Quantum noise in multipixel image processing

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We consider the general problem of the quantum noise in a multipixel measurement of an optical image. We first give a precise criterion in order to characterize intrinsic single-mode and multimode light. Then, using a transverse mode decomposition, for each type of possible linear combination of the pixels' outputs we give the exact expression of the detection mode, i.e., the mode carrying the noise. We give also the only way to reduce the noise in one or several simultaneous measurements.

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INTRODUCTION

Multipixel photodetectors such as diode arrays or charge-coupled device (CCD) sensors are now frequently used to record images. These sensors provide signals in which the useful information is mixed with random noise. A contribution to this noise originates from the quantum nature of light: the arrival of individual photons is a random process. Contrarily to technical noise, due to imperfections in the source, the optical system, or the detector, this quantum noise cannot be reduced by eliminating the defects in the measurement process. The purpose of this paper is to determine the precise origin of this noise and to analyze whether and how it can be reduced. With the analysis of the spatial distribution of this noise, we will single out the precise transverse modes whose fluctuations are at the origin of this quantum noise, and determine the parameters that have to be changed in order to reduce this noise.

As images are complex objects which carry a great deal of information, there are actually many ways to extract information from them, depending on the image user needs [1–3]. We will focus our attention on the extraction from the image of one or several continuous parameters, the variation of which modifies the light distribution in the image plane and not its total intensity. In such a case, the quantity of *a priori* information on the image is very important, as one assumes that the variation of the image under observation is due only to the variation of a searched parameter M. A second use to which our calculations can apply is the determination of predefined patterns in the image, such as given shapes, surfaces, borders, textures, and so on. It is a very difficult problem *per se*, and the incidence of quantum noise on it, to the best of our knowledge, has not been precisely studied so far. In contrast, we do not consider the search for the smallest possible details, where resolution is at stake. In this problem, there is very little to none *a priori* information and the problem of quantum limits to resolution has been already considered in other publications [4,5].

In most cases, the light used to carry the image comes from "classical sources," such as lamps or the usual lasers, in which the photons are randomly distributed in the image plane. This gives rise to a spatial shot noise which will yield a "standard quantum limit" in the measurement of a very small variation of M. It is now well known that "nonclassical light," such as squeezed light or sub-Poissonian light, is likely to reduce quantum fluctuations on a given measurement [6]. The aim of the last part of the present paper is to identify the best nonclassical light enabling us to reduce the quantum noise in the measurement of the quantity M performed in the image. It has been already shown [7] that nonclassical light in a single transverse mode, though very effective in reducing the noise for a measurement performed on the total beam, is of little use for a measurement performed on an image. One therefore needs multi-transverse-mode nonclassical light for our purpose. This is the reason why we devote the first section of this paper to a precise analysis of such a concept, before considering in the second section the problem of information extraction: we identify the exact noise source in the measurement of M, and show how to choose the best configuration which allows us to measure a variation of M with a sensitivity beyond the standard quantum limit.

I. "INTRINSIC" MULTIMODE LIGHT

We consider the propagation of light in the vacuum along the z direction, and call the transverse coordinate $\vec{r}$. We assume that the light frequency is ω_0 with a linewidth $\delta\omega$ much smaller than ω_0, and that it has a well defined polarization. One knows that it is possible to find several bases of transverse modes $\{u_i(\vec{r},z)\}$, such that each mode verifies the propagation equation of the field in vacuum projected onto the polarization axis,

$$\Delta(u_i e^{ikz}) + \frac{\omega_0^2}{c^2} u_i = 0; \tag{1}$$

it is an orthonormal basis,

$$\int u_i^*(z,\vec{r}) u_j(z,\vec{r}) d^2r = \delta_{ij}; \tag{2}$$

and it satisfies a completeness relation,

$$\sum_i u_i^*(z,\vec{r}) u_i(z,\vec{r}\,') = \delta(\vec{r} - \vec{r}\,'). \tag{3}$$

For instance, the usual Laguerre-Gauss TEM_{pq} basis satisfies these conditions. Considering a light beam, the electric field is written as the sum of the positive and negative frequencies components:

$$E(\vec{r},z,t) = E^{(+)}(\vec{r},z)e^{-i(\omega_0 t - kz)} + \text{c.c.} \tag{4}$$

