

Nonadiabatic geometric quantum computation using a single-loop scenario

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A single-loop scenario is proposed to realize nonadiabatic geometric quantum computation. Conventionally, a so-called multiloop approach is used to remove the dynamical phase accumulated in the operation process for geometric quantum gates. More intriguingly, we here illustrate in detail how to use a special single-loop method to remove the dynamical phase and thus to construct a set of universal quantum gates based on the nonadiabatic geometric phase shift. The present scheme is applicable to NMR systems and may be feasible in other physical systems.

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Quantum computers have been attracting more and more interests as they are illustrated to be capable of tackling efficiently certain problems that are intractable for classical computers [1]. Significant progress has recently been achieved in the field of quantum computing. Nevertheless, there are still many difficulties and challenges in physical implementation of quantum computation. The infidelity of quantum gates is one of them; to suppress the infidelity to a acceptable level is essential to construct workable quantum logical gates in a scalable quantum computer. Recently, a promising approach based on geometric phases [2–4] was proposed to achieve built-in fault-tolerant quantum gates with higher fidelities [5–10] since the geometric phase depends only on the global feature of the evolution path and is believed to be robust against local fluctuations. The geometric quantum computation (GQC) and its physical implementation were addressed for NMR systems [8,9], Josephson junctions [6,10], and trapped ions [7].

Theoretically, under the so-called adiabatic condition, one can construct a pure geometric phase quantum gate based on an adiabatic geometric phase [8]. However, the adiabatic condition is not satisfied in many realistic cases because the long operation time is required, and thus it is hard to experimentally realize quantum computation with adiabatic evolutions, particularly for solid-state systems whose decoherence time is quite short. To overcome this disadvantage, it was proposed to use the nonadiabatic cyclic geometric phase (AA phase) to construct geometric quantum gates [9,10]. These gates have not only the faster gate-operation time, but also intrinsic geometric features of the geometric phase. For a nonadiabatic cyclic evolution, the total phase difference between the final and initial states usually consists of both the geometric and dynamical phases. Therefore, to get the nonadiabatic geometric phase, we need to remove the dynamical component. An interesting idea is to choose the cyclic evolution in dark states [7]: dark states have a zero energy eigenvalue for the effective Hamiltonian, and thus its dynamical phase will always be zero during the evolution. Another useful method to remove the dynamical phase is a so-called

multiloop scheme [8,10,11], in which the evolution is driven by the Hamiltonian along several closed loops. The dynamical phases accumulated in different loops may be canceled, while the geometric phases are added.

In this paper, we propose a simple single-loop scheme to realize a set of universal quantum gates based nonadiabatic geometric phase shifts. In this scheme, the dynamic phase can be removed in the designed cyclic evolution, with only the geometric phase being accumulated in gate operations. Comparing with the existing multiloop geometric approach, the present scenario may simplify the gate operation and shorten the gate-operation time, which appears to be a distinct advantage for experimentally implementing geometric quantum computation.

Before we present our scheme, let us first summarize how to construct a single-qubit gate using cyclic evolutions [11]. For a qubit system, consider two orthogonal cyclic states $|\psi_+\rangle$ and $|\psi_-\rangle$, which satisfy the relation $U(\tau)|\psi_\pm\rangle = \exp(\pm i\gamma)|\psi_\pm\rangle$, where γ is the total phase accumulated and $U(\tau)$ is the evolution operator of a cyclic evolution with τ as the periodicity. We can write $|\psi_+\rangle = e^{-i(\phi/2)} \cos\chi/2 |\uparrow\rangle + e^{i(\phi/2)} \sin\chi/2 |\downarrow\rangle$ and $|\psi_-\rangle = -e^{-i(\phi/2)} \sin\chi/2 |\uparrow\rangle + e^{i(\phi/2)} \cos\chi/2 |\downarrow\rangle$, where (χ, ϕ) are the spherical coordinates of the state vector on the Bloch sphere (Fig. 1), $|\uparrow\rangle$ and $|\downarrow\rangle$ are the two eigenstates of the z component of the spin-1/2 operator ($\sigma_z/2$) and they constitute the computational basis for the qubit. For an arbitrary input state denoted as

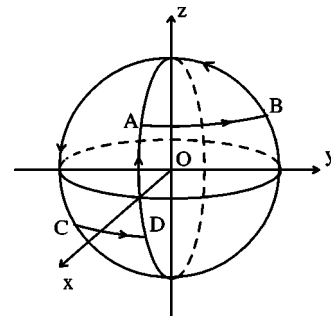


FIG. 1. The closed path ABCDA for the geometric single-qubit gate. The BC and DA are on the geodesic paths on which the dynamical phase is always zero.

