Characterization of nonclassical optical fields by photodetection statistics

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Experiments on nonclassical optical fields have recently been performed. One of the main properties reported is the antibunching effect of photoelectrons, a property that cannot be explained in the framework of the classical theory of optical fields. By carefully studying the random point process of the detection of the optical field, we show that bunching and antibunching effects can be fully explained by a concidence function. In the classical theory, this function is a correlation function which introduces necessarily a bunching effect. But this coincidence function has no reason to be in any case a correlation function. Therefore antibunching effects can simply be derived from the properties of the coincidence function. After having given a precise definition of this coincidence function, some of its properties are discussed and especially its relationship with bunching and antibunching effects. Similarly, its relationship with statistical properties of lifetimes and intervals between points of the process is established. Various examples are presented and analyzed. Several calculations and computer simulations highlight the theoretical results.

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I. INTRODUCTION

In the analysis of optical fields by photodetection experiments, the only available information is the set of time instants $\{t_i\}$ at which photons are transformed into photoelectrons. These instants are usually randomly distributed and constitute a point process (PP) [1–3]. Therefore the basic problem in photodetection experiments is to extract some properties of the optical field from the analysis of this PP.

This was already the point of view of Blanc-Lapierre 60 years ago in his study of the shot noise generated from low-intensity optical fields [4]. Because of the limitations of experimental devices at that time, it was shown that there is no memory effect and the consequence is that the photodetection PP is a Poisson process. Similar results were obtained by Rice at the same time [5].

Twenty years after, the realization of the laser stimulated a new interest in these problems. Indeed natural and laser lights do not exhibit the same PP's in photoelectron experiments. There were a great number of experiments to verify this fact and reviews of these studies can be found in [6–8]. In particular the *bunching effect* of photoelectrons was discovered [9], showing that the PP is not a Poisson process but a compound Poisson process (CPP), introduced empirically by Mandel [10].

Some years later, this result was shown by Glauber in his famous papers on quantum theory of optical detection [11,12]. In particular the bunching effect of photoelectrons of classical optical fields appears as a direct consequence of the fact that such fields generate a CPP.

These papers and some others introduced the possibility of nonclassical optical fields and in particular of fields introducing an *antibunching effect*. Thus the door was open to realize such nonclassical optical fields and various results were published [13,14].

However, the mathematical description of the PP's introduced by such fields is not always satisfactory and introduces sometimes some contradictions. Thus the purpose of this paper is to clarify some of these questions.

As the bunching or antibunching effects of photoelectrons are observed by coincidence devices with two photodetectors, a particular attention is devoted in Sec. II to bicoincidence functions describing these effects. One of the most important points exhibited is that these bicoincidence functions which look like correlations functions are not correlation functions as frequently assumed. Section III is devoted to the analysis of classical optical fields or to the analysis of CPP. The purpose is to present some characteristic properties of such PP's in such a way that violation of at least one of them leads to the conclusion that the optical field is not classical. The results of experiments concerning nonclassical optical fields are discussed in Sec. IV. In the last section we present some results of experiments concluding with nonclassical behavior and this discussion leads to a more careful analysis of some PP's.

II. COINCIDENCE AND BUNCHING FUNCTIONS

Consider a time PP, which is a random distribution of time instants t_i, and let $N(t,\tau)$ be the random variable (RV) equal to the number of points in the time interval $[t,t+\tau]$. We assume that this process is regular—i.e., that there is no accumulation point. This means that a small interval cannot contain more than one point of the process or, more explicitly, that

$$P[N(t,\Delta t) > 1] = o(\Delta t), \tag{2.1}$$

$$P[N(t,\Delta t) = 1] = \lambda(t)\Delta t + o(\Delta t), \tag{2.2}$$

where P means the probability and $\lambda(t)$ is the *density* of the PP. As a consequence we have $P[N(t,\Delta t)=0]=1-\lambda(t)\Delta t + o(\Delta t)$. If $\lambda(t)$ is constant, the process is said to be *station-*

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ary, and this is assumed in all what follows. Let us now introduce the concept of the coincidence function of a PP.