It is possible to expand the electric field positive frequency envelope in the transverse modes basis as

$$E^{(+)}(\vec{r},z) = \sum_i \mathcal{E}_i u_i(\vec{r},z). \tag{5}$$

A. Single-mode or multimode light: Classical approach

For a TEM_{pq} basis field expansion, when more than one $\mathcal{E}_i$ is nonzero, it seems at first sight natural to say that this field is multimode. However, if the $\mathcal{E}_i$ coefficients are fixed (i.e., we consider a *coherent superposition of modes* and not a statistical one), one can always define a new transverse mode

$$v_0 = \frac{1}{\sqrt{\sum_i |\mathcal{E}_i|^2}} \sum_i \mathcal{E}_i u_i \tag{6}$$

and construct a basis $\{v_i\}$ in which v_0 is the first element. In this basis, the field is proportional to v_0 which means it is single mode. We can conclude that for a coherent superposition of modes, there is no intrinsic definition of a multimode beam (i.e., a definition independent of the choice of the basis). We will restrict our analysis to spatial variables, but it can be applied to any physical dimension. For instance, in the time domain, a mode locked laser is single mode, as it is a coherent superposition of many temporal modes. If the temporal modes are incoherent with each other then the system is unambiguously multimode. More precisely, if the field is a stochastic superposition of modes, the v_0 mode cannot be defined and the multimode character has a clear meaning. We will exclude this case in the following.

B. Single-mode light: Quantum approach

In order to give the quantum description of the transverse plane of a light beam, it is very common to quantize the field starting from a transverse mode basis such as the one we just defined in the previous section. In order to obtain standard formulas, we consider that all measurements are performed in an exposure time T and associate to each vector of the mode basis a set of creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ such that the field $\mathcal{E}_i$ of the previous section is replaced by the operator $i\sqrt{\hbar\omega_0/2\epsilon_0 cT}\hat{a}_i$. With these notations we obtain the standard commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, and the positive field envelope operator can be written as [8]

$$\hat{E}^+(\vec{r},z) = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0 cT}}\hat{A}^+(\vec{r},z) \tag{7}$$

with

$$\hat{A}^{(+)}(\vec{r},z) = \sum_i \hat{a}_i(z) u_i(\vec{r},z), \tag{8}$$

so that $\hat{A}^{(+)\dagger}\hat{A}^{(+)}$ is a photon number per unit surface.

In order to give a proper definition of the single-mode case, let us write the most general state of the field in the Fock state basis $|n_1, \ldots, n_i, \ldots\rangle$, where n_i stands for the number of photons in the mode i:

$$|\psi\rangle = \sum_{n_1,\ldots,n_i,\ldots} C_{n_1,\ldots,n_i,\ldots} |n_1, \ldots, n_i, \ldots\rangle \tag{9}$$

and the mean value of the electric field is given by

$$\langle\psi|\hat{A}|\psi\rangle = \sum_i \left(\sum_{n_1,\ldots,n_i>1,\ldots} C^*_{n_1,\ldots,n_i-1,\ldots} C_{n_1,\ldots,n_i,\ldots} \right) \sqrt{n_i} u_i(\vec{r}). \tag{10}$$

Following the definition for the classical beams, we can give a definition of a single-mode beam.

Definition 1. A state is single mode if a mode basis $\{v_0, v_1, \ldots\}$ exists in which it can be written

$$|\psi\rangle = |\phi\rangle \otimes |0, \ldots, 0, \ldots\rangle$$

where $|\phi\rangle$ is the state of the field in the first transverse mode.

The question is now whether, in contrast with the classical states, quantum states exist that cannot be written as (1). To answer this question, we will demonstrate the following proposition.

Proposition 1. A quantum state of the field is single mode if and only if the actions on it of all the annihilation operators of a given basis give collinear vectors.

One can note that if this property stands for a given basis, it then stands for the action of any annihilation operator.

Let us assume first that our field $|\psi\rangle$ is single mode with respect to the basis $\{u_i, \hat{a}_i\}$; then

$$\hat{a}_0|\psi\rangle = |\psi_0\rangle \quad \text{and} \quad \hat{a}_i|\psi\rangle = 0 \quad \forall\ i \neq 0. \tag{11}$$

Consider now any linear combination of the operators

$$\hat{b} = \sum_i c_i \hat{a}_i \tag{12}$$

where $\sum_i |c_i|^2 = 1$ which ensures that $[\hat{b}, \hat{b}^\dagger] = 1$. The action of this operator on the field is given by

$$\hat{b}|\psi\rangle = \sum_i c_i \hat{a}_i |\psi\rangle = c_0 |\psi_0\rangle. \tag{13}$$

This demonstrates the first implication of our proposition: all the actions of annihilation operators on the field are proportional.