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$|\psi_{in}\rangle = a_+|\psi_+\rangle + a_-|\psi_-\rangle$ with $a_{\pm} = \langle\psi_{\pm}|\psi_{in}\rangle$, after the cyclic evolution for the $|\psi_+\rangle$ ($|\psi_-\rangle$) state, the output state is $|\psi_{out}\rangle = U(\gamma, \chi, \phi)|\psi_{in}\rangle$, where

$$U = \begin{pmatrix} e^{i\gamma} \cos^2 \frac{\chi}{2} + e^{-i\gamma} \sin^2 \frac{\chi}{2} & ie^{-i\phi} \sin \gamma \sin \chi \\ ie^{i\phi} \sin \gamma \sin \chi & e^{i\gamma} \sin^2 \frac{\chi}{2} + e^{-i\gamma} \cos^2 \frac{\chi}{2} \end{pmatrix}. \quad (1)$$

If ever we can let the γ be a pure geometric phase, this U gate is a geometric quantum gate because it depends only on the geometric phase γ (and the initial coordinates of the state $|\psi_+\rangle$) under the operation, even though an input state of a superposition of the two cyclic states may have a nonzero dynamical phase after the gate operation; this feature is a distinct merit in the proposed geometric quantum gates. The discussions on the robustness of the proposed geometric gates can be found in Ref. [12].

We now illustrate schematically how to realize the above pure geometric phase gate. In Fig. 1, we plot a cyclic evolution path (ABCD) on the Bloch sphere surface; a qubit state corresponds a point on it. Note that the state vector along the BC and DA curves takes the geodesic path on the Bloch sphere. The dynamical phases accumulated on these two curves are always zero. Since the AB and CD curves are symmetric with respect to the X-Y plane, when the state vector evolves along the two curves as indicated in Fig. 1, the dynamical phases should be canceled exactly in the presence of a z -axis magnetic field.

At this stage, we choose the point A in Fig. 1 to be the $|\psi_+\rangle$. In order to ensure that the state $|\psi_+\rangle$ ($|\psi_-\rangle$) evolves cyclically, with the accumulated dynamical phase being zero, we manipulate the magnetic field as follows. A constant magnetic field B is first applied along the z axis during the time $\tau_1 = (\pi/2\omega)$. The corresponding Hamiltonian in this period may be written as $H_1 = (\mu/2)\mathbf{B} \cdot \sigma = (\omega/2)\sigma_z$, where $\omega = \mu B$. Next from the point B, the magnetic field B_2 is chosen along the x axis during $\tau_2 = (\pi/\omega_2)$ with $\omega_2 = \mu B_2$, and the Hamiltonian becomes $H_2 = (\omega_2/2)\sigma_x$. Then from the point C, the magnetic field B is reapplied to the z axis during the period $\tau_3 = \pi/2\omega$, and $H_3 = (\omega/2)\sigma_z$. Finally, we choose the magnetic field B_2 along the y axis for $\tau_4 = \pi - 2\chi/\omega_2$, and thus $H_4 = -(\omega_2/2)\sigma_y$. In this series of τ_1 , τ_2 , τ_3 , and τ_4 , the $|\psi_+\rangle$ state evolves along the paths AB, BC, CD, and DA on the Bloch sphere, and finally returns to the starting point A to form a single loop. The dynamical phase accumulated in this cyclic evolution is written as

$$\gamma^d = - \int_0^{\tau_1} \langle\psi^{AB}|H_1|\psi^{AB}\rangle dt - \int_0^{\tau_3} \langle\psi^{CD}|H_3|\psi^{CD}\rangle dt. \quad (2)$$

Because $\langle\psi^{AB}|H_1|\psi^{AB}\rangle = -\langle\psi^{CD}|H_3|\psi^{CD}\rangle$, the accumulated dynamical phase $\gamma^d = 0$. Meanwhile, the geometric phase, which is half of the area enclosed in the path spanned by the Bloch vector, is found to be $-(\pi/2)$. As a result, the designed evolution operator for any input state reads

