only the nonnegativity of the relevant Wigner function that matters, but also the type of evolution induced in the observables by the Hamiltonian in question*. The fulfillment of the CHSH inequality in the present case, in which the evolution is induced by the harmonic oscillator Hamiltonian, is consistent with the discussion given in Sec. II, below Eq. (2.30). It is apt to notice that the present evolution is not analogous to rotation of the spins in the Bohm EPR version. The latter involves what was termed [25] orientational variation, which leads (depending on the preferred viewpoint) either to improper (dispersive) dynamical variables even when one starts with proper dynamical variables, or, alternatively, to a *non* non-negative Wigner function. In either case, BIQV's do not contradict Bell's theorem.

We now consider briefly the evolution due to the free Hamiltonian of Eq. (4.1b). Again, we study the evolution of the state $|\zeta\rangle$, Eq. (1.2), under H_f, during times t_1 for particle 1 and t_2 for particle 2. The corresponding Wigner function can be otained from Eq. (1.5), applying Eq. (2.33) with $a = 1$, $b = t_i$, $c = 0$ and $d = 1$, with the result

$$W_{\zeta(t_1,t_2)} = \frac{1}{\pi^2} \exp\{- \cosh(2\zeta)[(q_1 - t_1p_1)^2 + (q_2 - t_2p_2)^2 + p_1^2 + p_2^2] - 2 \sinh(2\zeta)[(q_1 - t_1p_1)(q_2 - t_2p_2) - p_1p_2]\}. \quad (4.9)$$

With the same definitions as above, we find

$$E(t_1,t_2) = \frac{2}{\pi} \arcsin[\alpha(t_1,t_2)\tanh 2\zeta], \quad (4.10a)$$

$$\alpha(t_1,t_2) = \frac{1 - t_1t_2}{\sqrt{(1 + t_1^2)(1 + t_2^2)}}. \quad (4.10b)$$

Alternatively, just as with the previous Hamiltonian H_0, one can find the Wigner representative of the time evolution of Π_x^i applying the general result (2.30) of Sec. II, with $a = 1$, $b = t_i$, $c = 0$, and $d = 1$.

We wish to analyze whether this case abides by the CHSH inequality, i.e., whether the inequality

$$|E(t_1,t_2) + E(t_1,t_2') + E(t_1',t_2) - E(t_1',t_2')| \leqslant 2, \quad (4.11)$$

or

$$|\arcsin[\alpha(t_1,t_2)\tanh 2\zeta] + \arcsin[\alpha(t_1,t_2')\tanh 2\zeta] + \arcsin[\alpha(t_1',t_2)\tanh 2\zeta] - \arcsin[\alpha(t_1',t_2')\tanh 2\zeta]| \leqslant \pi, \quad (4.12)$$

is satisfied. For instance, taking

$$t_1 = 0, \quad t_1' = T, \quad t_2 = 0, \quad t_2' = T, \quad (4.13)$$

and subsequently taking the limit $T \to \infty$, the left-hand side of Eq. (4.12) takes the value π, i.e., the inequality is saturated.

The fulfillment of the CHSH inequality in the present case, in which the evolution is induced by the free Hamiltonian, is, once again, consistent with the discussion given in Sec. II, below Eq. (2.30).

V. CONCLUSIONS AND REMARKS

In this study we took the Clauser, Horne, Shimony and Holt [11] inequality as the representative of the so-called

Bell's inequalities. Indeed this inequality is the often analyzed and experimentally tested one and is the one used by Bell himself in his study of the subject of this work: the relation of the non-negative Wigner function of the Einstein, Podolsky, and Rosen state to possible Bell's inequality violations.