To prove the other implication, consider now a field $|\psi\rangle$ on which the action of any annihilation operator $\hat{a}_i$ is proportional to $|\psi_0\rangle$. This is in particular true for the basis $\{u_i, \hat{a}_i\}$:

$$\hat{a}_i|\psi\rangle = \alpha_i |\psi_0\rangle. \tag{14}$$

If we assume that $\sum_i |\alpha_i|^2 = 1$ (which is always possible by changing the normalization of $|\psi_0\rangle$), we can define a new basis $\{v_i(\vec{r},z), \hat{b}_i\}$ such that

$$\hat{b}_0 = \sum_i \alpha_i^* \hat{a}_i, \quad v_0 = \sum_i \alpha_i^* u_i, \tag{15}$$

and complete the basis by defining a unitary matrix $[c_{ij}]$ such that

$$\hat{b}_i = \sum_j c_{ij}\hat{a}_j \quad \text{with } c_{0j} = \alpha_j^* \quad \text{and} \quad \sum_j c_{ij}c_{kj}^* = \delta_{ik}. \tag{16}$$

It is then straightforward to show that

$$\hat{b}_i|\psi\rangle = \delta_{0i}|\psi_0\rangle, \tag{17}$$

which concludes the demonstration.

In addition to the proposition, Eq. (15) gives the expression of the mode in which "lies" the mean field, knowing the action of a particular basis. We can also note that to show that a field is single mode, it is sufficient to show that all its projections on the annihilation operators of one particular basis are proportional.

To illustrate the proposition, if one considers the superposition of coherent states

$$|\psi\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_i\rangle \otimes \cdots \tag{18}$$

it is straightforward to show that the actions of all the annihilation operators on this state are proportional to the state itself; we have a single-mode beam. The basis in which it is single mode is the same as the one for the classical case, setting v_0 as in Eq. (6).

Using this proposition, we can also look for the different states that satisfy our definition of a single-mode quantum beam. As a state that cannot be written as follows in any mode basis:

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_i\rangle \otimes \cdots \tag{19}$$

is obviously not a single-mode beam, we will consider now such a factorized state of the field, on which the action of the annihilation operators gives

$$\hat{a}_i|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes (\hat{a}_i|\phi_i\rangle) \otimes \cdots. \tag{20}$$

Consequently, there are only two possibilities to have all these states proportional: either only one of the actions is different from zero, which means we are already in the basis in which the state is single mode; or all the states are coherent states.

We have described here all the possible single-mode states, and they agree with the intuitive description one might have. For instance, if one considers the superposition of several transverse modes, if at least one of them is a noncoherent state, one gets a quantum multimode state.

C. Multimode light: Quantum approach

A beam of light is said to be multimode, from a quantum point of view, when it is not single mode according to Definition 1. We can characterize such a beam by its degree n (this degree equals 1 for a single-mode beam).

Definition 2. For a beam $|\psi\rangle$, the minimum number of modes necessary to describe it (or the minimum number of nonvacuum modes in its modal decomposition), reached by choosing the appropriate basis, is called the degree n of a multimode beam. Any corresponding basis is called a minimum basis for the field $|\psi\rangle$.

The degree of a multimode beam can also be related to the generalization of Proposition 1 to an n-mode beam. Using the same technique, one can show that a quantum field is an n-mode beam if and only if the action on it of all the annihilation operators belongs to the same n-dimensioned subspace.

Whereas the previous paragraph gives a good definition of the degree of a multimode beam, it is not very convenient as one has no information on the basis in which the beam is exactly described by n modes. We can, however, define a particular basis, useful for calculations.

Proposition 2. For a beam $|\psi\rangle$ of degree n, it is always possible to find a basis $\{u_i,\hat{a}_i\}$ such that the mean value of the electric field is nonzero only in the first mode; and, it is a minimum basis for the field $|\psi\rangle$. We will call that basis an eigenbasis.

In order to demonstrate this proposition, let us consider a minimum basis $\{u_i,\hat{a}_i\}$ for the field $|\psi\rangle$. This basis is supposed to be ordered such that the n first modes are the relevant ones. We can then define a new basis $\{v_i,\hat{b}_i\}$ such that

$$v_0 = \frac{1}{\sqrt{\sum_{i=0}^{n-1}\langle\hat{a}_i\rangle^2}}\sum_{i=0}^{n-1}\langle\hat{a}_i\rangle u_i,$$

$$v_{i,0<i<n} = \sum_{j=0}^{n-1} c_{ij}u_j,$$

$$v_{i,i\geqslant n} = u_i, \tag{21}$$

where the coefficients $\{c_{ij}\}$ are chosen in order to get an orthonormal basis. Definitions similar to the one of Eq. (21) apply for the annihilation operators. The first vector of this basis is the same as the one defined for a classical beam in Eq. (6). In that basis, the mean field is single mode in a classical sense. However, the energy lying in all the other modes is not necessarily zero; only the electric field mean value is zero for these modes, and as the modes for $i\geqslant n$ were not changed, this new basis is still a minimum one for the field $|\psi\rangle$. This demonstrates the proposition. The demonstration illustrates the construction of a basis as defined in Proposition 2 from a minimum basis, even though thanks to the $\{c_{ij}\}$ coefficients an infinite number of bases are possible.