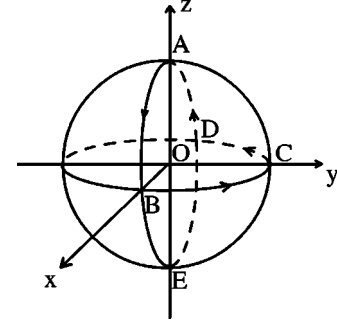


FIG. 2. The evolution paths on the Bloch sphere for a geometric two-qubit gate. When the qubit b is in the state $|\uparrow\rangle$, the state $|\psi_+\rangle_a$ completes a cyclic evolution on the ABCDA. If the qubit b is in the state $|\downarrow\rangle$, the $|\psi_+\rangle_a$ can be manipulated either to evolve along the ABE or to be unchanged.

$$U(\chi) = e^{-iH_4\tau_4} e^{-iH_3\tau_3} e^{-iH_2\tau_2} e^{-iH_1\tau_1} = \begin{pmatrix} -i \cos \chi & -i \sin \chi \\ -i \sin \chi & i \cos \chi \end{pmatrix}. \quad (3)$$

This is indeed a geometric gate with $\gamma = \gamma^g = -(\pi/2)$ in Eq. (1).

As is well known, to achieve a set of universal quantum gates, we need to construct two noncommutable single-qubit gates and one nontrivial two-qubit gate. Once we choose, for example, $\chi = \pi/4$ and $\chi = \pi/3$ in Eq. (3), respectively, it is straightforward to verify that $U(\chi_1 = \pi/4)$ and $U(\chi_2 = \pi/3)$ are noncommuting. Therefore, the two noncommutable single-qubit gates can be constructed based on nonadiabatic geometric phases.

It is also interesting to note that the loops corresponding to $\chi = 0$ and $\chi = \pi/2$ are very special, on which dynamical phases are always zero; thus they are intrinsically geometric closed paths. Obviously, the corresponding two geometric quantum gates ($-i\sigma_z$ and $-i\sigma_x$) are also noncommuting as well.

We now turn to achieve a nontrivial two-qubit gate. Let us consider a typical two-qubit system, like NMR [13], described by the following Hamiltonian with a simple interaction between two qubits:

$$H = (\omega_a \sigma_z^a + \omega_b \sigma_z^b + \pi J \sigma_z^a \sigma_z^b)/2, \quad (4)$$

where a and b denote two qubits, respectively, and J is a coupling constant. If we apply an accessory field ω'_a to the qubit a with $\omega'_a = (\omega_a - \pi J)$, then the effective Hamiltonian of the qubit a will become $H_a = (\omega_a - \omega'_a \pm \pi J) \sigma_z^a / 2 = (\pi J \pm \pi J) \sigma_z^a / 2$, in which \pm corresponds to up and down states of the qubit b . When the qubit b is in the state $|\uparrow\rangle_b$, $H_a = \pi J \sigma_z^a$; while if the qubit b is in the state $|\downarrow\rangle_b$, $H_a = 0$. This important property can be used to realize a controlled two-qubit gate based on nonadiabatic geometric phases using the similar scenario as that used in Ref. [13].

We first consider the controlled qubit b to be in the state $|\uparrow\rangle_b$. As shown in Fig. 2, we choose the arctic point on the Bloch sphere as the $|\psi_+\rangle$ state of qubit a (point A in Fig. 2), i.e., $|\psi_+\rangle_a = |\uparrow\rangle_a$. In the first step, a magnetic field B is applied on the qubit a along the y -axis and the interaction is turned

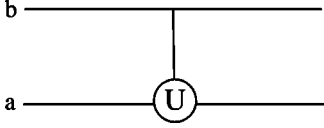


FIG. 3. Schematic diagram of a nonadiabatic two-qubit gate. The output state of the qubit a depends on the state of controlled qubit b .

off. The effective Hamiltonian of the qubit a is $H(1) = \omega\sigma_y/2$. After the time $\tau_1 = \pi/2\omega$, the state of qubit a is in the state $(\sqrt{2}/2)(|\uparrow\rangle_a + |\downarrow\rangle_a)$ (point B). Then, the magnetic field is removed and the interaction is turned on for the time $\tau_2 = 1/2J$. The effective Hamiltonian of the qubit a in this period is $H(2) = \pi J\sigma_z^a$. After this evolution along the path BCD in Fig. 2, the state changes to $(-\sqrt{2}i/2)(|\downarrow\rangle_a - |\uparrow\rangle_a)$. Next, we turn off the interaction again and apply the magnetic field along the y axis as in the first step for the time $\tau_3 = \pi/2\omega$, the final state of qubit a becomes the state $e^{-i\pi/2}|\uparrow\rangle_a$. From the whole process described above, it is clearly seen that $|\psi_+\rangle_a$ has experienced a cyclic evolution with the closed path ABCDA on the Bloch sphere:

$$|\uparrow\rangle_a \rightarrow (\sqrt{2}/2)(|\downarrow\rangle_a + |\uparrow\rangle_a) \rightarrow (-\sqrt{2}i/2)(|\downarrow\rangle_a - |\uparrow\rangle_a) \rightarrow e^{-i\pi/2}|\uparrow\rangle_a.$$

The total phase $\gamma = -\pi/2$ is just the geometric phase shift accumulated because the evolution path is geodesic and the dynamical phase is zero.