We subjected the reader to a lengthy derivation and explanation of what we considered points worthy of clarification. These were the delineation of what is meant by proper and improper dynamical variables in the context of the Wigner function as a probability distribution in phase space, the canonical variables of the latter playing the role of the local hidden variables, and showed that a proper observable (=dynamical variable) is non-dispersive. Thus only proper dynamical variables can be considered as accountable for by a local hidden-variable theory with the phase space variables (q,p) being the local hidden variables. A proper dynamical variable is one whose Wigner function representative gives the eigenvalues of the corresponding quantal dynamical variable which the local hidden-variable theory aims at underpinning.

We now summarize the relations that we have found between the properties of Wigner's representative of the operators and states on the one hand and the fulfillment of CHSH inequality on the other.

From the standpoint of the Heisenberg picture, i.e., when we work with a fixed wave function (which translates here to a fixed density) and transform the operators, it is not enough, for the fulfillment of CHSH inequality, that the Wigner representative of the wave function be non-negative: we saw that we can ensure the fulfillment of CHSH inequality when Wigner's function of the state is non-negative *and* the operators $\hat{A}$, $\hat{B}$, $\hat{A}'$ and $\hat{B}'$ and their Wigner representatives correspond to proper dynamical variables. That is, as follows.

(1) $W_\psi > 0$ and A,A',B,B' are proper dynamical variables. Then Bell's inequality is fulfilled.

In Sec. III we discussed "orientational" transformations that do *not* keep the observables as proper dynamical variables and found violations of Bell's inequality.

In Schrödinger's picture, i.e., when we consider the operators $\hat{A}$ and $\hat{B}$ and transform the wave function as explained in relation with Eq. (2.34), we have the following possibilities.

(2) $W_\psi, W_{\psi_1'}, W_{\psi_2'}, W_{\psi_{12}'} > 0$ and $\hat{A}$ and $\hat{B}$ are proper dynamical variables. For example: $\hat{A}=\mathrm{sgn}(\hat{q}_1)$, $\hat{B}=\mathrm{sgn}(\hat{q}_2)$, or the other examples discussed in Sec. II.

The only transformation V that keeps $W_\psi, W_{\psi_1'}, W_{\psi_2'}, W_{\psi_{12}'} > 0$ is a linear canonical transformation. At least for these $\hat{A}$ and $\hat{B}$, this case can be reduced to (1) above and thus Bell's inequality is fulfilled. This case is the closest one to the original problem studied by Bell [1].

(3) $W_\psi > 0$, but $W_{\psi_1'}, W_{\psi_2'}, W_{\psi_{12}'} \ngtr 0$, and $\hat{A}$ and $\hat{B}$ are proper dynamical variables. For example: $\hat{A}=\mathrm{sgn}(\hat{q}_1)$; $\hat{B}=\mathrm{sgn}(\hat{q}_2)$. We may have the two following situations.

(a) If V is an "orientational" transformation as explained in Sec. III, then Bell's inequality violations are possible.

(b) As another example, take V so that $\overline{q}=q^3$. This transformation does not preserve the non-negativity of Wigner's representative of the state (the wave function is no longer Gaussian); nonetheless, looked at in the Heisenberg picture, it keeps the above-mentioned $\hat{A}$ and $\hat{B}$ as proper dynamical variables. Thus Bell's inequality is fulfilled.

As a consequence of the above discussion, we can state the following. Suppose that we have a Gaussian wave function and that $\hat{A}$ and $\hat{B}$ are proper dynamical variables [for example: $\hat{\hat{A}}=\mathrm{sgn}(\hat{q}_1)$, $\hat{B}=\mathrm{sgn}(\hat{q}_2)$, or the other examples discussed in Sec. II]; suppose also that, in the Heisenberg picture, the transformations of the type (2.35) do *not* keep the observables as proper dynamical variables, and that we discover that *Bell's inequality violations are allowed*. We conclude that, in the Schrödinger picture, the transformed states of the type ψ_1' of Eq. (2.36) cannot be Gaussian and thus their Wigner representatives $W_{\psi_1'}$ cease to be non-negative. In fact, should the ψ_1''s be Gaussian, their Wigner representatives $W_{\psi_1'}$ would be non-negative and by case (2) above, Bell's inequality would be fulfilled, at least for the above $\hat{A}$ and $\hat{B}$.