The existence of this basis is also a confirmation of the intuitive idea of the difference between single-mode and multimode quantum light. Indeed, for a single-mode beam, the spatial variation of the noise is the same as the one of the mean field. For a multimode beam, the previous description shows that some of the modes orthogonal to the mean field are sources of noise but do not contribute to the mean field. This implies that the variation of the noise is independent of the one of the mean field. This property can be used to experimentally characterize the multimode character of light. For instance, one can show the quantum multimode character of the light using a variable spatial filter. This idea has been implemented to study the semiconductor lasers output by cutting the field with a razor blade [9], and, more recently, we have shown that spatial quantum behavior of a spatially multimode optical parametric oscillator can be demonstrated using an iris whose aperture size is continuously varied [10].

We have defined the theoretical basis required to develop a study on optical image measurements. The following section on information extraction will indeed strongly rely on the propositions and definitions of the first part.

II. DIFFERENCE MEASUREMENTS

A. Description

A widely used technique in optics, and more generally in physics, to improve the signal to noise ratio in a measurement is to perform a *difference measurement*. It consists in producing two identical signals from the light source used in the experiment. When one monitors the difference between these two signals, one gets of course a zero mean signal, but one also cancels all the common mode noises, for example, the one arising from the classical intensity fluctuations of the source. The remaining noise arises from the noise sources affecting the two channels differently.

One simple way to produce two identical beams is to use a 50% beam splitter. In this case, the vacuum noise coming from the unused side of the splitter is such a not-common-mode noise and remains in the difference measurement: whatever the actual excess noise of the beam impinging on the beam splitter, the remaining noise corresponds to the shot noise of this beam.

This simple technique of noise cancellation is used, for example, to measure very small absorptions [11] by inserting the absorbing medium in one of the arms of the difference setup, or very small frequency shifts, by inserting a Fabry-Pérot cavity in one of the arms. It is also extensively used in multipixel measurements, with either split detectors or quadrant detectors, to measure submicrometer displacements, for example of nanoscale fluorophores in biological samples [12] and in atomic force microscopy [13], and ultrasmall absorptions by the mirage effect [14].

The problem of the determination of the origin of quantum noise on a split detector and of its reduction has been already investigated theoretically [7] and experimentally [15–17]. We will here extend these considerations to more general configurations.

More formally, we consider the measurement by a detector consisting of a set of pixels, each one occupying a transverse area D_i. The pixels cover the whole transverse plane, with no overlap between them. Each photodetector delivers a power given by

$$\hat{I}(D_i) = \int_{D_i} 2\epsilon_0 c \hat{E}^\dagger(\vec{r})\hat{E}(\vec{r}) d^2 r. \tag{22}$$

This can also be written as the photon number measured during the exposure time T of the detector:

$$\hat{N}(D_i) = \int_{D_i} \hat{A}^\dagger(\vec{r})\hat{A}(\vec{r}) d^2 r. \tag{23}$$

In this section, the measurement M is defined by

$$\hat{M}(\{\sigma_i\}) = \sum_i \sigma_i \hat{I}(D_i) \quad \text{such that } \sigma_i = \pm 1 \tag{24}$$

or again in terms of number of photons per second:

$$\hat{N}(\{\sigma_i\}) = \sum_i \sigma_i \hat{N}(D_i), \tag{25}$$

where $\sigma_i = \pm 1$ corresponds to the electronic gain of detector i.

Considering a light beam in state $|\psi\rangle$, the measurement is a difference measurement for that beam if its mean value is zero, i.e., if

$$\langle \hat{N}(\{\sigma_i\}) \rangle = 0. \tag{26}$$

B. One difference measurement

If one considers one difference measurement performed with a coherent state, which has spatially uncorrelated quantum fluctuations, the noise arising from the measurement will not depend on the choice of $\{\sigma_i\}$ if $\sigma_i = \pm 1$, and will be equal to the square root of the total number of photons. This is what is called the standard quantum noise. In the general case, in order to compute the noise, an analysis equivalent to the one performed in the case of a small displacement measurement, as done in Ref. [7], is necessary. We recall it here and extend it to the general case of transverse modes of any shape, in order to show the following proposition.

