Casimir force acting on magnetodielectric bodies embedded in media

Christian Raabe and Dirk-Gunnar Welsch
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany
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Within the framework of macroscopic quantum electrodynamics, general expressions for the Casimir force acting on linearly and causally responding magnetodielectric bodies that can be embedded in another linear and causal magnetodielectric medium are derived. Consistency with microscopic harmonic-oscillator models of the matter is shown. The theory is applied to planar structures, and proper generalizations of Casimir's and Lifshitz-type formulas are given.

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I. INTRODUCTION

It is well known that the introduction of momentum and energy of the macroscopic electromagnetic field requires careful consideration, even for linear media. In fact, insertion of constitutive relations into Gauss's and Ampere's laws prevents one, in general, from deriving local balance equations of a similar type as in the microscopic theory. However, with quite restrictive (and most questionable) assumptions about the material under consideration, these difficulties can be formally overcome. Therefore, textbooks typically resort to approximate formulas that are based on assumptions such as quasimonochromatic fields and lossless media [1,2]. Although the limitations inherent in such theories are rather obvious, they are nevertheless applied beyond their range of validity.

A typical example is the Casimir effect, which is closely related to the changes in the vacuum electromagnetic-field energy and/or momentum flow (stress) induced by the presence of inhomogeneous matter. With regard to the calculation of the Casimir force acting on macroscopic bodies that are embedded in a medium, the question of what are the correct expressions for these quantities becomes crucial. Frequently, expressions that seem reasonable at first glance—such as Minkowski's stress tensor—have been taken for granted without justification. As we shall see, this has led to incorrect extensions of the well-known Lifshitz formula for the Casimir force between two dielectric half-spaces separated by vacuum to the case where the interspace is not empty but also filled with material [3–5] (see also the textbooks [6–8] and references therein).

In this paper, we reconsider, within the framework of macroscopic quantum electrodynamics, the problem of the calculation of Casimir forces, by regarding the Lorentz force density as the fundamental quantity. The Lorentz force acting on some (macroscopic) spatial region containing (electrically neutral) matter is of course the corresponding volume integral of the Lorentz force density, where the relevant charge and current densities may be thought of as being expressed in terms of the polarization and the magnetization of the matter. As a consequence, the Casimir force on a macroscopic body and, equivalently, the stress on its surface can be expressed—in close analogy with microscopic electrodynamics—in terms of the electric and induction fields, irrespective of any specific constitutive relations. In particular, if the body linearly responds to the electric and induction fields and the medium the body is embedded in is also a linear one, then the Casimir force can be expressed solely in terms of the classical (retarded) Green tensor, which in turn is determined by the response functions of the magnetodielectric matter under consideration.

We show that the Casimir force formula found in this way is consistent with microscopic theories based on harmonic-oscillator models of dispersing and absorbing dielectric matter. The formula is valid under very general conditions, and enables one to study in a consistent way the Casimir force on linearly responding, dispersing, and absorbing magnetodielectric bodies that are not necessarily placed in vacuum but may also be surrounded by a dispersing and absorbing linear magnetodielectric medium. Since magnetodielectric matter is included in the theory, it is possible to consider also left-handed materials [9]. To illustrate the theory, we apply it to a planar geometry and derive a proper extension of Lifshitz-type formulas, with emphasis also on the extension of Casimir's original formula.

The paper is organized as follows. In Sec. II, the stress tensor associated with the (macroscopic) Lorentz force is introduced. The Casimir force is calculated in Sec. III, and Sec. IV makes contact with the microscopic harmonic-oscillator model. The application of the theory to planar structures is given in Sec. V, and a summary and some concluding remarks are given in Sec. VI.

II. LORENTZ FORCE AND STRESS TENSOR

Let us begin with the classical Maxwell equations for the electric and induction fields $\mathbf{E}$ and $\mathbf{B}$ in the presence of matter,

$$\boldsymbol{\nabla}\mathbf{B} = 0, \tag{1}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}, \tag{2}$$

$$\varepsilon_0\,\boldsymbol{\nabla}\,\mathbf{E} = \rho, \tag{3}$$

$$\mu_0^{-1}\nabla\times\mathbf{B}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j}. \tag{4}$$

In this paper "dot products" are written without the dot, and dyadic products are denoted by $\otimes$. In Eqs. (3) and (4), ρ and $\mathbf{j}$ cover *all* charges and currents of the system under consideration. Within the framework of a macroscopic description, the "internal" charges and currents associated with the particles that form some neutral material system are commonly described in terms of polarization and magnetization fields $\mathbf{P}$ and $\mathbf{M}$, respectively. The remaining "external" charges and currents—if any—are kept explicitly, i.e.,

$$\rho=\rho_{\rm int}+\rho_{\rm ext}, \tag{5}$$

$$\mathbf{j}=\mathbf{j}_{\rm int}+\mathbf{j}_{\rm ext}, \tag{6}$$

where

$$\rho_{\rm int}=-\nabla\mathbf{P}, \tag{7}$$

$$\mathbf{j}_{\rm int}=\frac{\partial\mathbf{P}}{\partial t}+\nabla\times\mathbf{M}. \tag{8}$$

As long as constitutive equations (relating the polarization and magnetization fields to the electric and induction fields) are not introduced, Eqs. (1)–(8), which may also be interpreted microscopically, are generally valid. Clearly, the defining Eqs. (7) and (8) of $\mathbf{P}$ and $\mathbf{M}$, respectively, can be satisfied for any choice of (conserved) "internal" sources, the corresponding integrability condition being just

$$\frac{\partial\rho_{\rm int}}{\partial t}+\nabla\mathbf{j}_{\rm int}=0. \tag{9}$$

Note that the "internal" sources typically comprise bound charges and the associated currents—a concept that is commonly used together with spatial averaging in macroscopic electrodynamics [10].

As known, the Lorentz force density

$$\mathbf{f}=\rho\mathbf{E}+\mathbf{j}\times\mathbf{B} \tag{10}$$

can be rewritten with the help of Eqs. (1)–(4) as

$$\mathbf{f}=\nabla\boldsymbol{T}-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}), \tag{11}$$

where the stress tensor

$$\boldsymbol{T}=\varepsilon_0\mathbf{E}\otimes\mathbf{E}+\mu_0^{-1}\mathbf{B}\otimes\mathbf{B}-\frac{1}{2}(\varepsilon_0E^2+\mu_0^{-1}B^2)\boldsymbol{1} \tag{12}$$

has been introduced ($\boldsymbol{1}$, unit tensor). Clearly, Eqs. (10) and (11) are universally valid, regardless of whether the charge and current densities have been decomposed according to Eqs. (5)–(8) or not. Note that essentially the same position has been recently taken up [11] in the (re)analysis of measurements of the electromagnetic force that acts on dielectric [12–16] or magnetodielectric [17] (see also Refs. [18,19]) disks exposed to crossed electric and magnetic fields. (For a different perspective, see also Ref. [20].)

The integral of the Lorentz force density $\mathbf{f}$ over some space region (volume V) gives of course the total electromagnetic force $\mathbf{F}$ acting on the matter inside it,

$$\mathbf{F}=\int_V d^3r\mathbf{f}. \tag{13}$$

Using Eq. (11), we have

$$\mathbf{F}=\int_{\partial V}d\mathbf{a}\boldsymbol{T}-\varepsilon_0\frac{d}{dt}\int_V d^3r\mathbf{E}\times\mathbf{B}, \tag{14}$$

which is obviously also true if the space region is occupied by a macroscopic body, with the charges and currents being "internal" ones described by polarization and magnetization fields. In particular, if the volume integral on the right-hand side of this equation does not depend on time, then the total force reduces to the surface integral

$$\mathbf{F}=\int_{\partial V}d\mathbf{F}, \tag{15}$$

where

$$d\mathbf{F}=d\mathbf{a}\boldsymbol{T}=\boldsymbol{T}d\mathbf{a} \tag{16}$$

may be regarded as the infinitesimal force element acting on an infinitesimal surface element $d\mathbf{a}$. Note that a constant term in the stress tensor does not contribute to the integral in Eq. (15) and can therefore be omitted. In the calculation of the Casimir force in Sec. III, it will be necessary to make use of this fact.

Expressing in Eq. (16) the stress tensor $\boldsymbol{T}$ in terms of Minkowski's stress tensor $\boldsymbol{T}^{(\rm M)}$ (which agrees with Abraham's stress tensor [7]),

$$\begin{aligned}\boldsymbol{T}^{(M)}&=\mathbf{D}\otimes\mathbf{E}+\mathbf{H}\otimes\mathbf{B}-\frac{1}{2}(\mathbf{DE}+\mathbf{HB})\boldsymbol{1}\\&=\boldsymbol{T}+\mathbf{P}\otimes\mathbf{E}-\mathbf{M}\otimes\mathbf{B}-\frac{1}{2}(\mathbf{PE}-\mathbf{MB})\boldsymbol{1},\end{aligned} \tag{17}$$

one finds that

$$d\mathbf{F}=d\mathbf{a}\boldsymbol{T}^{(M)}-d\mathbf{a}\left[\mathbf{P}\otimes\mathbf{E}-\mathbf{M}\otimes\mathbf{B}-\frac{1}{2}(\mathbf{PE}-\mathbf{MB})\right], \tag{18}$$

from which it is seen that in general

$$d\mathbf{F}\neq d\mathbf{a}\boldsymbol{T}^{(M)}. \tag{19}$$

That is to say, the use of Minkowski's stress tensor is expected not to yield the correct force in general, whereas the use of $\boldsymbol{T}$, which is formally the same as the stress tensor in microscopic electrodynamics, is *always* correct.

Let

$$\mathbf{P}=\mathbf{P}_{\rm ind}+\mathbf{P}_{\rm N}, \tag{20}$$

$$\mathbf{M}=\mathbf{M}_{\rm ind}+\mathbf{M}_{\rm N} \tag{21}$$

be the decompositions of the polarization and the magnetization into induced parts $\mathbf{P}_{\rm ind}$, $\mathbf{M}_{\rm ind}$ and noise parts $\mathbf{P}_{\rm N},\mathbf{M}_{\rm N}$, where the noise parts are closely related to dissipation. Substituting in Eq. (18) for $\mathbf{P}$ and $\mathbf{M}$ the expressions (20) and

(21), respectively, we see that force calculations that are based on Minkowski's stress tensor are expected to be incorrect with respect to both the induced parts and the noise parts of the polarization and the magnetization in general. Clearly, if—and only if—the aim is to calculate the force acting on bodies that are placed in a free-space region, then both $\boldsymbol{T}$ and $\boldsymbol{T}^{(\mathrm{M})}$ lead to the same result.

The idea to regard [according to Eqs. (10)–(13)] the Lorentz force acting on the totality of charges and currents belonging to a system under consideration as the fundamental quantity is neither new [11,21–23] nor particularly hard to agree with. Despite this, the use of Minkowski's stress tensor or related quantities has still been common in the calculation of electromagnetic forces. In this context, let us make a few general remarks. The momentum that may be introduced on the basis of Eq. (11) is related to the Noether symmetry expressing homogeneity of space. It must be distinguished from the pseudomomentum related to (strict) homogeneity of the material. In connection with the so-called Minkowski-Abraham controversy, Refs. [21,24] analyze in a Lagrangian framework the meaning of different momentumlike quantities by consideration of explicit (classical) dynamical models of a homogeneous dielectric. In Ref. [21], the homogeneous dielectric is assumed to be lossless and treated in some multipolar, long-wavelength approximation (for an inclusion of magnetic properties, see [25]). In Ref. [24], the homogeneous dielectric is described by a single-resonance Drude-Lorentz model. All the calculations show that Eq. (11) [together with Eqs. (10) and (12)] is really the momentum balance of the macroscopic electromagnetic field.