A. Coincidence

In coincidence measurements we consider two (or more than two) small time intervals at different time instants. A coincidence event is characterized by the fact that there is one point of the PP in each of these intervals. In what follows we restrict the analysis to bicoincidence described by a *coincidence function* $\gamma(t,t')$, $t\neq t'$, defined by

$$P\{[N(t,\Delta t)=1][N(t',\Delta t')=1]\}=\gamma(t,t')\Delta t\Delta t'+o(\Delta t,\Delta t'). \tag{2.3}$$

Since the PP is stationary, this coincidence function only depends on $\tau=t-t'$ and is equivalent to a function $c(\cdot)$ defined for $\tau\neq 0$ by

$$c(\tau)=c(-\tau)=\gamma(t,t-\tau),\quad \tau\neq 0. \tag{2.4}$$

In general the RV's $N(t,\Delta t)$ and $N(t',\Delta t')$ become uncorrelated for large values of τ, which is expressed as

$$\lim_{\tau\to\infty} c(\tau)=\lambda^2. \tag{2.5}$$

On the other hand, as Eq. (2.4) cannot be used for $\tau=0$, we state by definition $c(0)=\lim_{\tau\to 0} c(\tau)$.

Note that the basic property of *Poisson processes* is that the variables $N(t,\Delta t)$ and $N(t',\Delta t')$ are independent. This yields $c(\tau)=\lambda^2$ for any τ and this is the simplest example of coincidence function.

According to Eq. (2.1) the RV's $N(t,\Delta t)$ and $N(t-\tau,\Delta t')$ take only the values 0 or 1 for sufficiently small intervals Δt and $\Delta t'$, and this yields

$$\mathsf{E}[N(t,\Delta t)N(t-\tau,\Delta t')]=c(\tau)\Delta t\Delta t'+o(\Delta t,\Delta t'), \tag{2.6}$$

where E means the expectation value. From this relation it seems natural to identify $c(\tau)$ to the correlation function of a continuous time signal sometimes called intensity of the PP. This is often done [15,16]. However, it is important to note that, despite the appearance in its definition, the coincidence function $c(\tau)$ *is not a correlation function*. Indeed a correlation function $r(\tau)$ must satisfy some constraints and, more precisely, must be positive definite, which is characterized by the fact that its Fourier transform $S(f)$ is positive (power spectrum). As a consequence $|r(\tau)|\leq r(0)$. This has no reason to be true for $c(\tau)$ and there are various examples of PP where $c(\tau)>c(0)$. It is even possible, as seen later, to have PP's for which there exists τ_0 such that $c(\tau)=0$ for $|\tau|<\tau_0$. Furthermore, it can be shown that $c(\tau)$ is not arbitrary and must satisfy some conditions that will be discussed in another paper.

B. Bunching effect

The coincidence function has a direct application in the description of the *bunching* (or antibunching) effect in a PP. This effect is related to the fact that the presence of a point of the process at $t=0$ can modify the probability $P[N(t,\Delta t)=1]$. This does of course not appear in the case of a Poisson PP, because the fundamental property of Poisson processes is to be memoryless.

However, since the fact that there is a point at $t=0$ is not a statistical event, we shall introduce a small interval $\Delta t'$ at $t=0$ and use the conditional probability

$$\hat{F}(t,\Delta t,\Delta t')\triangleq P\{[N(t,\Delta t)=1]|[N(0,\Delta t')=1]\}. \tag{2.7}$$

The bunching effect is described by the limit of this function when $\Delta t'\to 0$ or

$$F(t,\Delta t)\triangleq \lim_{\Delta t'\to 0}\hat{F}(t,\Delta t,\Delta t'). \tag{2.8}$$