Proposition 3. The noise on a difference measurement performed on a beam $|\psi\rangle$ originates from a single mode, orthogonal to the mean field: the "flipped mode." In order to reduce the noise in that measurement, it is necessary and sufficient to inject a squeezed state in this flipped mode.

In order to perform the general noise calculation, let us define the two "detectors":

$$D_+ = \bigcup_{i,\sigma_i=+1} D_i,$$

$$D_- = \bigcup_{i,\sigma_i=-1} D_i, \tag{27}$$

which gives

$$\hat{N}_- = \hat{N}(D_+) - \hat{N}(D_-)$$
$$= \sum_{i,j} \hat{a}_i^\dagger \hat{a}_j \left[\int_{D_+} u_i^*(\vec{r}) u_j(\vec{r}) d^2 r - \int_{D_-} u_i^*(\vec{r}) u_j(\vec{r}) d^2 r \right]. \tag{28}$$

Considering small quantum fluctuations for which $\delta\hat{a}_i = \hat{a}_i - \langle \hat{a}_i \rangle$, the fluctuations of $\hat{N}_-$ are

$$\delta\hat{N}_- = \hat{N}_- - \langle \hat{N}_- \rangle = \sum_i [\delta\hat{a}_i^\dagger C_-^i + \delta\hat{a}_i C_-^{i*}], \tag{29}$$

with C_-^i defined as

$$C_-^i = \sum_j \langle \hat{a}_j \rangle \left[\int_{D_+} u_i^*(\vec{r}) u_j(\vec{r}) d^2 r - \int_{D_-} u_i^*(\vec{r}) u_j(\vec{r}) d^2 r \right]$$
$$= \int_{D_+} u_i^*(\vec{r}) A_\psi(\vec{r}) d^2 r - \int_{D_-} u_i^*(\vec{r}) A_\psi(\vec{r}) d^2 r$$

and where $A_\psi(\vec{r})$ is the mean value of the electric field $\langle \psi | \hat{A}(\vec{r}) | \psi \rangle$. The C_-^i coefficients are the partial overlap integrals between the modes u_i and the mean field.

We can now compute the noise related to this measurement:

$$\langle \delta \hat{N}_-^2 \rangle = \sum_i |C_-^i|^2 + \left[\sum_{i,j} \langle \delta \hat{a}_i^\dagger \delta \hat{a}_j^\dagger \rangle C_-^i C_-^j + \langle \delta \hat{a}_i^\dagger \delta \hat{a}_j \rangle C_-^i C_-^{j*} + \text{c.c.} \right]. \quad (30)$$

Using the completeness relation, the first term of the last equation can be shown to be equal to the total number of incident photons per second, N_0. This shows that the noise related to this measurement arises *a priori* from all the modes.

We will now demonstrate that the noise comes in fact from a single mode when we write $\langle \delta \hat{N}_-^2 \rangle$ in the appropriate basis. We indicate by v_0 the mode of the mean field as defined in the previous part:

$$v_0(\vec{r}) = \frac{1}{\sqrt{N_0}} A_\psi(\vec{r}). \quad (31)$$

If v_0 is the first mode of a basis, the mean value of the electric field in all the other modes will be zero, as shown in the previous section. We define now the mode v_1, which we will refer to as the flipped mode of v_0, such that

$$v_1(\vec{r}) = \begin{cases} v_0(\vec{r}) & \text{if } r \in D_+, \\ -v_0(\vec{r}) & \text{if } r \in D_-. \end{cases} \quad (32)$$

As we have assumed that the mean value of the measurement is zero, v_1 is orthogonal to v_0, which means that we can find a basis $\{v_i, \hat{b}_i\}$ where v_0 and v_1 are the two first modes. In that basis, the overlap integrals become

$$C_-^i = \sqrt{N_0} \left[\int_{D_+} v_i^*(\vec{r}) v_0(\vec{r}) d^2 r - \int_{D_-} v_i^*(\vec{r}) v_0(\vec{r}) d^2 r \right]$$
$$= \sqrt{N_0} \int_D v_i^*(\vec{r}) v_1(\vec{r}) d^2 r = \sqrt{N_0} \delta_{i,1}. \quad (33)$$

These integrals are different from zero only for the flipped mode. The noise of Eq. (30) becomes

$$\langle \delta \hat{N}_-^2 \rangle = N_0 \langle (\delta \hat{b}_1^\dagger + \delta \hat{b}_1)^2 \rangle, \quad (34)$$

which shows that the noise arises only from the quadrature of the flipped mode of v_0 in phase with the mean field mode. For this reason, we call this mode the eigenmode of the measurement. Another standard notation is

$$\langle \delta \hat{N}_-^2 \rangle = N_0 \langle \delta X_1^{+2} \rangle, \quad (35)$$

where $X_1^+ = \hat{b}_1 + \hat{b}_1^\dagger$ is the quadrature of the flipped mode, and N_0 represents the shot noise. Consequently, having a squeezed state in that mode is necessary and sufficient to reduce the noise related to the measurement.