III. CASIMIR FORCE ON BODIES EMBEDDED IN DISPERSING AND ABSORBING MEDIA

In classical electrodynamics, electrically neutral material bodies at zero temperature which do not carry a permanent polarization and/or magnetization are not subject to a Lorentz force in the absence of external electromagnetic fields. As known, the situation changes in quantum electrodynamics, since the vacuum fluctuations of the electromagnetic field can give rise to a nonvanishing Lorentz force—the Casimir force. Its experimental demonstration has therefore been regarded as a confirmation of quantum theory.

To translate the classical formulas given in Sec. II into the language of quantum theory, let us consider linear, inhomogeneous media that locally respond to the electromagnetic field and can thus be characterized by a spatially varying complex permittivity $\varepsilon(\mathbf{r},\omega)$ and a spatially varying complex permeability $\mu(\mathbf{r},\omega)$. Following Ref. [26], we may write the medium-assisted electric and induction field operators in the form of

$$\hat{\mathbf{E}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) + \text{H.c.}, \tag{22}$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) + \text{H.c.}, \tag{23}$$

where

$$\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) = i\mu_0\omega \int d^3r' \boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\underline{\hat{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r}',\omega), \tag{24}$$

$$\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) = \mu_0 \boldsymbol{\nabla} \times \int d^3r' \boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\underline{\hat{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r}',\omega). \tag{25}$$

Here, $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$ is the classical Green tensor, which has to be determined from the equation

$$\boldsymbol{\nabla} \times \kappa(\mathbf{r},\omega) \boldsymbol{\nabla} \times \boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) - \frac{\omega^2}{c^2}\varepsilon(\mathbf{r},\omega)\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) = \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \tag{26}$$

together with the boundary condition at infinity, and $\underline{\hat{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r},\omega)$ is defined by

$$\underline{\hat{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r},\omega) = -i\omega\underline{\hat{\mathbf{P}}}_{\mathrm{N}}(\mathbf{r},\omega) + \boldsymbol{\nabla} \times \underline{\hat{\mathbf{M}}}_{\mathrm{N}}(\mathbf{r},\omega), \tag{27}$$

where $\underline{\hat{\mathbf{P}}}_{\mathrm{N}}(\mathbf{r},\omega)$ and $\underline{\hat{\mathbf{M}}}_{\mathrm{N}}(\mathbf{r},\omega)$ are, respectively, the (fluctuating) noise parts of the polarization $\underline{\hat{\mathbf{P}}}(\mathbf{r},\omega)$ and the magnetization $\underline{\hat{\mathbf{P}}}(\mathbf{r},\omega)$ in the frequency domain,

$$\underline{\hat{\mathbf{P}}}(\mathbf{r},\omega) = \varepsilon_0[\varepsilon(\mathbf{r},\omega)-1]\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) + \underline{\hat{\mathbf{P}}}_{\mathrm{N}}(\mathbf{r},\omega), \tag{28}$$

$$\underline{\hat{\mathbf{M}}}(\mathbf{r},\omega) = \kappa_0[1-\kappa(\mathbf{r},\omega)]\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) + \underline{\hat{\mathbf{M}}}_{\mathrm{N}}(\mathbf{r},\omega) \tag{29}$$

$[\kappa_0=\mu_0^{-1}, \kappa(\mathbf{r},\omega)=\mu^{-1}(\mathbf{r},\omega)]$.

The Green tensor [as well as $\varepsilon(\mathbf{r},\omega)$ and $\kappa(\mathbf{r},\omega)$] is holomorphic in the upper ω half-plane and has the "reality" property

$$\boldsymbol{G}(\mathbf{r},\mathbf{r}',-\omega^*) = \boldsymbol{G}^*(\mathbf{r},\mathbf{r}',\omega). \tag{30}$$

Moreover, it obeys the reciprocity relation

$$\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) = \boldsymbol{G}^{\mathsf{T}}(\mathbf{r}',\mathbf{r},\omega) \tag{31}$$

(the superscript T denotes matrix transposition) and the integral relation

$$\int d^3s \left\{ [\boldsymbol{G}(\mathbf{r},\mathbf{s},\omega) \times \overleftarrow{\boldsymbol{\nabla}}_s]\mathrm{Im}\,\kappa(\mathbf{s},\omega)[\boldsymbol{\nabla}_s \times \boldsymbol{G}^*(\mathbf{s},\mathbf{r}',\omega)] + \frac{\omega^2}{c^2}\boldsymbol{G}(\mathbf{r},\mathbf{s},\omega)\mathrm{Im}\,\varepsilon(\mathbf{s},\omega)\boldsymbol{G}^*(\mathbf{s},\mathbf{r}',\omega) \right\} = \mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega), \tag{32}$$

where the notation

$$\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) \times \overleftarrow{\boldsymbol{\nabla}}' = -[\boldsymbol{\nabla}' \times \boldsymbol{G}^{\mathsf{T}}(\mathbf{r},\mathbf{r}',\omega)]^{\mathsf{T}} \tag{33}$$

has been used.

According to Ref. [26], $\underline{\hat{\mathbf{P}}}_{\mathrm{N}}(\mathbf{r},\omega)$ and $\underline{\hat{\mathbf{M}}}_{\mathrm{N}}(\mathbf{r},\omega)$ can be related to bosonic fields $\hat{\mathbf{f}}_e(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}_m(\mathbf{r},\omega)$, respectively, in such a way that the correct (equal-time) commutation relations of the electromagnetic field operators are satisfied,

$$\underline{\hat{\mathbf{P}}}_{\mathrm{N}}(\mathbf{r},\omega) = i[\hbar\varepsilon_0\,\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)/\pi]^{1/2}\hat{\mathbf{f}}_e(\mathbf{r},\omega), \tag{34}$$

$$\underline{\hat{\mathbf{M}}}_{\mathrm{N}}(\mathbf{r},\omega) = [-\hbar\kappa_0\,\mathrm{Im}\,\kappa(\mathbf{r},\omega)/\pi]^{1/2}\hat{\mathbf{f}}_m(\mathbf{r},\omega), \tag{35}$$

$$[\hat{f}_{\lambda k}(\mathbf{r},\omega),\hat{f}^{\dagger}_{\lambda' l}(\mathbf{r},\omega)] = \delta_{kl}\delta_{\lambda\lambda'}\delta(\mathbf{r}-\mathbf{r}')\delta(\omega-\omega') \quad (36)$$

($\lambda = e,m$). Note that the $\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)$ play the role of the dynamical (canonical) variables of the theory. Combining Eqs. (22)–(25) with Eqs. (27), (34), and (35) yields the electric and induction fields in terms of the dynamical variables.

The charge and current densities that are subject to the Lorentz force are given by

$$\hat{\rho}(\mathbf{r}) = \int_0^\infty d\omega \hat{\underline{\rho}}(\mathbf{r},\omega) + \text{H.c.}, \quad (37)$$

$$\hat{\mathbf{j}}(\mathbf{r}) = \int_0^\infty d\omega \hat{\underline{\mathbf{j}}}(\mathbf{r},\omega) + \text{H.c.}, \quad (38)$$

where, according to Eqs. (5)–(8) ($\hat{\rho}_{\text{ext}}=0, \hat{\mathbf{j}}_{\text{ext}}=0$) together with Eqs. (27)–(29),

$$\hat{\underline{\rho}}(\mathbf{r},\omega) = -\varepsilon_0 \boldsymbol{\nabla}\{[\varepsilon(\mathbf{r},\omega)-1]\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega)\} + (i\omega)^{-1}\boldsymbol{\nabla}\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega) \quad (39)$$

and

$$\hat{\underline{\mathbf{j}}}(\mathbf{r},\omega) = -i\omega\varepsilon_0[\varepsilon(\mathbf{r},\omega)-1]\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) + \boldsymbol{\nabla}\times\{\kappa_0[1-\kappa(\mathbf{r},\omega)]\hat{\underline{\mathbf{B}}}(\mathbf{r},\omega)\} + \hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega). \quad (40)$$

Using the definitions of $\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega)$, $\hat{\underline{\mathbf{B}}}(\mathbf{r},\omega)$, and $\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega)$ together with the bosonic commutation relations for the fundamental fields $\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)$, one can prove (Appendix A) that

$$[\hat{\rho}(\mathbf{r}),\hat{\mathbf{E}}(\mathbf{r}')] = \mathbf{0}, \quad (41)$$

$$[\hat{\rho}(\mathbf{r}),\hat{\mathbf{B}}(\mathbf{r}')] = \mathbf{0}, \quad (42)$$

$$[\hat{j}_k(\mathbf{r}),\hat{B}_l(\mathbf{r}')] = 0, \quad (43)$$

and

$$[\hat{j}_k(\mathbf{r}),\hat{E}^\perp_l(\mathbf{r}')] = i\hbar\Omega^2_\varepsilon(\mathbf{r})\delta^\perp_{kl}(\mathbf{r}-\mathbf{r}'), \quad (44)$$

where the position-dependent plasma frequency $\Omega_\varepsilon(\mathbf{r})$ is defined by the asymptotic behavior of the permittivity for large ω in the upper half-plane according to $\varepsilon(\mathbf{r},\omega)\simeq 1 - \Omega^2_\varepsilon(\mathbf{r})/\omega^2$. The commutation relations (41)–(44) clearly show that $\hat{\rho}(\mathbf{r})$ and $\hat{\mathbf{j}}(\mathbf{r})$ really represent matter quantities. It is worth noting that Eq. (44) exactly corresponds to the equation obtained when—on the basis of a microscopic description—the current density is explicitly specified in terms of particle velocities (Appendix A).

If the field-matter system is in a number state [defined with respect to the number (density) operators $\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)$] such as the ground state, or an incoherent mixture of them such as a thermal state, then all one-time averages are evidently time-independent. Recalling the bosonic character of the fundamental fields $\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)$ [and $\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)$] and assuming them to be excited in thermal states, we easily obtain, in close analogy to Ref. [27],

$$\langle\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)\otimes\hat{\mathbf{f}}^\dagger_{\lambda'}(\mathbf{r}',\omega')\rangle = \frac{1}{2}\left[\coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)+1\right] \times \delta_{\lambda\lambda'}\delta(\omega-\omega')\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}'), \quad (45)$$

$$\langle\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)\otimes\hat{\mathbf{f}}_{\lambda'}(\mathbf{r}',\omega')\rangle = \frac{1}{2}\left[\coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)-1\right] \times \delta_{\lambda\lambda'}\delta(\omega-\omega')\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}'), \quad (46)$$

$$\langle\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)\otimes\hat{\mathbf{f}}_{\lambda'}(\mathbf{r}',\omega')\rangle = \mathbf{0}. \quad (47)$$

Making use of Eq. (27) together with Eqs. (34) and (35), we find that the correlation functions (45)–(47) imply the correlation functions

$$\langle\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega)\otimes\hat{\underline{\mathbf{j}}}^\dagger_{\text{N}}(\mathbf{r}',\omega')\rangle = \frac{\hbar}{2\pi\mu_0}\delta(\omega-\omega')\left[\coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)+1\right] \times\left\{\frac{\omega^2}{c^2}\sqrt{\text{Im}\,\varepsilon(\mathbf{r},\omega)}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\text{Im}\,\varepsilon(\mathbf{r}',\omega')} + \boldsymbol{\nabla}\times[\sqrt{\text{Im}\,\kappa(\mathbf{r},\omega)}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\text{Im}\,\kappa(\mathbf{r}',\omega')}]\times\overleftarrow{\boldsymbol{\nabla}}'\right\}, \quad (48)$$

$$\langle\hat{\underline{\mathbf{j}}}^\dagger_{\text{N}}(\mathbf{r},\omega)\otimes\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r}',\omega')\rangle = \frac{\hbar}{2\pi\mu_0}\delta(\omega-\omega')\left[\coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)-1\right] \times\left\{\frac{\omega^2}{c^2}\sqrt{\text{Im}\,\varepsilon(\mathbf{r},\omega)}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\text{Im}\,\varepsilon(\mathbf{r}',\omega')} + \boldsymbol{\nabla}\times[\sqrt{\text{Im}\,\kappa(\mathbf{r},\omega)}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\text{Im}\,\kappa(\mathbf{r}',\omega')}]\times\overleftarrow{\boldsymbol{\nabla}}'\right\}, \quad (49)$$

and

$$\langle\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega)\otimes\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r}',\omega')\rangle = \mathbf{0}. \quad (50)$$