The conditional probability $\hat{F}(t,\Delta t,\Delta t')$ can be expressed as

$$\hat{F}(t,\Delta t,\Delta t')=\frac{P\{[N(t,\Delta t)=1][N(0,\Delta t')=1]\}}{P\{[N(0,\Delta t')=1]\}}, \tag{2.9}$$

and it results from Eqs. (2.2) and (2.3) that $F(t,\Delta t)=b(t)\Delta t$ with

$$b(t)\triangleq\frac{c(t)}{\lambda}. \tag{2.10}$$

This *bunching function* $b(t)$ has the same properties of the coincidence function $c(t)$, and especially Eq. (2.5) yields

$$\lim_{t\to\infty} b(t)=\lambda. \tag{2.11}$$

For a Poisson process, $b(t)=\lambda$. Then we shall say that there is a bunching effect at t if $b(t)>\lambda$ and an antibunching effect in the opposite case. Note that the bunching effect is not necessarily an intrinsic property of a PP because for the same PP instants t can exist with bunching effects and the other with antibunching effects. If $b(t)>\lambda$ for any t, we say that the PP is of a *permanent* bunching effect.

C. Counting

In counting experiments we study the statistical properties of the RV $N(t,T)$. In the stationary case, they are the same of those of $N(0,T)$. We shall now see that the second-order properties can be deduced from the coincidence function introduced above.

Consider a partition of the interval $[0,T]$ in subintervals $[t_i,t_i+\Delta t_i]$. Thus we can write

$$N(0,T)=\sum_i N(t_i,\Delta t_i), \tag{2.12}$$

and, passing to the limit, $N(0,T)$ is defined by the stochastic integral

$$N(0,T)=\int_0^T dN(\theta), \tag{2.13}$$

where the increments $dN(\theta)$ are the RV's $N(\theta,d\theta)$.

We deduce from Eq. (2.2) that $\mathsf{E}[dN(\theta)]=\lambda d\theta$, which yields the mean value $\mathsf{E}[N(0,T)]=\lambda T$. Similarly we can write

$$\mathsf{E}[N^2(0,T)] = \int_0^T \int_0^T \mathsf{E}[dN(\theta)dN(\theta')]. \tag{2.14}$$

For $\theta \neq \theta'$ we deduce from Eq. (2.6) that $\mathsf{E}[dN(\theta)dN(\theta')] = c(\theta-\theta')d\theta d\theta'$. But it is necessary to use an expression valid for any θ, θ'. For this we note that it results from Eqs. (2.1) and (2.2) that $\mathsf{E}[dN^2(\theta)]=\lambda d\theta$. This allows us to write the complete expression

$$\mathsf{E}[dN(\theta)dN(\theta')] = [c(\theta-\theta') + \lambda\delta(\theta-\theta')]d\theta d\theta', \tag{2.15}$$

where $c(\cdot)$ is the coincidence function and $\delta(\cdot)$ the Dirac distribution. This must be inserted into Eq. (2.14) to obtain $\mathsf{E}[N^2(0,T)]$.

Actually we are more interested in the variance of $N(0,T)$ defined by $\sigma_N^2 = \mathsf{E}[N^2(0,T)] - \mathsf{E}^2[N(0,T)]$. It results from the previous equations that this variance can be expressed as

$$\sigma_N^2(T) = \lambda T + g(T), \tag{2.16}$$

where the function g is

$$g(T) = \int_0^T \int_0^T c(\theta-\theta')d\theta d\theta' - \lambda^2 T^2. \tag{2.17}$$

For Poisson processes, $c(t)=\lambda^2$, which yields $g(T)=0$. We find again the well-known result that the variance of a Poisson RV is equal to its mean. By convention we shall say that if $g(T)<0$ we have a sub-Poisson behavior and if $g(T)>0$ a super-Poisson behavior. Note that this property of $N(0,T)$ can depend on T and, as for the bunching effect, is not necessarily an intrinsic property of the PP. Note also that while the bunching effect is due to the behavior at two time instants t_i and t_j, the variance corresponds to the whole interval T.