This calculation shows that, for a difference measurement, the noise in the measurement is exactly the one of the flipped mode. Changing the noise properties of the flipped mode is then the only way to change the noise in the measurement. We have a necessary and sufficient condition to improve the measurement compared to the standard quantum limit.

This demonstration imposes the noise properties of only one quadrature of the flipped mode, but there is no condition on the other quadrature, and all the other modes can be in any state. Then, there is not only one practical solution.

C. Multiple difference measurement

We have demonstrated which mode one needs to squeeze in order to perform one difference measurement on a beam. We can now expand this analysis in the case of several difference measurements. Let us consider n difference measurements of the type of Eq. (26). We will assume that these measurements are independent, which means that none of them is a linear combination of the others. One can show that the corresponding flipped modes are then also linearly independent. We have shown that in order to improve simultaneously the sensitivity of all these measurements it is necessary, and sufficient, to squeeze all these flipped modes. Practically these modes are in general not orthogonal, but one can find an orthogonal basis of the subspace generated by these modes. Injecting squeezed vacuum states in each of these modes will result in squeezed states in each of the flipped modes.

Regarding the degree of the beam necessary to improve simultaneously all the measurements, it is clear that in order to perfectly squeeze all the flipped modes, a beam of degree $n+1$ is necessary (and sufficient). We can summarize all the considerations of Sec. II into a proposition.

Proposition 4. In order to reduce the noise simultaneously in n independent difference measurements it is necessary and sufficient to use a beam of degree at least $n+1$ that can be described in a transverse mode basis $\{\hat{a}_i, u_i\}$ such that u_0 is proportional to the electric field profile of the beam; $\{u_i\}_{0<i\leq n}$ is the basis of the space vector generated by the flipped modes of the measurements; and all these modes are perfectly squeezed.

III. LINEAR MEASUREMENT

Difference measurements are obviously not the only ones performed in image processing [1–3]. The extraction of the pertinent information arises generally from the numerical computation of a function $F(I(D_1), I(D_2), \ldots, I(D_n))$ from the intensities $I(D_i)$ $(i=1, \ldots, n)$ measured on each pixel. To simplify the following discussion, we will restrict ourselves to the case when this function is *linear* with respect to the intensities $I(D_i)$, as is a case often encountered in real situations, for example, when one wants to determine the spatial Fourier components of the image, or when the variations of the parameter to measure are small enough so that the function F can be linearized.

In the formalism of Eqs. (24) and (25), using a linear function corresponds to letting the gain σ_i of the detectors take any real value and not only ±1:

$$\hat{M}(\{\sigma_j\}) = \sum_j \sigma_j \hat{I}(D_j),$$

$$\hat{N}_\sigma = \sum_j \sigma_j \hat{N}(D_j). \tag{36}$$

We emphasize that, contrary to the previous section, the mean value of the measurement is not necessarily zero. In that case, we will show the following proposition.

Proposition 5. Consider a field state $|\psi\rangle$ described in an eigenbasis $\{\hat{b}_i, v_i\}$, and consider a linear measurement performed with an array of detectors D_i, each detector having a gain σ_i. The noise on the measurement, $\hat{N}_\sigma = \Sigma_j \sigma_j \hat{N}(D_j)$, arises only from the generalized flipped mode w defined by

$$\forall\ \vec{r}, \vec{r} \in D_i \Rightarrow w_1(\vec{r}) = \frac{1}{f}\sigma_i v_0(\vec{r}) \tag{37}$$

where f is a normalization factor.

Here, there is not much sense in defining the positive and negative gain domains. We can anyway extend the notion of overlap integral between a basis vector and the mean field:

$$C_\sigma^i = \sum_j \sigma_j \int_{D_j} u_i^*(\vec{r}) A_\psi(\vec{r}) d^2 r, \tag{38}$$

which leads to a formula equivalent to Eq. (30)

$$\langle \delta \hat{N}_\sigma^2 \rangle = \sum_i |C_\sigma^i|^2 + \left[\sum_{i,j} \langle \delta \hat{a}_i^\dagger \delta \hat{a}_j^\dagger \rangle C_\sigma^i C_\sigma^j + \langle \delta \hat{a}_i^\dagger \delta \hat{a}_j \rangle C_\sigma^i C_\sigma^{j*} + \text{c.c.} \right]. \tag{39}$$