Using Eqs. (22) and (23) together with Eqs. (24) and (25) and employing Eqs. (31), (32), and (48)–(50), we can calculate the thermal-equilibrium correlation functions of the electric field and the induction field to obtain

$$\langle\hat{\mathbf{E}}(\mathbf{r})\otimes\hat{\mathbf{E}}(\mathbf{r}')\rangle = \frac{\hbar\mu_0}{\pi}\int_0^\infty d\omega\omega^2\coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)\text{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega), \quad (51)$$

$$\langle\hat{\mathbf{B}}(\mathbf{r})\otimes\hat{\mathbf{B}}(\mathbf{r}')\rangle = -\frac{\hbar\mu_0}{\pi}\int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) \times\boldsymbol{\nabla}\times \mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}'. \quad (52)$$

Taking the limit $T\to 0$ (i.e., replacing the hyperbolic cotangent with unity) yields the respective ground-state correlation functions. Note that the correlation functions (51) and (52) inherit the reciprocity property according to Eq. (31).

Now we calculate the expectation value of the Lorentz force [which is Hermitian—recall Eqs. (41) and (43)],

$$\mathbf{F} = \int_V d^3r\langle\hat{\rho}\hat{\mathbf{E}} + \hat{\mathbf{j}}\times\hat{\mathbf{B}}\rangle, \quad (53)$$

where $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$, respectively, are defined by Eqs. (22) and (23) together with Eqs. (24) and (25), and $\hat{\rho}$ and $\hat{\mathbf{j}}$, respectively, are defined by Eqs. (37) and (38) together with Eqs. (39) and (40). Following the line suggested by classical electrodynamics, paying proper attention to operator symmetrization as well as regularization, and taking into account that the time derivative in the (quantum-mechanical version of) Eq. (14) does not contribute to the force, we find (Appendix B) that Eqs. (15) and (16) apply, where the (time-independent) stress tensor can be obtained, in agreement with the classical Eq. (12), from the quantum-mechanical expectation value

$$\boldsymbol{T}(\mathbf{r},\mathbf{r}') = \varepsilon_0\langle\hat{\mathbf{E}}(\mathbf{r})\otimes\hat{\mathbf{E}}(\mathbf{r}')\rangle + \mu_0^{-1}\langle\hat{\mathbf{B}}(\mathbf{r})\otimes\hat{\mathbf{B}}(\mathbf{r}')\rangle - \frac{1}{2}\boldsymbol{1}[\varepsilon_0\langle\hat{\mathbf{E}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r}')\rangle + \mu_0^{-1}\langle\hat{\mathbf{B}}(\mathbf{r})\hat{\mathbf{B}}(\mathbf{r}')\rangle] \quad (54)$$

in the limit $\mathbf{r}'\to\mathbf{r}$, where divergent bulk contributions are to be removed before taking the limit [recall the remark below Eq. (16)]. This is always possible if the body under study is embedded in a material environment that is homogeneous at least in the vicinity of the body. If this is not the case, special care and additional considerations are necessary, and it may happen that physically interpretable results can hardly be extracted. Note that in the calculation of the surface integral in Eq. (15) the "outer" values of the integrand should be used if ∂V is the interface between an *inhomogeneous* body embedded in a homogeneous environment (see Appendix B).

Inserting Eqs. (51) and (52) into Eq. (54) finally yields the stress tensor as

$$\boldsymbol{T}(\mathbf{r},\mathbf{r}) = \lim_{\mathbf{r}'\to\mathbf{r}}\left[\boldsymbol{\theta}(\mathbf{r},\mathbf{r}') - \frac{1}{2}\boldsymbol{1}\mathrm{Tr}\,\boldsymbol{\theta}(\mathbf{r},\mathbf{r}')\right], \quad (55)$$

where

$$\boldsymbol{\theta}(\mathbf{r},\mathbf{r}') = \frac{\hbar}{\pi}\int_0^\infty d\omega\coth\left(\frac{\hbar\omega}{2k_B T}\right)\left[\frac{\omega^2}{c^2}\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) - \boldsymbol{\nabla}\times\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}'\right]. \quad (56)$$

As expected, the permittivity $\varepsilon(\mathbf{r},\omega)$ and the permeability $\mu(\mathbf{r},\omega)$ do not appear explicitly in Eq. (56), but only via the Green tensor $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$. Having removed divergent bulk contributions, we may take the imaginary part of the whole integral instead of the integrand in Eq. (56) and rotate the integration contour in the usual way toward the imaginary frequency axis, on which the Green tensor is real [recall Eq. (30)]. In the zero-temperature limit, the result is simply

$$\boldsymbol{\theta}(\mathbf{r},\mathbf{r}') = -\frac{\hbar}{\pi}\int_0^\infty d\xi\left[\frac{\xi^2}{c^2}\boldsymbol{G}(\mathbf{r},\mathbf{r}',i\xi) + \boldsymbol{\nabla}\times\boldsymbol{G}(\mathbf{r},\mathbf{r}',i\xi)\times\overleftarrow{\boldsymbol{\nabla}}'\right]. \quad (57)$$

For nonzero temperatures, a sum over the poles of the hyperbolic cotangent (corresponding to the Matsubara frequencies) arises instead. It should be mentioned that the zero-frequency contribution to the resulting series can be problematic if the expression in the square brackets in Eq. (57) has a singularity there, which is the case when permittivities of Drude type (exhibiting a pole at zero frequency) are employed. In fact, this unpleasant feature expresses the conceptual limitations of a spatially local description of the material response to the electromagnetic field, which disregards spatial dispersion. It is known that, for materials with (almost) freely movable charge carriers, this can become an issue especially at low frequencies (large free path lengths). Extension of the quantization scheme to nonlocally responding materials would render it possible to include such materials in the calculation of Casimir forces in a consistent way.

IV. HARMONIC-OSCILLATOR MEDIUM

It is maybe illustrative to make contact with microscopic approaches to the problem. The simplest and most widely used model for describing linearly polarizable media is quite certainly the harmonic-oscillator model (inclusion of magnetic properties into the model is still scarce). To account for dissipation, the medium oscillators that are relevant to the linear interaction with the electromagnetic field—shortly referred to as medium oscillators—are also linearly coupled to (infinitely many) heat bath oscillators (e.g., phonon modes). The effect of the heat bath can then be adequately taken into account by including friction terms and associated noise forces in the equations of motion of the medium oscillators. On a coarse-grained time scale, the friction terms are commonly regarded as being local in time (Markov approximation), so that they can be characterized by simple damping constants. It should be noted that the requirement of limited time resolution implies that different, not strictly equivalent noise forces are acceptable in that regime (for details of damping theory and oscillator models, see, e.g., Refs. [28–31]).

In the context of the one-dimensional theory of the Casimir force on absorbing bodies, the harmonic-oscillator model has been used to study the interaction of damped medium oscillators with the transverse part of the quantized one-dimensional electromagnetic field, with special emphasis on homogeneous media [32]. Extending the one-dimensional theory to three dimensions, we begin with the Heisenberg equations of motion of the system in the form of

$$\dot{\hat{\mathbf{p}}}(\mathbf{r},t) = -m\omega_0^2\hat{\mathbf{s}}(\mathbf{r},t) - m\gamma\dot{\hat{\mathbf{s}}}(\mathbf{r},t) + e\hat{\mathbf{E}}(\mathbf{r},t) + \hat{\mathbf{F}}_{\mathrm{N}}(\mathbf{r},t), \tag{58}$$

$$\dot{\hat{\mathbf{s}}}(\mathbf{r},t) = \hat{\mathbf{p}}(\mathbf{r},t)/m, \tag{59}$$

$$\boldsymbol{\nabla}\times\hat{\mathbf{B}}(\mathbf{r},t) - \frac{1}{c^2}\dot{\hat{\mathbf{E}}}(\mathbf{r},t) = \mu_0\hat{\mathbf{j}}(\mathbf{r},t), \tag{60}$$

$$\boldsymbol{\nabla}\times\hat{\mathbf{E}}(\mathbf{r},t) = -\dot{\hat{\mathbf{B}}}(\mathbf{r},t), \tag{61}$$

where $\hat{\mathbf{s}}(\mathbf{r},t)$ and $\hat{\mathbf{p}}(\mathbf{r},t)$ are, respectively, the coordinate field and the momentum field of the medium oscillators, and

$$\hat{\mathbf{j}}(\mathbf{r},t) = e\,\eta(\mathbf{r})\dot{\hat{\mathbf{s}}}(\mathbf{r},t) \tag{62}$$

is the (model) current [$\eta(\mathbf{r})$, number density of the medium oscillators]. Further, $\hat{\mathbf{F}}_{\mathrm{N}}(\mathbf{r},t)$ is the Langevin noise force acting on the damped harmonic oscillators (γ, damping constant). In addition to the equations of motion Eqs. (58)–(61) and the definition (62), the commutation relations [28]

$$[\hat{\mathbf{s}}(\mathbf{r},t),\hat{\mathbf{p}}(\mathbf{r}',t)] = \frac{i\hbar}{\eta(\mathbf{r})}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \tag{63}$$

and

$$[\hat{\mathbf{F}}_{\mathrm{N}}(\mathbf{r},t),\hat{\mathbf{F}}_{\mathrm{N}}(\mathbf{r}',t')] = \frac{2m\gamma}{\eta(\mathbf{r})}i\hbar\,\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\frac{\partial\delta(t-t')}{\partial t} \tag{64}$$

together with the standard commutators of the electromagnetic field are required to specify the model. The commutator (64) ensures that Eq. (63) is preserved in time. Note that the proof given in Ref. [28] extends to the inhomogeneous case $\eta=\eta(\mathbf{r})$ considered here. It should be stressed that the number density $\eta(\mathbf{r})$ is not allowed to have zeros (nor infinities), otherwise Eqs. (63) and (64) were not well-defined. Further, note that

$$\hat{\mathbf{E}}(\mathbf{r},t) = \hat{\mathbf{E}}^{\parallel}(\mathbf{r},t) + \hat{\mathbf{E}}^{\perp}(\mathbf{r},t) \tag{65}$$

is the full electric field consisting of both longitudinal and transverse parts [38],

$$\hat{\mathbf{E}}^{\parallel(\perp)}(\mathbf{r},t) = \int d^3r'\,\boldsymbol{\delta}^{\parallel(\perp)}(\mathbf{r}-\mathbf{r}')\hat{\mathbf{E}}(\mathbf{r}',t). \tag{66}$$

The transverse part may be associated with a vector potential in the Coulomb gauge and expanded into orthogonal modes in the usual way. By contrast, the longitudinal part is not a dynamical electromagnetic field variable but is (nonlocally) determined by the oscillator field as

$$\hat{\mathbf{E}}^{\parallel}(\mathbf{r},t) = -e[\eta(\mathbf{r})\hat{\mathbf{s}}(\mathbf{r},t)]^{\parallel}/\varepsilon_0, \tag{67}$$

implying the conserved (model) charge density

$$\hat{\rho}(\mathbf{r},t) = -e\,\boldsymbol{\nabla}[\eta(\mathbf{r})\hat{\mathbf{s}}(\mathbf{r},t)], \tag{68}$$

which is consistent with Eqs. (60) and (62). Needless to say, Eqs. (62) and (68) do not actually represent the sources on a truly microscopic level but rather on a mesoscopic one, since the term "continuously varying field" applied to matter consisting of well-distinguishable constituents already indicates some averaging. However, complying with established terminology, we refer to this mesoscopic description as being microscopic throughout the paper. Note that *ab initio* calculations on a truly microscopic level would lead to time-ordered products in the treatment of the complicated interaction problem even in linear electrodynamics, because of the interaction with the dissipative system. However, if this interaction is treated in Born and (quasi-)Markov approximations, then the closed equations derived in this way no longer contain any time-ordered products.