Next we consider the qubit b to be in the state $|\downarrow\rangle_b$. As we indicated before, the effective Hamiltonian of the qubit a in the above second period τ_2 is zero [$H(2) = 0$]. Thus, the $|\psi_+\rangle_a$ takes the evolution $|\uparrow\rangle_a \rightarrow (\sqrt{2}/2)(|\downarrow\rangle_a + |\uparrow\rangle_a) \rightarrow |\downarrow\rangle_a$ under the above operation process. The evolution path on the Bloch sphere corresponds the ABE path, on which no dynamical phase is accumulated. As a result, we have the time evolution operator $U_2(\tau)$ for the present two-qubit system, such that $U_2(\tau)|\uparrow\rangle_a|\uparrow\rangle_b = e^{-i\pi/2}|\uparrow\rangle_a|\uparrow\rangle_b$, $U_2(\tau)|\downarrow\rangle_a|\uparrow\rangle_b = e^{i\pi/2}|\downarrow\rangle_a|\uparrow\rangle_b$, $U_2(\tau)|\uparrow\rangle_a|\downarrow\rangle_b = |\downarrow\rangle_a|\downarrow\rangle_b$, and $U_2(\tau)|\downarrow\rangle_a|\downarrow\rangle_b = -|\uparrow\rangle_a|\downarrow\rangle_b$. In the basis of $|\uparrow\rangle_a|\uparrow\rangle_b$, $|\downarrow\rangle_a|\uparrow\rangle_b$, $|\uparrow\rangle_a|\downarrow\rangle_b$, $|\downarrow\rangle_a|\downarrow\rangle_b$, the matrix form of the evolution operator of this two-qubit system is written as

$$U_2 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5)$$

Since the output state of the qubit a depends on the state of the qubit b , as shown in Fig. 3, it is seen that the above U_2 obviously denotes a nontrivial two-qubit gate.

Alternatively, if the effective field on the qubit a can be turned off once the qubit b in the $|\downarrow\rangle$ state, $H_a = 0$ in the whole process, as that can be manipulated in the NMR ex-

periment [14] where a line-selective pulse is used (to ensure that the effective field can be applied on the qubit a by satisfying the resonant condition only if the qubit b in the state $|\uparrow\rangle$). Therefore, for the operation (on the qubit a for the case $|\uparrow\rangle_b$) corresponding to the path ABCDA in Fig. 2, we have

$$U'_2 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

This is a nontrivial conditional geometric phase gate (two qubit) [8,11]. Moreover, when the qubit a is manipulated as in Fig. 1 (for the case $|\uparrow\rangle_b$), we can achieve a more general controlled U gate as

$$U''_2 = \begin{pmatrix} U(\chi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

Finally, we wish to clarify that the robustness of the proposed geometric gates will depend on the state accurately undergoing a cyclic excursion, which may be perturbed by the fluctuations in rather strong control fields. Nevertheless, in most experimental systems, the effective fields can be controlled with high accuracy, particularly in NMR-like systems, where the effective fields to appear in Hamiltonian (4) are just the oscillating frequencies of the nucleus and the applied pulses; these frequencies can be controlled very accurately. Therefore, serious errors in the control fields may be avoided in many experimental systems. Even though there may be some unavoids noises in control fields, the proposed geometric gates are still robust against certain types of noises due to the nonuniformity of the control parameter, as illustrated in Ref. [12].

In conclusion, we have proposed a single-loop scheme to construct a set of universal quantum gates based on nonadiabatic geometric phases. Comparing with the existing multi-loop methods, our scheme using the singleloop to remove the dynamical phase is interesting and valuable in physical implementation of geometric quantum computation because it may simplify the gate operation and shorten the gate-operation time. The present scheme is applicable to NMR systems and may be feasible in other physical systems, which would stimulate experimental interests in implementing nonadiabatic geometric quantum computation.

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