D. Lifetimes

The lifetime L_k of order k is the RV equal to the distance between a point t_i of the PP and the kth point of this process posterior to t_i. Because of the assumption of stationarity, the probability distribution of L_k does not depend on t_i, and then we can assume that $t_i=0$. For all processes considered here the RV's L_k are continuous and characterized by their probability distribution function (PDF) $f_k(t)$. The quantity $f_k(t)\Delta t$ is by definition the probability to have one point of the PP in $[t,t+\Delta t]$ and $k-1$ points in $[0,t]$, conditionally to one point at $t=0$. But it results from Eq. (2.10) that $b(t)\Delta t$ is the probability to have one point in $[t,t+\Delta t]$ conditional to one point at 0. As a result we have

$$b(t) = \frac{c(t)}{\lambda} = \sum_{k=1}^{\infty} f_k(t). \tag{2.18}$$

This equation yields the relationship between the coincidence function $b(t)$ and the set of PDFs $f_k(t)$ of the lifetimes of all orders. It is clear that it will play an important role for all processes defined from their lifetimes, especially renewal processes.

E. First-order approximation

In some specific situations it is possible to approximate Eq. (2.18) by its first term, $b(t)\approx f_1(t)$. In this case coincidence measurements are equivalent to first-order lifetime measurements. Thus, instead of using a coincidence device, it is sometimes more appropriate to use a time to amplitude converter (TAC) to reach the coincidence function $c(t)$ [15–18].

Let us discuss some conditions allowing this approximation. Let τ_0 be the radius of variation of the coincidence function $c(t)$. This means that for $t>\tau_0$ $c(t)\approx\lambda^2$, which, according to Eq. (2.5), is its asymptotic value. For a stationary PP the mean value of the distance between two points is the inverse of the density λ. Then, if $\lambda\tau_0 \ll 1$, the probability to have more than one point in the interval $[0,t]$ with $t<\tau_0$ is very low. As a consequence in the domain of variation of $c(t)$ the bunching function is approximately equal to the PDF of the lifetime of order 1. We shall see that there are PP's where this can always be satisfied, and this especially the case of CPP's. On the other hand, there are PP's where this approximation can never be used and this is discussed in the following. As a consequence it must be pointed out that results of experiments involving classical [16] and nonclassical states of radiation as well [15] must be carefully interpreted.

In conclusion of this section we note that the coincidence function $c(t)$ defined by Eq. (2.4), which is not, in general, a correlation function, allows one to calculate the bunching effect and also second-order counting statistics of an arbitrary PP. Conversely this function can be deduced from the PDF's of the various lifetimes of the process or from the counting probabilities.

III. PHOTODETECTION OF CLASSICAL OPTICAL FIELDS: COMPOUND POISSON PROCESSES

As indicated previously, it results from the quantum theory of optical detection that the PP appearing in classical fields is a compound Poisson process, sometimes called a doubly stochastic PP [2,3]. This is the reason why such PP's are sometimes called in what follows classical PP's.

These CPP's have various specific properties analyzed below. This introduces a possible test for deciding whether or not a given optical field (or a PP) is classical. Indeed, if the PP obtained from this field in photodetection experiments does not satisfy at least one of these properties, it cannot be a CPP, and the optical field analyzed cannot be classical.

A. General properties

The CPP's are Poisson PP's in which the density $\lambda(t)$ is a stationary random function [3]. It can be denoted $\lambda(t,\omega)$. This means that for a given ω—say, ω_0—the PP is a nonstationary Poisson process defined by the density $\lambda(t,\omega_0)$.

For classical optical fields it is possible to define a light intensity $I(t)$ which is a positive and, in general, random function noted $I(t,\omega)$. It results from the quantum theory of optical photodetection that for classical fields the PP of photodetection is a CPP with a density $\lambda(t,\omega)$ proportional to the this light intensity $I(t,\omega)$.