As the system evolves toward its dressed ground state as $t\to\infty$, the model can be shown (Appendix C) to lead to the equal-time electromagnetic field correlation functions

$$\lim_{t\to\infty}\langle\hat{\mathbf{E}}(\mathbf{r},t)\otimes\hat{\mathbf{E}}(\mathbf{r}',t)\rangle = \frac{\hbar\mu_0}{\pi}\int_0^\infty d\omega\,\omega^2\,\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) \tag{69}$$

and

$$\lim_{t\to\infty}\langle\hat{\mathbf{B}}(\mathbf{r},t)\otimes\hat{\mathbf{B}}(\mathbf{r}',t)\rangle = -\frac{\hbar\mu_0}{\pi}\int_0^\infty d\omega\,\boldsymbol{\nabla}\times\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}' \tag{70}$$

if the heat bath that interacts with the medium oscillators is assumed to have zero temperature. Here, $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$ is the Green tensor that is the solution to Eq. (26), with $\kappa(\mathbf{r},\omega)$ and $\varepsilon(\mathbf{r},\omega)$, respectively, being the model-specific quantities $\kappa(\mathbf{r},\omega)\equiv 1$ and

$$\varepsilon(\mathbf{r},\omega) = 1 + \frac{e^2\eta(\mathbf{r})}{\varepsilon_0 m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}. \tag{71}$$

Obviously, Eqs. (69) and (70), which directly follow from the microscopic model under consideration, correspond exactly to Eqs. (51) and (52) in the zero-temperature limit.

The (steady-state) Lorentz force acting on the harmonic-oscillator matter in some space region V is given by

$$\mathbf{F} = \lim_{t\to\infty}\int_V d^3r\langle\hat{\rho}(\mathbf{r},t)\hat{\mathbf{E}}(\mathbf{r},t) + \hat{\mathbf{j}}(\mathbf{r},t)\times\hat{\mathbf{B}}(\mathbf{r},t)\rangle, \tag{72}$$

with $\hat{\rho}(\mathbf{r},t)$ and $\hat{\mathbf{j}}(\mathbf{r},t)$ from Eqs. (68) and (62), respectively. At this stage it not difficult to see that the procedure outlined in Appendix B yields $\mathbf{F}$ in the form of Eq. (15) together with Eq. (16), where the stress tensor has exactly the form of Eq. (55) together with Eq. (57). This result shows that the microscopic approach to the Casimir force fully confirms the macroscopic approach as given in Sec. III, where the calculations were based on the quantized macroscopic electromagnetic field, with the matter phenomenologically described in terms of Kramers-Kronig consistent response functions. Thus, the Casimir force acting on a macroscopic piece of matter may be viewed as "just" the (quantum) Lorentz force on the constituting charges and currents, which, in a macroscopic description, can be expressed in terms of the (induced and noise) polarization and magnetization—a conceptually straightforward and satisfactory point of view.

V. CASIMIR FORCE IN PLANAR STRUCTURES

Let us apply the theory to a planar magnetodielectric structure defined according to

$$\varepsilon(\mathbf{r},\omega) = \begin{cases} \varepsilon_-(z,\omega), & z < 0, \\ \varepsilon_j(\omega), & 0 < z < d_j, \\ \varepsilon_+(z,\omega), & z > d_j, \end{cases} \tag{73}$$

$$\mu(\mathbf{r},\omega) = \begin{cases} \mu_-(z,\omega), & z < 0, \\ \mu_j(\omega), & 0 < z < d_j, \\ \mu_+(z,\omega), & z > d_j. \end{cases} \tag{74}$$

To determine the Casimir stress in the interspace $0<z<d_j$, we need the Green tensor in Eq. (56) for both spatial arguments within the interspace $(0<z=z'<d_j)$. The Green tensor is well known and can be taken, e.g., from Ref. [33]. Since the transverse projection $\mathbf{q}$ of the wave vector is conserved and the polarizations $\sigma=s,p$ decouple, the scattering part of the Green tensor within the interspace can be expressed in terms of reflection coefficients $r_{j\pm}^{\sigma}=r_{j\pm}^{\sigma}(\omega,q)$ referring to reflection of waves at the right $(+)$ and left $(-)$ wall, respectively, as seen from the interspace. Explicit (recurrence) expressions for the reflection coefficients are available if the walls are multislab magnetodielectrics like Bragg mirrors [27,33,34]. (For continuous wall profiles, Riccati-type equations have to be solved [33].) In the simplest case of two homogeneous, semi-infinite walls, the coefficients $r_{j\pm}^{\sigma}$ reduce to the well-known Fresnel amplitudes. In the case first treated by Lifshitz [35], the interspace is empty and the walls are nonmagnetic.

A. Casimir stress within a nonempty interspace

For the sake of generality, we first leave the wall structure unspecified. By modifying the expression for the scattering part of the Green tensor given in Ref. [34] to also account for magnetic properties, from Eq. (55) together with Eq. (56) (without the bulk part of the Green tensor) it then follows that the relevant stress tensor element $T_{zz}(\mathbf{r},\mathbf{r})$ in the interspace $0<z<d_j$ can be given in the form of

$$T_{zz}(\mathbf{r},\mathbf{r}) = -\frac{\hbar}{8\pi^2}\int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \mathrm{Re}\int_0^\infty dqq\frac{\mu_j(\omega)}{\beta_j(\omega,q)}g_j(z,\omega,q) \tag{75}$$

$(q=|\mathbf{q}|)$, where the function $g_j(z,\omega,q)$, which in general depends on the position z within the interspace, reads

$$\begin{aligned} g_j(z,\omega,q) = {} & 2[\beta_j^2(1+n_j^{-2}) - q^2(1-n_j^{-2})]D_{js}^{-1}r_{j+}^s r_{j-}^s e^{2i\beta_j d_j} \\ & + 2[\beta_j^2(1+n_j^{-2}) + q^2(1-n_j^{-2})]D_{jp}^{-1}r_{j+}^p r_{j-}^p e^{2i\beta_j d_j} \\ & - (\beta_j^2+q^2)(1-n_j^{-2})D_{js}^{-1}[r_{j-}^s e^{2i\beta_j z} + r_{j+}^s e^{2i\beta_j(d_j-z)}] \\ & + (\beta_j^2+q^2)(1-n_j^{-2})D_{jp}^{-1}[r_-^p e^{2i\beta_j z} + r_{j+}^p e^{2i\beta_j(d_j-z)}], \end{aligned} \tag{76}$$

with the definitions

$$n_j^2 = n_j^2(\omega) = \varepsilon_j(\omega)\mu_j(\omega), \tag{77}$$

$$\beta_j = \beta_j(\omega,q) = (\omega^2 n_j^2/c^2 - q^2)^{1/2}, \tag{78}$$

$$D_{j\sigma} = D_{j\sigma}(\omega,q) = 1 - r_{j+}^{\sigma}r_{j-}^{\sigma}e^{2i\beta_j d_j}. \tag{79}$$

Note that the equations $D_{j\sigma}(\omega,q)=0$ determine, for real $\mathbf{q}$, the frequencies of the guided waves in the planar structure, which are of major interest in all "mode summation" approaches. (In the presence of material losses, however, these waves have complex frequencies and are not ordinary normal modes.) For practical reasons, it may be advantageous to transform the integral over real frequencies in Eq. (75) into an integral along the imaginary frequency axis by means of contour integral techniques [cf. Eqs. (55) and (57)]. In particular, in the zero-temperature limit, Eq. (75) may be rewritten as

$$T_{zz}(\mathbf{r},\mathbf{r}) = \frac{\hbar}{8\pi^2}\int_0^\infty d\xi\int_0^\infty dqq\frac{\mu_j(i\xi)}{i\beta_j(i\xi,q)}g_j(z,i\xi,q). \tag{80}$$

From the derivation it is obvious that the stress formula (75) [together with Eq. (76)] allows for an arbitrary linear, causal interspace medium. By contrast, Minkowski's stress tensor [Eq. (17)] leads to [4,27] $(\mu_j\equiv 1)$

$$T_{zz}^{(\mathrm{M})}(\mathbf{r},\mathbf{r}) = -\frac{\hbar}{2\pi^2}\int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \mathrm{Re}\int_0^\infty dqq\beta_j\sum_{\sigma=s,p}\frac{r_{j+}^{\sigma}r_{j-}^{\sigma}e^{2i\beta_j d_j}}{D_{j\sigma}}. \tag{81}$$

From Eq. (76) it is easily seen that for an empty interspace, i.e., $\varepsilon_j=\mu_j=1$, $g_j(z,\omega,q)$ becomes independent of z and simplifies to

$$g_j(z,\omega,q) \to g_j(\omega,q) = 4\beta_j^2\sum_{\sigma=s,p}\frac{r_{j+}^{\sigma}r_{j-}^{\sigma}e^{2i\beta_j d_j}}{D_{j\sigma}}. \tag{82}$$

In this case, and *only* in this case, Eq. (75) reduces to Eq. (81), from which in the case of semi-infinite (homogeneous) dielectric walls Lifshitz's well-known formula [35] can be recovered. As already mentioned, formulas of the type of Eq. (81) [which need not necessarily be derived within the stress tensor approach to the Casimir force] have been claimed to apply also to the case where the interspace is filled with dielectric material [3,5], at least if the material is nonabsorbing [4] (see also the textbooks [6–8] and references therein). Since $T_{zz}^{(\mathrm{M})}(\mathbf{r},\mathbf{r})$ does not depend on the position z within the interspace, application of Eq. (81) implies the very paradoxical result that the force acting on any slice of material selected within the interspace vanishes identically, regardless of the presence and arrangement of the remaining material (in particular, regardless of the yet unspecified walls). This unphysical result clearly shows that Eq. (81) cannot be valid if the interspace is not empty, not even if it may be justified to regard the interspace medium as nonabsorbing. In contrast, the stress $T_{zz}(\mathbf{r},\mathbf{r})$ obtained from Eq. (75) [together with Eq. (76)] is not uniform within an interspace if the interspace is filled with a medium. Hence it gives rise, in general, to a nonvanishing force on a slice of interspace material, and no paradox appears.

Let us return to the stress formula (75) [together with Eq. (76)]. It is not difficult to see that, for a nonempty interspace, the q integral in Eq. (75) fails to converge at $z=0$ and $z=d_j$, i.e., on the interfaces where the different materials are in immediate contact with each other. Mathematically, the reason for this divergence can be seen in the fact that the reflection coefficients obtained under the assumption of *infinite* lateral extension of the system do not approach zero as q tends to infinity. However, large values of q correspond to very oblique traveling waves. In any real planar setup of finite lateral extension, such high-q waves clearly do not contribute to the q integral at all; they are not reflected but walk off instead. Note that a divergence of exactly the same type already appears also in the standard case of an empty interspace in the limit $d_j \to 0$. In order to (approximately) take into account the finite lateral extension of an actual planar setup, an appropriately chosen cutoff value (depending on the lateral system size) for the reflection coefficients at high q values could be introduced, thereby rendering the q integral finite. Of course, a more satisfactory approach would be to abandon the translational invariance from the outset, which, however, leads to serious mathematical difficulties since waves with different polarizations and transverse wave vectors are then no longer decoupled.