Finally, we want to remark that, although the word "local" was repeated several times, locality as such was not an issue in the present discussion: Bell's locality condition is automatically fulfilled, as the Wigner function of any dynamical variable that depends on distinct phase space coordinates factorizes. In the derivation of Bell's inequality one makes two explicit assumptions: (1) the independence of A on the setting of B (and vice versa) and (2) each dynamical variable has values for the observables, whether or not they can be simultaneously measured. Our argument leads us to assert that in the present context it is really the second assumption which leads to Bell's inequality—whose violation, therefore, implies that the theory disallows it. This point was noted in the past [27–32]. In point of fact, two often quoted examples for underpinning noncommuting dynamical variables with local hidden variables—Bell's [2] and Wigner's [27]—are manifestly so, although these examples are, perhaps, somewhat artificial. In the present work—which in its essence follows Bell's suggestion [1]—we outlined a canonical theory which automatically abides by the locality requirement (the phase space variables are local), and Bell's inequality is abided by in cases where the dynamical variables are proper ones, even when they are noncommuting.

Our main conclusion is that in the present context the validity of Bell's inequality that we have considered hinges on the assumption of having definite values for all the dynamical variables—thus endowing them with physical reality—and not the issue of locality. Of course one might ponder what one would mean by a local hidden-variables theory without a definite value for all the dynamical variables; however, this is a separate issue.

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APPENDIX A: EVALUATION OF $E(t_1,t_2)$ FOR THE HARMONIC HAMILTONIAN

We first evaluate $P_{-+}(t_1,t_2)$ [cf. Eq. (4.6)]. After the integration over the p's and letting $q_1 \to -q_1$ we find

$$P_{-+}(t_1,t_2) = \frac{1}{\pi \cosh(2\zeta)\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}} \times \int_0^\infty dq_1 dq_2 \exp[-\cosh(2\zeta)\Gamma(\theta,\zeta) \times (q_1^2 + q_2^2 - 2q_1q_2\tanh(2\zeta)\cos\theta)]. \quad \text{(A1)}$$

Here $\theta=(t_1+t_2)$ and $\Gamma(\theta,\zeta)=\mathbf{(}1-\tanh(2\zeta)\mathbf{)}/\mathbf{(}1-\tanh(2\zeta)\cos^2\theta\mathbf{)}$. This integral is evaluated directly to give

$$P_{-+}(t_1,t_2) = \frac{1}{2\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}}\right)\right].$$

Similar calculation gives

$$P_{++}(t_1,t_2) = \frac{1}{2\pi}\left[\frac{\pi}{2} + \arctan\left(\frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}}\right)\right].$$

The equalities $P_{++}(t,t')=P_{--}(t,t')$ and $P_{+-}(t,t')=P_{-+}(t,t')$ are easily verifiable; hence we have

$$E(t_1,t_2) = 2P_{++}(\theta) - 2P_{-+}(\theta) = 1 - \frac{2}{\pi}\chi, \quad \text{(A2)}$$

with $\tanh(2\zeta)\cos\theta \equiv \cos\chi$, $\theta=t_1+t_2$.

APPENDIX B: THE WIGNER FUNCTION OF $\Pi_x(t)$ FOR $H=H_0$

The Wigner function for $\Pi_x(t)$ is given by

$$W_{\Pi_x(t)}(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_0^\infty dq e^{-ipy}\langle x+y/2|e^{iH_0t} \times [|q\rangle\langle q| - |-q\rangle\langle -q|]e^{-iH_0t}|x-y/2\rangle. \quad \text{(B1)}$$

Inserting the harmonic oscillator propagators [37] and performing the y integration gives $\text{sgn}(x\cos t + p\sin t)$.

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