Recalling that $A_\psi(\vec{r}) = \sqrt{N_0} v_0(\vec{r})$, we can also extend the notion of the flipped mode, and define a *detection mode* by

$$\forall\ \vec{r}, \vec{r} \in D_i \Rightarrow w_1(\vec{r}) = \frac{1}{f}\sigma_i v_0(\vec{r}), \tag{40}$$

where f ensures the normalization of w_1:

$$f^2 = \sum_j \sigma_j^2 \int_{D_j} v_0^*(\vec{r}) v_0(\vec{r}) d^2 r. \tag{41}$$

However, as the mean value of the measurement can be different from zero, the detection mode w_1 is not in general orthogonal to the mean field mode v_0. In order to calculate the noise in the measurement, it is necessary to construct a basis that contains the detection mode w_1. As the mean value of the electric field in this mode is different from zero, it is not possible to obtain an eigenbasis with w_1, but we can still choose w_0 such that the mean field mode v_0 is a linear combination of w_0 and w_1. Choosing all the other modes w_i (with $i \geqslant 2$) in order to obtain an orthonormal basis, we obtain a basis such that the mean field is distributed in the two first modes, the detection mode is w_1, and the mean value of the electric field in all the other modes is zero. We can then perform a calculation similar to the one of the previous section, which gives

$$C_\sigma^i = \sqrt{N_0} f \int_D w_i(\vec{r}) * w_1(\vec{r}) d^2 r = \sqrt{N_0} f \delta_{i,1}. \tag{42}$$

Once again the detection mode is the only one that is relevant for the calculation of the noise related to the measurement. Taking into account that the normalization giving rise to the shot noise has changed,

$$\sum_i |C_\sigma^i|^2 = |C_\sigma^1|^2 = N_0 f^2, \tag{43}$$

the noise formula becomes

$$\langle \delta \hat{N}_\sigma^2 \rangle = f^2 N_0 \langle (\delta \hat{c}_1^\dagger + \delta \hat{c}_1)^2 \rangle, \tag{44}$$

where the $\{\hat{c}_i\}$ are the annihilation operators associated with the transverse mode basis $\{w_i\}$.

The f^2 factor is a global effect of the gain, and modifies both the measured signal and shot noise level. In any case, if the flipped mode is perfectly squeezed, we can still perform a perfect measurement. However, the experimental configuration is much more complicated as, in general, the mean value of the electric field in mode w_1 is different from 0, which means that, as is shown in the Appendix, generating the good mode is difficult. An appropriate approach would be to describe the field back into an eigenbasis, and check how to set the noise of the different modes in that basis. We will see in the Appendix how this can be done in a simple case. The important result of this part is that whatever the measurement we perform the noise arises only from one mode. Changing the noise of this mode allows us to improve the sensitivity of the measurement. As in the previous section, it is also possible in that general case to perform several simultaneous measurements, and to identify the subspace of modes responsible for the noise.

It is interesting to note that, in the particular case of a measurement where the gains are adapted to have $\langle \hat{M}(\{\sigma_j\}) \rangle = 0$, the mode v_0 coincides with w_0. Indeed, v_0 is here orthogonal to w_1:

$$\int_D w_1^*(\vec{r}) v_0(\vec{r}) d^2 r = \sum_j \frac{\sigma_j}{f} \int_{D_j} v_0^*(\vec{r}) v_0(\vec{r}) d^2 r \propto \left\langle \sum_j \sigma_j \hat{N}(D_j) \right\rangle = 0; \tag{45}$$

hence the basis is an eigenbasis of the field. Again, that case is relevant experimentally as it means that one can act on the noise without perturbing the mean field mode.

CONCLUSION

We have shown in this article how to properly define the degree of multimode character of a light beam. We have used the basis decomposition associated with that definition in order to single out, in a linear transverse measurement, the transverse mode carrying the noise. We have shown that it is possible to go beyond the standard quantum noise limit by injecting in that mode squeezed light, and that this can be done simultaneously for several independent measurements.

It order to implement the theory developed here to complex experimental configurations we have shown that it was preferable that the various detection modes be orthogonal to the mean field (i.e., they do not contribute to the mean electric field), and it is necessary to mix them without introducing losses. For instance, one can use the proposal we have detailed in [17] and used to mix two nonclassical beams in orthogonal transverse modes, and a mean coherent field, in order to improve the sensitivity of the transverse position measurement of a laser beam.

In this paper, we have analyzed in great detail the origin of quantum noise in a multipixel measurement. What remains to be considered now is the signal, and not only the noise in the measurement. This will be the natural continuation of our work, and we will describe in a future publication what is the influence of the gain configuration on the signal to noise ratio and how to optimize a given measurement in an optical image.