Since, according to Eq. (15), the Casimir force acting on a body is given by the integral of the stress tensor over the surface enclosing the body, the stress tensor on its own is of less importance. What is really important is the integral force value over a closed surface. To obtain the force (per unit area) acting on a (multilayered) plate of infinite lateral extension, the stress on the two sides of the plate must be taken into account. As the example in Sec. V B shows, it may then happen that the parts of the stress tensor that diverge when the plate is approached from the two sides cancel each other out. In such a case, the Casimir force (per unit area) on a plate remains well defined even if its lateral extension is assumed to be infinite.

B. Casimir force on a plate in a nonempty cavity

In order to make contact with recent work on the Casimir force on bodies embedded in media [4], let us calculate the force acting at zero temperature on a homogeneous plate in a nonempty planar cavity, according to the five-region setup as sketched in Fig. 1. The cavity walls are labeled by $l=0$ and $l=4$, the plate by $l=2$, and the cavity regions that are filled with the medium the plate is embedded in are labeled by $l=1$ and $l=3$, with $\varepsilon(\omega)\equiv\varepsilon_1(\omega)=\varepsilon_3(\omega)$ and $\mu(\omega)\equiv\mu_1(\omega)=\mu_3(\omega)$. The total (volume) force per unit transverse area acting on the plate can be obtained by (vectorial) addition of the two force contributions from the two sides of the plate. Application of Eq. (80) then yields the total force per unit transverse area in the form of

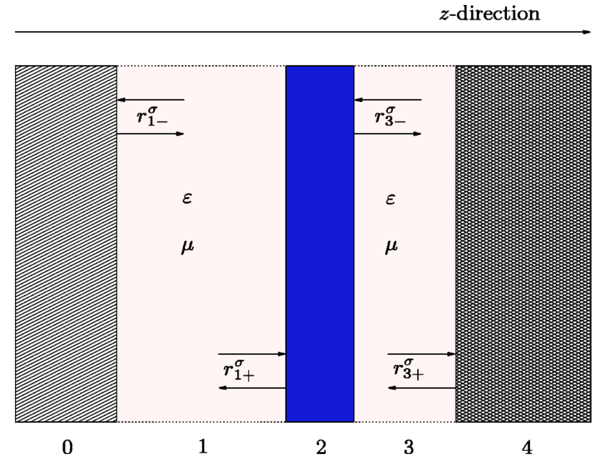

FIG. 1. Homogeneous plate embedded in a nonempty cavity. The cavity medium on the right and left sides of the plate is the same.

$$F=\frac{\hbar}{8\pi^2}\int_0^\infty d\xi\int_0^\infty dqq\frac{\mu(i\xi)}{i\beta(i\xi,q)}[g_3(0,i\xi,q)-g_1(d_1,i\xi,q)] \tag{83}$$

$[\beta(i\xi,q)\equiv\beta_1(i\xi,q)=\beta_3(i\xi,q)]$.

For a quantitative comparison with specific results obtained in Ref. [4] on the basis of Minkowski's stress tensor, we make the following simplifying assumptions. We assume that (i) all the reflection coefficients can be regarded as being almost constant, and (ii) the reflection coefficients r^σ_{1+} and r^σ_{3-} can be approximated by the (same) single-interface (Fresnel) reflection coefficient $r^\sigma_{1/2}$. Physically, these assumptions mean that (i) the distances d_1 and d_3 between the plate and the cavity walls must not be too small, and (ii) the plate must be thick enough. Moreover, the approximation scheme implies that the permittivity and the permeability of the medium the plate is embedded in can be replaced with their static values briefly referred to as ε and μ in the following, with $n=\sqrt{\varepsilon\mu}$ being the static refractive index. From Eq. (76) it then follows that the difference of the functions $g_3(0,i\xi,q)$ and $g_1(d_1,i\xi,q)$ appearing in Eq. (83) can be approximated according to

$$\begin{aligned}&g_3(0,i\xi,q)-g_1(d_1,i\xi,q)\\&\quad\simeq\sum_{\sigma=s,p}\Bigg\{2\left(\frac{1}{D_{3\sigma}}-\frac{1}{D_{1\sigma}}\right)\left[\beta^2\left(1+\frac{1}{n^2}\right)\right.\\&\quad\left.+\Delta_\sigma q^2\left(1-\frac{1}{n^2}\right)\right]+\Delta_\sigma(\beta^2+q^2)\left(1-\frac{1}{n^2}\right)\\&\quad\times\left[\frac{r^\sigma_{1/2}+r^\sigma_{3+}e^{2i\beta d_3}}{D_{3\sigma}}-\frac{r^\sigma_{1/2}+r^\sigma_{1-}e^{2i\beta d_1}}{D_{1\sigma}}\right]\Bigg\}\end{aligned} \tag{84}$$

$(\Delta_\sigma=\delta_{\sigma p}-\delta_{\sigma s})$, where

$$\frac{r_{3+}^{\sigma}e^{2i\beta d_3}}{D_{3\sigma}} - \frac{r_{1-}^{\sigma}e^{2i\beta d_1}}{D_{1\sigma}} = \frac{1-D_{3\sigma}}{r_{3-}^{\sigma}D_{3\sigma}} - \frac{1-D_{1\sigma}}{r_{1+}^{\sigma}D_{1\sigma}} \simeq \frac{1}{r_{1/2}^{\sigma}}\left(\frac{1}{D_{3\sigma}} - \frac{1}{D_{1\sigma}}\right). \quad (85)$$

Substituting Eq. (84) together with Eq. (85) into Eq. (83), we (approximately) obtain

$$F = \frac{\hbar}{8\pi^2}\int_0^\infty d\xi \int_0^\infty dqq\frac{\mu}{i\beta}\sum_{\sigma=s,p}\left(\frac{1}{D_{3\sigma}} - \frac{1}{D_{1\sigma}}\right) \times \left\{2\beta^2\left(1+\frac{1}{n^2}\right) - \Delta_\sigma\frac{\xi^2}{c^2}(n^2-1)\left(r_{1/2}^{\sigma} + \frac{1}{r_{1/2}^{\sigma}}\right) + 2\Delta_\sigma q^2\left(1-\frac{1}{n^2}\right)\right\}. \quad (86)$$

From an inspection of Eq. (86) it is seen that there is no divergence; the integrals are well behaved. It is worth noting that even without application of the approximation scheme, the integrals in the basic formula (83) do not diverge. The reason is that, for a chosen value of ξ, the coefficients $r_{3-}^{\sigma}(i\xi,q)$ and $r_{1+}^{\sigma}(i\xi,q)$ tend exponentially to the same single-interface Fresnel coefficient $r_{1/2}^{\sigma}(i\xi,q)$ as q goes to infinity, as may be seen from relations like

$$r_{1+}^{\sigma} = \frac{r_{1/2}^{\sigma} + e^{2i\beta_2 d_2}r_{2+}^{\sigma}}{1 + r_{1/2}^{\sigma}e^{2i\beta_2 d_2}r_{2+}^{\sigma}} \to r_{1/2}^{\sigma} \quad \text{if } q\to\infty, \quad (87)$$

$$r_{3-}^{\sigma} = \frac{r_{3/2}^{\sigma} + e^{2i\beta_2 d_2}r_{2-}^{\sigma}}{1 + r_{3/2}^{\sigma}e^{2i\beta_2 d_2}r_{2-}^{\sigma}} \to r_{3/2}^{\sigma} \quad \text{if } q\to\infty \quad (88)$$

together with the relation $r_{3/2}^{\sigma}=r_{1/2}^{\sigma}$ (valid for arbitrary values of ξ and q). Note that $i\beta_2\to-\infty$ if $q\to\infty$. As a consequence, the divergent contributions to the q integral in Eq. (83), which would arise from $g_3(0,i\xi,q)$ and $g_1(d_1,i\xi,q)$ separately, combine in a convergent fashion. Thus, for the setup under study, a q cutoff need not be introduced.

Let us return to Eq. (86). If the two walls and the plate are almost perfectly reflecting, i.e., $r_{1-}^{\sigma}\simeq r_{3+}^{\sigma}\simeq\Delta_\sigma$, $r_{1/2}^{\sigma}\simeq\Delta_\sigma$, then standard evaluation of the integrals leads to $(n=\sqrt{\varepsilon\mu})$

$$F = \frac{\hbar c\pi^2}{240}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{2}{3} + \frac{1}{3\varepsilon\mu}\right)\left(\frac{1}{d_3^4} - \frac{1}{d_1^4}\right). \quad (89)$$

In particular, if only one wall is present, say the left one, then Eq. (89) reduces to $(d_3\to\infty,\ d_1\equiv d)$

$$F = -\frac{\hbar c\pi^2}{240}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{2}{3} + \frac{1}{3\varepsilon\mu}\right)\frac{1}{d^4}, \quad (90)$$

which is the generalization of Casimir's well known formula [36] for the force between two almost perfectly reflecting plates separated by vacuum [$\mu=\varepsilon=1$ in Eq. (90)] to the case where the interspace between the plates is filled with a medium of static permeability μ and static permittivity ε.

In order to compare Eq. (89) with the force formula obtained on the basis of Minkowski's stress tensor, we note that the use of Minkowski's stress tensor for a nonmagnetic medium leads to [see Eqs. (3.6) and (3.7) in Ref. [4]]

$$F^{(\mathrm{M})} = -\frac{\hbar}{\pi^2}\int_0^\infty d\xi\int_0^\infty dqqi\beta\left(\frac{1}{e^{-2i\beta d_3}-1} - \frac{1}{e^{-2i\beta d_1}-1}\right) \quad (91)$$

in place of Eq. (86) with $\mu=1$. For an almost perfectly reflecting plate in a cavity with almost perfectly reflecting walls, standard evaluation of the integrals in Eq. (91) then yields, in place of Eq. (89),

$$F^{(\mathrm{M})} = \frac{\hbar c\pi^2}{240}\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{d_3^4} - \frac{1}{d_1^4}\right), \quad (92)$$

which in the limit $d_3\to\infty$ reduces to $(d_1\equiv d)$

$$F^{(\mathrm{M})} = -\frac{\hbar c\pi^2}{240}\frac{1}{\sqrt{\varepsilon}}\frac{1}{d^4}. \quad (93)$$

Note that Eq. (93) corresponds to the result derived in Ref. [37] by means of mode summation methods. Comparing Eq. (89) with Eq. (92) [or Eq. (90) with Eq. (93)], we see that

$$|F| \leqslant |F^{(\mathrm{M})}|, \quad (94)$$

i.e., the absolute value of the force is $(n>1)$ always smaller than that predicted from Minkowski's stress tensor. Introduction of a (polarizable) medium into the interspace is obviously associated with some screening of the plate, thereby reducing the force acting on it. Since the internal charges and currents of the interspace medium are not fully taken into account in a theory that is based on Minkowski's stress tensor or an equivalent formalism, the screening effect is underestimated and consequently the force calculated in this way is overestimated. Although the assumptions made to derive the results given above are rather restrictive, the comparison of Eq. (89) with Eq. (92) clearly shows that the correct inclusion of the medium into the theory can give rise to noticeable effects (see Fig. 2).

A consequence of the approximation scheme employed in this section is the appearance of the real values of the static permittivity and the static permeability of the interspace material in Eq. (89). However, the basic Eq. (83) is of course valid for arbitrary linear magnetodielectric media with Kramers-Kronig consistent permittivities and permeabilities. The influence of material dispersion and absorption comes into play when the distances d_1 and/or d_3 are decreased. The behavior of the permeability and the permittivity at nonzero frequencies becomes then important.

VI. SUMMARY AND CONCLUSIONS

On the basis of (i) the quantized macroscopic electromagnetic field in the presence of causal linear magnetodielectric media without spatial dispersion and (ii) the Lorentz force acting on the internal charges and currents of the medium, we have derived general expressions for the Casimir force acting on magnetodielectric bodies embedded in a common magnetodielectric medium. All the matter has been allowed for being dispersing and absorbing. Specializing to planar structures, we have generalized Lifshitz-type formulas (being valid for empty interspaces) to the case where the interspaces

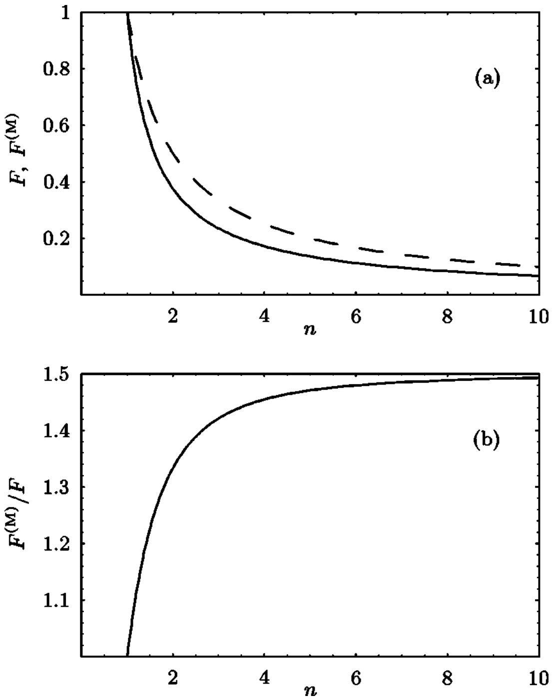

FIG. 2. (a) The Casimir force F given by Eq. (89) (solid curve) is shown as a function of the medium refractive index $n=\sqrt{\varepsilon}(\mu=1)$ for chosen distances d_1 and d_3. For comparison, $F^{(\mathrm{M})}$ given by Eq. (92) (dashed curve) is shown. (b) The ratio $F^{(\mathrm{M})}/F$ is shown as a function of the medium refractive index.

are filled with a magnetodielectric medium. In this context, we have analyzed the failure implied by basing the calculation of the Casimir force on Minkowski's stress tensor—a method that has been used in the literature but has never been proven correct. Interestingly, Lifshitz himself did not address nonempty interspaces in his seminal article [35].

For comparison reasons, we have studied in some detail the Casimir force acting on a homogeneous plate embedded in a medium in a planar cavity. Applying standard approximations such as high reflection, we have explicitly demonstrated that when the plate is embedded in a medium, then the force can noticeably differ from the result obtained on the basis of Minkowski's stress tensor. By the way, we have given the correct extension of Casimir's original formula for the force between two perfectly reflecting plates to the case where the interspace between the plates is filled with a medium.

In order to make contact with microscopic theories, we have also described the matter microscopically, by employing the model of damped harmonic oscillators, which is widely used for treating dielectric matter. Solving the quantum-mechanical equations of motion of the overall system (with the heat bath assumed in its ground state), we have calculated the Lorentz force acting on a chosen matter element. The result obtained in this way exactly corresponds to the general result obtained from the macroscopic approach. This clearly shows that the use of Minkowski's stress tensor to calculate the Casimir force is wrong in general, even if the matter may be regarded as being nonabsorbing.

Note added. Instead of Eq. (84), it may be advantageous to use the exact equation

$$g_3(0,i\xi,q)-g_1(d_1,i\xi,q) = \sum_{\sigma=s,p}\left\{2\left[\beta^2\left(1+\frac{1}{n^2}\right)+\Delta_\sigma q^2\left(1-\frac{1}{n^2}\right)\right]r^\sigma + \Delta_\sigma(\beta^2+q^2)\left(1-\frac{1}{n^2}\right)(1+r^{\sigma 2}-t^{\sigma 2})\right\} \times \frac{r^\sigma_{3+}e^{2i\beta d_3}-r^\sigma_{1-}e^{2i\beta d_1}}{N^\sigma}, \quad (95)$$

where

$$N^\sigma = 1-r^\sigma(r^\sigma_{1-}e^{2i\beta d_1}+r^\sigma_{3+}e^{2i\beta d_3})+(r^{\sigma 2}-t^{\sigma 2})r^\sigma_{1-}r^\sigma_{3+}e^{2i\beta(d_1+d_3)}, \quad (96)$$

with $r^\sigma \equiv r^\sigma_{1/3}=r^\sigma_{3/1}$ and $t^\sigma \equiv t^\sigma_{1/3}=t^\sigma_{3/1}$ being single-plate reflection and transmission coefficients, respectively. We thank Marin-Slobodan Tomaš for this suggestion.

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APPENDIX A: PROOF OF EQS. (41)–(44)

Using Eqs. (24)–(26), we express $\hat{\underline{\rho}}(\mathbf{r},\omega)$ and $\hat{\underline{\mathbf{j}}}(\mathbf{r},\omega)$ as defined by Eqs. (39) and (40), respectively, in terms of $\hat{\underline{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r},\omega)$ to obtain

$$\hat{\underline{\rho}}(\mathbf{r},\omega)=\frac{i\omega}{c^2}\boldsymbol{\nabla}\int d^3r'\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\hat{\underline{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r}',\omega), \quad (\mathrm{A1})$$

$$\hat{\underline{\mathbf{j}}}(\mathbf{r},\omega)=\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\right)\int d^3r'\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\hat{\underline{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r}',\omega). \quad (\mathrm{A2})$$

By combining Eqs. (27), (34), and (35) with the standard bosonic commutation relations for the fundamental fields $\hat{\mathbf{f}}_\lambda(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}^\dagger_\lambda(\mathbf{r},\omega)$, it is not difficult to show that $\hat{\underline{\mathbf{j}}}_{\mathrm{N}}(\mathbf{r},\omega)$ and $\hat{\underline{\mathbf{j}}}^\dagger_{\mathrm{N}}(\mathbf{r},\omega)$ obey the commutation relation

$$[\hat{\underline{j}}_{\mathrm{N}k}(\mathbf{r},\omega),\hat{\underline{j}}^\dagger_{\mathrm{N}l}(\mathbf{r}',\omega')] = \frac{\hbar}{\mu_0\pi}\delta(\omega-\omega')\left[\frac{\omega^2}{c^2}\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)}\times\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega')}+\boldsymbol{\nabla}\times\sqrt{\mathrm{Im}\,\kappa(\mathbf{r},\omega)}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\sqrt{\mathrm{Im}\,\kappa(\mathbf{r}',\omega')}\times\overleftarrow{\boldsymbol{\nabla}}'\right]_{kl}. \quad (\mathrm{A3})$$

From Eqs. (24), (25), (A1), and (A2) together with the commutation relation (A3), we derive, on recalling the Green-tensor relations (31) and (32),

$$[\underline{\hat{\rho}}(\mathbf{r},\omega),\underline{\hat{\mathbf{E}}}^{\dagger}(\mathbf{r}',\omega')] = \frac{\hbar}{\pi}\frac{\omega^2}{c^2}\delta(\omega-\omega')\,\boldsymbol{\nabla}\ \mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) = -[\underline{\hat{\rho}}^{\dagger}(\mathbf{r},\omega),\underline{\hat{\mathbf{E}}}(\mathbf{r}',\omega')], \tag{A4}$$

$$[\underline{\hat{j}}_k(\mathbf{r},\omega),\underline{\hat{B}}_l^{\dagger}(\mathbf{r}',\omega')] = -\frac{\hbar}{\pi}\delta(\omega-\omega')\left[\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\right)\times\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}'\right]_{kl} = -[\underline{\hat{j}}_k^{\dagger}(\mathbf{r},\omega),\underline{\hat{B}}_l(\mathbf{r}',\omega')], \tag{A5}$$

$$[\underline{\hat{\rho}}(\mathbf{r},\omega),\underline{\hat{\mathbf{B}}}^{\dagger}(\mathbf{r}',\omega')] = -\frac{\hbar}{\pi}\frac{i\omega}{c^2}\delta(\omega-\omega')\,\boldsymbol{\nabla}\,\mathrm{Im}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}' = [\underline{\hat{\rho}}^{\dagger}(\mathbf{r},\omega),\underline{\hat{\mathbf{B}}}(\mathbf{r}',\omega')], \tag{A6}$$

$$[\underline{\hat{j}}_k(\mathbf{r},\omega),\underline{\hat{E}}_l^{\perp\dagger}(\mathbf{r}',\omega')] = -\frac{\hbar}{\pi}i\omega\delta(\omega-\omega')\left[\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\right)\times\mathrm{Im}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\right]_{kl} = [\underline{\hat{j}}_k^{\dagger}(\mathbf{r},\omega),\underline{\hat{E}}_l^{\perp}(\mathbf{r}',\omega')], \tag{A7}$$

where

$$\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega) = \int d^3s\,\boldsymbol{G}(\mathbf{r},\mathbf{s},\omega)\,\boldsymbol{\delta}^{\perp}(\mathbf{s}-\mathbf{r}'). \tag{A8}$$

Note that in Eq. (A6), $\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)$ may be replaced with $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$, because of the operation $\times\overleftarrow{\boldsymbol{\nabla}}'$.

Equations (A4) and (A5) obviously imply the commutation relations

$$[\hat{\rho}(\mathbf{r}),\hat{\mathbf{E}}(\mathbf{r}')] = \int_0^{\infty}d\omega\int_0^{\infty}d\omega'\{[\underline{\hat{\rho}}(\mathbf{r},\omega),\underline{\hat{\mathbf{E}}}^{\dagger}(\mathbf{r}',\omega')] + [\underline{\hat{\rho}}^{\dagger}(\mathbf{r},\omega),\underline{\hat{\mathbf{E}}}(\mathbf{r}',\omega')]\} = \mathbf{0} \tag{A9}$$

and

$$[\hat{j}_k(\mathbf{r}),\hat{B}_l(\mathbf{r}')] = \int_0^{\infty}d\omega\int_0^{\infty}d\omega'\{[\underline{\hat{j}}_k(\mathbf{r},\omega),\underline{\hat{\mathbf{B}}}_l^{\dagger}(\mathbf{r}',\omega')] + [\underline{\hat{j}}_k^{\dagger}(\mathbf{r},\omega),\underline{\hat{B}}_l(\mathbf{r}',\omega')]\} = \mathbf{0}, \tag{A10}$$

and hence Eqs. (41) and (43) are seen to hold. Note in particular that the commutation relation $[\hat{\rho}(\mathbf{r}),\hat{\mathbf{E}}^{\perp}(\mathbf{r}')]=0$ is valid. From Eqs. (A6) and (A7), respectively, it follows that

$$[\hat{\rho}(\mathbf{r}),\hat{\mathbf{B}}(\mathbf{r}')] = -\frac{2i\hbar}{\pi c^2}\int_0^{\infty}d\omega\omega\,\boldsymbol{\nabla}\ \mathrm{Im}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}' \tag{A11}$$

and

$$[\hat{j}_k(\mathbf{r}),\hat{E}_l^{\perp}(\mathbf{r}')] = -\frac{2i\hbar}{\pi}\int_0^{\infty}d\omega\omega\left[\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\right)\mathrm{Im}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\right]_{kl}. \tag{A12}$$