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APPENDIX: TWO-ZONE MEASUREMENT

In this article, we have exhibited the mode structure of the light in a multipixel measurement, using a basis that contains the detection mode. However, when the mean value of the measurement is different from zero, we have shown that this detection mode has a mean electric field value different from zero. In that configuration, it is very difficult experimentally to address the detection mode without modifying the mean field distribution. We have shown that the only basis pertinent for such a task is an eigenmode basis. We will show here what is the structure of that basis for a two-zone measurement of nonzero mean value.

Using the notations of the previous sections, we consider two detectors D_+ and D_- whose gains are, respectively, +1 and −1. We recall here the mode structure defined in the main text of this article. v_0 is the transverse mode carrying the mean field of the beam and w_1 is the detection mode as defined in Eq. (40) [which, in this case, is equivalent to the flipped mode of Eq. (32)]. w_0 is the mode orthogonal to w_1 in the subspace generated by v_0 and w_1. Let us call the partial integrals of v_0 on each zone i_+ and i_-,

$$i_+ = \int_{D_+} v_0^*(\vec{r}) v_0(\vec{r}) d^2 r \quad \text{and} \quad i_- = \int_{D_-} v_0^*(\vec{r}) v_0(\vec{r}) d^2 r.$$

A simple calculation gives

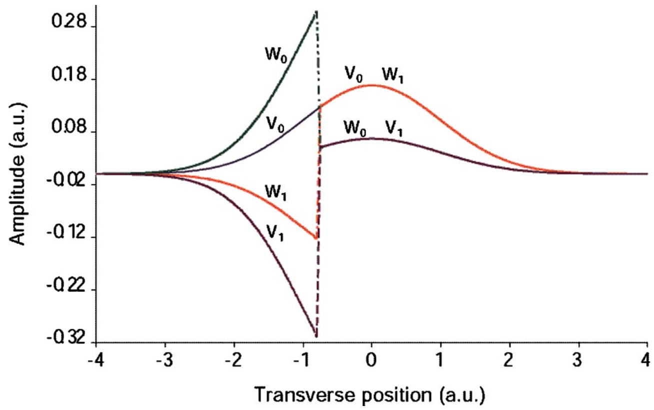

FIG. 1. Electric field profile of the constituent modes used to form the nonclassical multimode beam.

$$w_0(\vec{r}) = \begin{cases} \sqrt{\dfrac{i_-}{i_+}} v_0(\vec{r}) & \text{if } r \in D_+, \\ \sqrt{\dfrac{i_+}{i_-}} v_0(\vec{r}) & \text{if } r \in D_-. \end{cases} \tag{A1}$$

The first mode of an eigenbasis for the field is v_0. The second one, v_1, is defined as the mode orthogonal to v_0 in the subspace generated by w_0 and w_1. Its expression is found to be v_1 such that

$$v_1(\vec{r}) = \begin{cases} w_0(\vec{r}) & \text{if } r \in D_+, \\ -w_0(\vec{r}) & \text{if } r \in D_-. \end{cases} \tag{A2}$$

As w_0 is orthogonal to w_1, which is the flipped mode of v_0, one can show that v_0 is orthogonal to v_1, which is the flipped mode of w_0 (see Fig. 1). In order to calculate the noise in the measurement using that basis, the flipped mode is expressed as a linear combination of the two first modes of the eigenbasis:

$$w_1 = \alpha v_0 + \beta v_1, \tag{A3}$$

where $\alpha = i_+ - i_-$ and $\beta = 2\sqrt{i_+ i_-}$, which leads to

$$\langle(\delta\hat{c}_1^\dagger + \delta\hat{c}_1)^2\rangle = \alpha^2 \langle(\delta\hat{b}_0^\dagger + \delta\hat{b}_0)^2\rangle + \beta^2 \langle(\delta\hat{b}_1^\dagger + \delta\hat{b}_1)^2\rangle + 2\alpha\beta\langle(\delta\hat{b}_0^\dagger + \delta\hat{b}_0)(\delta\hat{b}_1^\dagger + \delta\hat{b}_1)\rangle. \tag{A4}$$

Expressed in an eigenbasis that does not contain the detection mode, we see that the noise arises from the individual noise of the two first modes and from their correlation function. In that basis, in order to reduce the noise we have several solutions: either the two first modes are perfectly squeezed, or they are perfectly correlated, or any solution in between. Anyway, we can assume that if we want to make a lot of different measurements, it is very difficult to produce correlation between the mean field and the different vacuum modes; hence the easiest solution is to have the mean field squeezed, and the corresponding vacuum squeezed. The same argument as before applies, and we show that we still need an extra mode for each piece of extra information.

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