To further evaluate the integrals in Eqs. (A11) and (A12), we recall the asymptotic behavior of $\varepsilon(\mathbf{r},\omega)$ and $\kappa(\mathbf{r},\omega)$ for large ω in the upper half-plane, viz.,

$$\varepsilon(\mathbf{r},\omega) \simeq 1-\frac{\Omega_{\varepsilon}^2(\mathbf{r})}{\omega^2}, \tag{A13}$$

$$\kappa(\mathbf{r},\omega) \simeq 1+\frac{\Omega_{\kappa}^2(\mathbf{r})}{\omega^2}. \tag{A14}$$

Substituting Eqs. (A13) and (A14) into Eq. (26), we easily see that the Green tensor asymptotically behaves like

$$\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) \simeq -\frac{c^2}{\omega^2}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \tag{A15}$$

for large ω in the upper half-plane. Thus, on recalling Eq. (30) and the holomorphic behavior of the Green tensor, we may evaluate the integral in Eq. (A11) to prove Eq. (42),

$$\begin{aligned}[\hat{\rho}(\mathbf{r}),\hat{\mathbf{B}}(\mathbf{r}')] &= -\frac{\hbar}{\pi c^2}\mathrm{P}\int_{-\infty}^{\infty}d\omega\omega\,\boldsymbol{\nabla}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}' \\ &= -\frac{\hbar}{\pi c^2}\int_{C}d\omega\omega\,\boldsymbol{\nabla}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\times\overleftarrow{\boldsymbol{\nabla}}' \\ &= -i\hbar\,\boldsymbol{\nabla}\,\boldsymbol{\delta}^{\perp}(\mathbf{r}-\mathbf{r}')\times\overleftarrow{\boldsymbol{\nabla}}' = \mathbf{0}\end{aligned} \tag{A16}$$

(P denotes principal value). Here, we have replaced the principal value integral along the real frequency axis by a contour ($\mathcal{C}$) integral over an infinitely large semicircle in the upper half-plane and have used Eq. (A15). Note that there is no extra pole contribution from $\omega=0$ [26]. To evaluate Eq. (A12), we take into account that, according to Eq. (26), the relation

$$\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\right)\mathrm{Im}\,\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega) = \mathrm{Im}\left\{\frac{\omega^2}{c^2}[\varepsilon(\mathbf{r},\omega)-1]\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega) + \boldsymbol{\nabla}\times[1-\kappa(\mathbf{r},\omega)]\,\boldsymbol{\nabla}\times\boldsymbol{G}^{\perp}(\mathbf{r},\mathbf{r}',\omega)\right\} \tag{A17}$$

may be used on the real ω axis. Inserting this relation into Eq. (A12) and recalling general properties of $\varepsilon(\mathbf{r},\omega)$ and $\kappa(\mathbf{r},\omega)$, we see that the evaluation of Eq. (A12) can be done in exactly the same way as the evaluation of Eq. (A11).

Thus, making use of Eqs. (A13)–(A15), we derive

$$[\hat{j}_k(\mathbf{r}),\hat{E}_l^{\perp}(\mathbf{r}')] = -\frac{\hbar}{\pi}\int_C d\omega\omega\frac{\omega^2}{c^2}[\varepsilon(\mathbf{r},\omega)-1]G_{kl}^{\perp}(\mathbf{r},\mathbf{r}',\omega) = i\hbar\Omega_\varepsilon^2(\mathbf{r})\delta_{kl}^{\perp}(\mathbf{r}-\mathbf{r}'), \tag{A18}$$

which is Eq. (44).

For a consistency check of the commutation relation (A18), let us consider a set of atoms, with each of them having one valence electron (e, charge; m, mass). Let $\mathbf{r}_A$ be the (fixed) positions and $\hat{\mathbf{s}}_A$ the relative coordinates of the electrons. The microscopic (electron) current density is then given by

$$\hat{\mathbf{j}}(\mathbf{r}) = e\sum_A \dot{\hat{\mathbf{s}}}_A\delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A). \tag{A19}$$

By assuming minimal coupling and Coulomb gauge, the canonical momenta of the electrons commute with the vector potential $\hat{\mathbf{A}}(\mathbf{r})$, whose conjugate momentum field is $-\varepsilon_0\hat{\mathbf{E}}^{\perp}(\mathbf{r})$. Hence, we derive

$$\begin{aligned}[\hat{j}_k(\mathbf{r}),\hat{E}_l^{\perp}(\mathbf{r}')] &= -\frac{e^2}{m}\sum_A \delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A)[\hat{A}_k(\mathbf{r}_A+\hat{\mathbf{s}}_A),\hat{E}_l^{\perp}(\mathbf{r}')]\\ &= \frac{i\hbar}{\varepsilon_0}\frac{e^2}{m}\sum_A \delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A)\delta_{kl}^{\perp}(\mathbf{r}_A+\hat{\mathbf{s}}_A-\mathbf{r}')\\ &= \frac{i\hbar}{\varepsilon_0}\frac{e^2}{m}\sum_A \delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A)\delta_{kl}^{\perp}(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{A20}$$

In the macroscopic theory, the sum of the δ functions in Eq. (A20) is expected to be replaced according to

$$\sum_A \delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A) \mapsto \sum_A \Delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A), \tag{A21}$$

where $\Delta(\mathbf{r})$ is a well-behaved function with unit integral, $\int d^3r\Delta(\mathbf{r})=1$. Further, in order to produce reasonable coarse-graining, $\Delta(\mathbf{r})$ must be sufficiently flat so that the change of $\Delta(\mathbf{r})$ on atomic length scales can be regarded as being negligibly small. With the $\hat{\mathbf{s}}_A$ acting on well localized electronic bound states, we may hence write

$$\Delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A) \simeq \Delta(\mathbf{r}-\mathbf{r}_A). \tag{A22}$$

Thus,

$$\sum_A \Delta(\mathbf{r}-\mathbf{r}_A-\hat{\mathbf{s}}_A) \simeq \sum_A \Delta(\mathbf{r}-\mathbf{r}_A) = \eta(\mathbf{r}), \tag{A23}$$

where $\eta(\mathbf{r})$ is the number density $\eta(\mathbf{r})$ of the atoms, and the macroscopic version of Eq. (A20) reads

$$[\hat{j}_k(\mathbf{r}),\hat{E}_l^{\perp}(\mathbf{r}')] = \frac{i\hbar}{\varepsilon_0}\frac{e^2}{m}\eta(\mathbf{r})\delta_{kl}^{\perp}(\mathbf{r}-\mathbf{r}'). \tag{A24}$$

From a comparison of Eq. (A24) with Eq. (A18), the relation

$$\Omega_\varepsilon^2(\mathbf{r}) = \frac{e^2\eta(\mathbf{r})}{\varepsilon_0 m} \tag{A25}$$

is suggested to be valid, which is in full agreement with the harmonic-oscillator model permittivity given by Eq. (71).

APPENDIX B: QUANTUM LORENTZ FORCE

Using Maxwell's equations (1)–(4) (promoted to operator equations) together with the commutation relations (41) and (43) and relations of the type

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{r})\times[\nabla\times\hat{\mathbf{E}}(\mathbf{r})] &= -[\nabla\times\hat{\mathbf{E}}(\mathbf{r})]\times\hat{\mathbf{E}}(\mathbf{r})\\ &= \nabla\left[\frac{1}{2}\hat{\mathbf{E}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r}) - \hat{\mathbf{E}}(\mathbf{r})\otimes\hat{\mathbf{E}}(\mathbf{r})\right]\\ &\quad - [\nabla\hat{\mathbf{E}}(\mathbf{r})]\hat{\mathbf{E}}(\mathbf{r}),\end{aligned} \tag{B1}$$

we derive

$$\hat{\rho}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r}) + \hat{\mathbf{j}}(\mathbf{r})\times\hat{\mathbf{B}}(\mathbf{r}) - [\nabla\hat{\boldsymbol{T}}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}} - [\nabla'\hat{\boldsymbol{T}}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}} = \varepsilon_0\begin{cases}\dfrac{\partial}{\partial t}[\hat{\mathbf{B}}(\mathbf{r}')\times\hat{\mathbf{E}}(\mathbf{r})]_{\mathbf{r}'=\mathbf{r}},\\ -\dfrac{\partial}{\partial t}[\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{B}}(\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}},\end{cases} \tag{B2}$$

where

$$\hat{\boldsymbol{T}}(\mathbf{r},\mathbf{r}') = \varepsilon_0\left[\hat{\mathbf{E}}(\mathbf{r})\otimes\hat{\mathbf{E}}(\mathbf{r}') - \frac{1}{2}\boldsymbol{1}\,\hat{\mathbf{E}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r}')\right] + \mu_0^{-1}\left[\hat{\mathbf{B}}(\mathbf{r})\otimes\hat{\mathbf{B}}(\mathbf{r}') - \frac{1}{2}\boldsymbol{1}\,\hat{\mathbf{B}}(\mathbf{r})\hat{\mathbf{B}}(\mathbf{r}')\right] \tag{B3}$$

is a reciprocal operator function of two spatial variables,

$$\hat{\boldsymbol{T}}(\mathbf{r},\mathbf{r}') = \hat{\boldsymbol{T}}^{\mathsf{T}}(\mathbf{r}',\mathbf{r}), \tag{B4}$$

because of the commutation relations

$$[\hat{\mathbf{E}}(\mathbf{r}),\hat{\mathbf{E}}(\mathbf{r}')] = \mathbf{0} = [\hat{\mathbf{B}}(\mathbf{r}),\hat{\mathbf{B}}(\mathbf{r}')]. \tag{B5}$$

Since the left-hand side of Eq. (B2) is Hermitian, so is either of the two alternative right-hand sides, which means that symmetrization is not necessary. Thus, Eq. (11) is also valid as an operator equation, and the steady-state Eqs. (15) and (16) apply, with the stress tensor being defined by Eq. (54) in the limit $\mathbf{r}'\to\mathbf{r}$.

To perform the limit, we write the force acting on some space region V in the form of

$$\mathbf{F} = \lim_{\epsilon\to 0}\int_V d^3r\int d^3r'\,\delta_\epsilon(\mathbf{r}-\mathbf{r}') \times \{[\nabla\boldsymbol{T}(\mathbf{r},\mathbf{r}')] + [\nabla'\boldsymbol{T}(\mathbf{r},\mathbf{r}')]\}, \tag{B6}$$

where $\delta_\epsilon(\mathbf{r}-\mathbf{r}')$ approaches $\delta(\mathbf{r}-\mathbf{r}')$ as ϵ tends to zero. For instance, one could choose

$$\delta_\epsilon(\mathbf{r}) = (4\pi\epsilon^2)^{-1}\delta(|\mathbf{r}|-\epsilon), \tag{B7}$$

which corresponds to an average over a spherical surface of radius ϵ. Let us first consider the case in which the material properties are homogeneous everywhere, except at the surface of the volume V, where they may change abruptly. The function $\boldsymbol{T}(\mathbf{r},\mathbf{r}')$ can then be uniquely decomposed into a bulk part, which is divergent at $\mathbf{r}'=\mathbf{r}$, and a scattering part, which is well behaved at $\mathbf{r}'=\mathbf{r}$, and we may write

$$\nabla \boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r}) = [\nabla \boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}} + [\nabla' \boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}. \tag{B8}$$

For the scattering part, the limit $\epsilon \to 0$ simply restores the δ function, so Eq. (B6) becomes

$$\mathbf{F} = \int_V d^3r \, \nabla \boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r}) + \lim_{\epsilon\to 0}\int_V d^3r \int d^3r' \, \delta_\epsilon(\mathbf{r}-\mathbf{r}') \times \{[\nabla \boldsymbol{T}^{(\text{bulk})}(\mathbf{r},\mathbf{r}')] + [\nabla' \boldsymbol{T}^{(\text{bulk})}(\mathbf{r},\mathbf{r}')]\}. \tag{B9}$$

The second term on the right-hand side of Eq. (B9), which arises from the bulk part, vanishes, as can be seen from the following argument [39]. Since the bulk part is a function of $\mathbf{r}-\mathbf{r}'$, it follows that

$$\int_V d^3r \int d^3r' \, \delta_\epsilon(\mathbf{r}-\mathbf{r}') \, \nabla \boldsymbol{T}^{(\text{bulk})}(\mathbf{r},\mathbf{r}') = V\mathbf{b}(\epsilon), \tag{B10}$$

where $\mathbf{b}$ is some vector that depends only on the parameter ϵ, and in this way selects, somewhat artificially, a particular direction in space. However, the bulk part corresponds to a setup where the whole space is filled with homogeneous and isotropic material, implying that such a preferred direction does not exist, and we can conclude that $\lim_{\epsilon\to 0}\mathbf{b}(\epsilon)=0$. To apply the divergence theorem to the first term on the right-hand side of Eq. (B9) and transform the volume integral into a surface integral, we note that if the material properties change discontinuously at the surface ∂V of the chosen volume V [cf. Eqs. (39) and (40)], then $\boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r})$ is also discontinuous there. In view of the macroscopic description, it is clear that the material properties can be regarded as changing continuously across a sufficiently thin boundary layer. To include the net change across such a boundary layer, the "outer" values of the integrand should be taken (indicated by ∂V_+),

$$\mathbf{F} = \int_{\partial V_+} d\mathbf{a} \boldsymbol{T}^{(\text{scat})}(\mathbf{r},\mathbf{r}). \tag{B11}$$

In order to establish the validity of Eq. (B11) for the more general case of varying material properties inside the chosen space region (whose vicinity is again assumed to be homogeneous), one has to return to Eq. (B6) and decompose V including a thin boundary layer as described above into sufficiently small, nonintersecting cells V_i. Summing over all cells, one can then show, by using similar arguments as above, that in the limit of vanishingly small cells, $V_i \to 0$, Eq. (B11) is obtained.

APPENDIX C: PROOF OF EQS. (69) AND (70)

Combination of Eq. (58) with Eq. (59) yields the second-order differential equation

$$m\ddot{\hat{\mathbf{s}}}(\mathbf{r},t) = -m\omega_0^2\hat{\mathbf{s}}(\mathbf{r},t) - m\gamma\dot{\hat{\mathbf{s}}}(\mathbf{r},t) + e\hat{\mathbf{E}}(\mathbf{r},t) + \hat{\mathbf{F}}_{\text{N}}(\mathbf{r},t), \tag{C1}$$

and combination of Eq. (60) with Eqs. (61) and (62) leads to

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\mathbf{E}}(\mathbf{r},t) + \nabla\times\nabla\times\hat{\mathbf{E}}(\mathbf{r},t) = -e\mu_0\eta(\mathbf{r})\ddot{\hat{\mathbf{s}}}(\mathbf{r},t). \tag{C2}$$

We are interested in the solution to Eqs. (C1) and (C2) which is reached in the limit $t\to\infty$, thereby being independent of the initial conditions. We may represent it in terms of Fourier integrals according to

$$f(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\underline{f}(\omega) \Leftrightarrow \underline{f}(\omega) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{i\omega t}f(t). \tag{C3}$$

Note that the ω integrals should be treated as principal value integrals (with respect to $\omega=0$) if necessary. From Eqs. (C1) and (C2) it follows that the Fourier transforms $\hat{\underline{\mathbf{s}}}(\mathbf{r},\omega)$ and $\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega)$ of $\hat{\mathbf{s}}(\mathbf{r},t)$ and $\hat{\mathbf{E}}(\mathbf{r},t)$, respectively, are determined by

$$\hat{\underline{\mathbf{s}}}(\mathbf{r},\omega) = [m\omega_0^2 - m\omega^2 - im\gamma\omega]^{-1}[\hat{\underline{\mathbf{F}}}_{\text{N}}(\mathbf{r},\omega) + e\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega)] \tag{C4}$$

and

$$\nabla\times\nabla\times\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) = e\mu_0\eta(\mathbf{r})\omega^2\hat{\underline{\mathbf{s}}}(\mathbf{r},\omega). \tag{C5}$$

Substituting $\hat{\underline{\mathbf{s}}}(\mathbf{r},\omega)$ from Eq. (C4) into Eq. (C5) and rearranging, we obtain

$$\nabla\times\nabla\times\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\varepsilon(\mathbf{r},\omega)\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) = i\mu_0\omega\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega), \tag{C6}$$

where $\varepsilon(\mathbf{r},\omega)$, which is given by Eq. (71), defines the permittivity of the harmonic-oscillator medium, and

$$\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon_0}{e}[\varepsilon(\mathbf{r},\omega)-1]\hat{\underline{\mathbf{F}}}_{\text{N}}(\mathbf{r},\omega) \tag{C7}$$

is the current density associated with the Langevin force. The unique inversion of Eq. (C6) is

$$\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega) = i\mu_0\omega\int d^3r' \boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)\hat{\underline{\mathbf{j}}}_{\text{N}}(\mathbf{r}',\omega), \tag{C8}$$

where $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$ is the Green tensor that, for $\kappa(\mathbf{r},\omega)=1$ and $\varepsilon(\mathbf{r},\omega)$ from Eq. (71), solves Eq. (26) together with the boundary condition at infinity.

To prove Eq. (69), we write

$$\lim_{t\to\infty}\langle\hat{\mathbf{E}}(\mathbf{r},t)\otimes\hat{\mathbf{E}}(\mathbf{r}',t)\rangle = \lim_{t\to\infty}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega' e^{-i(\omega+\omega')t} \times\langle\hat{\underline{\mathbf{E}}}(\mathbf{r},\omega)\otimes\hat{\underline{\mathbf{E}}}(\mathbf{r}',\omega')\rangle, \tag{C9}$$

where, according to Eq. (C8) [together with Eq. (31)],

$$\langle \underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) \otimes \underline{\hat{\mathbf{E}}}(\mathbf{r}',\omega')\rangle = -\mu_0^2\omega\omega' \int d^3s \int d^3s' \boldsymbol{G}(\mathbf{r},\mathbf{s},\omega) \times \langle \underline{\hat{\mathbf{J}}}_{\mathrm{N}}(\mathbf{s},\omega) \otimes \underline{\hat{\mathbf{J}}}_{\mathrm{N}}(\mathbf{s}',\omega')\rangle \boldsymbol{G}(\mathbf{s}',\mathbf{r}',\omega'). \quad \text{(C10)}$$

If the heat bath is in the vacuum state, then

$$\langle \underline{\hat{\mathbf{F}}}_{\mathrm{N}}(\mathbf{r},\omega) \otimes \underline{\hat{\mathbf{F}}}_{\mathrm{N}}(\mathbf{r}',\omega')\rangle = \frac{m\gamma\hbar}{\pi\eta(\mathbf{r})}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\int_0^\infty d\omega''\omega''\delta(\omega''-\omega)\delta(\omega''+\omega') \quad \text{(C11)}$$

holds [28], and we find, on recalling Eq. (C7),

$$\langle \underline{\hat{\mathbf{J}}}_{\mathrm{N}}(\mathbf{s},\omega) \otimes \underline{\hat{\mathbf{J}}}_{\mathrm{N}}(\mathbf{s}',\omega')\rangle = -\frac{\omega\omega'\varepsilon_0^2}{e^2}[\varepsilon(\mathbf{s},\omega)-1][\varepsilon(\mathbf{s}',\omega')-1] \times \frac{m\gamma\hbar}{\pi\eta(\mathbf{s})}\boldsymbol{\delta}(\mathbf{s}-\mathbf{s}')\int_0^\infty d\omega''\omega'' \times \delta(\omega''-\omega)\delta(\omega''+\omega'). \quad \text{(C12)}$$

Combining Eqs. (C9), (C10), and (C12), we derive

$$\lim_{t\to\infty}\langle \hat{\mathbf{E}}(\mathbf{r},t) \otimes \hat{\mathbf{E}}(\mathbf{r}',t)\rangle = \frac{m\gamma\hbar}{\pi e^2 c^4}\int_0^\infty d\omega\omega^5 \int \mathrm{d}^3s \boldsymbol{G}(\mathbf{r},\mathbf{s},\omega) \times \frac{[\varepsilon(\mathbf{s},\omega)-1][\varepsilon(\mathbf{s},-\omega)-1]}{\eta(\mathbf{s})}\boldsymbol{G}(\mathbf{s},\mathbf{r}',-\omega). \quad \text{(C13)}$$

From Eq. (71) it follows that the relation

$$\frac{[\varepsilon(\mathbf{s},\omega)-1][\varepsilon(\mathbf{s},-\omega)-1]}{\eta(\mathbf{s})} = \frac{e^2}{\varepsilon_0 m\gamma}\frac{\mathrm{Im}\,\varepsilon(\mathbf{s},\omega)}{\omega} \quad \text{(C14)}$$

is valid for real ω. Hence, we may rewrite Eq. (C13) as

$$\lim_{t\to\infty}\langle \hat{\mathbf{E}}(\mathbf{r},t) \otimes \hat{\mathbf{E}}(\mathbf{r}',t)\rangle = \frac{\hbar\mu_0}{\pi}\int_0^\infty d\omega\frac{\omega^4}{c^2}\int d^3s\boldsymbol{G}(\mathbf{r},\mathbf{s},\omega) \times \mathrm{Im}\,\varepsilon(\mathbf{s},\omega)\boldsymbol{G}(\mathbf{s},\mathbf{r}',-\omega), \quad \text{(C15)}$$

which by means of Eqs. (30) and (32) eventually leads to Eq. (69).

To calculate

$$\lim_{t\to\infty}\langle \hat{\mathbf{B}}(\mathbf{r},t) \otimes \hat{\mathbf{B}}(\mathbf{r}',t)\rangle = \lim_{t\to\infty}\int_{-\infty}^\infty d\omega\int_{-\infty}^\infty d\omega' e^{-i(\omega+\omega')t} \times \langle \underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) \otimes \underline{\hat{\mathbf{B}}}(\mathbf{r}',\omega')\rangle, \quad \text{(C16)}$$

we express $\langle \underline{\hat{\mathbf{B}}}(\mathbf{r},\omega)\otimes\underline{\hat{\mathbf{B}}}(\mathbf{r}',\omega')\rangle$ in terms of $\langle \underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) \otimes\underline{\hat{\mathbf{E}}}(\mathbf{r}',\omega')\rangle$, by using Eq. (61) in the Fourier domain,

$$\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) = (i\omega)^{-1}\boldsymbol{\nabla}\times\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega). \quad \text{(C17)}$$

By means of Eqs. (C10), (C12), and (C14) [together with Eqs. (30) and (32)] it is now not difficult to prove Eq. (70). Note that there are no problems at $\omega=0$.

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[38] The longitudinal and transverse tensorial δ functions, respectively, are given by $\boldsymbol{\delta}^{\|}(\mathbf{r})=[(\boldsymbol{\nabla}\otimes\boldsymbol{\nabla})/\Delta]\delta(\mathbf{r})$ and $\boldsymbol{\delta}^{\perp}(\mathbf{r})=\boldsymbol{\delta}(\mathbf{r})-\boldsymbol{\delta}^{\|}(\mathbf{r})$.
[39] The argument may be viewed as an application of Curie's principle: "asymmetrical" effects are not produced by "symmetrical" causes.