For a given ω the coincidence function appearing in Eq. (2.3) becomes

$$c(t,t',\omega) = \lambda(t,\omega)\lambda(t',\omega), \tag{3.1}$$

which is a conditional probability. By taking the expectation with respect to ω, we obtain a marginal probability defining the coincidence function by

$$c(t,t') = \mathsf{E}[\lambda(t,\omega)\lambda(t',\omega)]. \tag{3.2}$$

As it was assumed that $\lambda(t)$ is stationary, this function depends only on $t-t'$. Let $\gamma_\lambda(\tau)$ be the correlation function of $\lambda(t)$. Using this function in Eq. (2.4) yields

$$c(\tau) = \gamma_\lambda(\tau) + \lambda^2, \tag{3.3}$$

where $\lambda = \mathsf{E}[\lambda(t)]$ is the mean value of $\lambda(t)$, the density of the PP. As a consequence we deduce Eq. (2.5) because it is known that the correlation function $\gamma(\tau)$ of an ergodic random function tends to zero when τ tends to infinity. Inserting Eq. (3.3) into Eq. (2.17) yields

$$g(T) = \int_0^T \int_0^T \gamma_\lambda(\theta - \theta')d\theta d\theta'. \tag{3.4}$$

As $\gamma(t)$ is a positive definite function, this quantity is positive whatever T is, and we conclude that a CPP is always of super-Poisson type. In fact $g(T)$ has a physical meaning and is simply the variance of the RV $\int_0^T \lambda(\theta)d\theta$.

Finally the life times used in Eq. (2.18) are the expectation values of the lifetimes of a nonstationary Poisson process—i.e.,

$$f_k(t) = \frac{1}{\lambda}\mathsf{E}\left[\exp[-m(t)]\lambda(0)\lambda(t)\frac{m(t)^{k-1}}{(k-1)!}\right], \tag{3.5}$$

where

$$m(t) = \int_0^t \lambda(\theta)d\theta. \tag{3.6}$$

It results from these equations that $f_k(0)=0$ for $k>1$. Let us summarize some properties of CPP.

1. Counting

Because a CPP is always a super-Poisson process, we can deduce from Eqs. (2.5) and (3.4) that, for a CPP, we have

$$\sigma_N^2(T) > m(T), \quad \forall\, T, \tag{3.7}$$

where $m(t)$ is the mean value of $N(t,T)$ equal to λT.

2. Lifetimes

The PDF $f_k(t)$ for $k>1$ is zero for $t=0$ or $f_k(0)=0$, because $m(0)=0$. If a given PP does not satisfy *at least one* of these properties, it cannot be a CPP.

3. Bunching effects

A CPP has necessarily a bunching effect in the neighbourhood of the origin, and this bunching effect can be permanent.

Indeed we deduce from Eqs. (2.10) and (3.3) that $b(t) = (1/\lambda)\gamma_\lambda(t)+\lambda$. Then the condition $b(t)>\lambda$ ensuring the bunching effect is realized as soon as $\gamma_\lambda(t)>0$. This can appear for any value of t, and this is, for example, the case of the exponential correlation function.

If that is not the case, then there exists a t_0 such that there is a bunching effect for $t<t_0$. This results from the facts that the correlation function $\gamma_\lambda(\tau)$ is continuous, its maximum value is reached for $\tau=0$, and $\gamma_\lambda(0)>0$ because it is the variance of $\lambda(t)$.

B. Modulation and first-order approximation

Suppose that the random density characterizing a CPP can be modulated or expressed as $\lambda(t)=\alpha A(t)$ where α is a nonrandom constant related to a possible modulation effect. The coincidence function (3.3) can be expressed as

$$c(t) = \alpha^2[\gamma_A(t) + A^2], \tag{3.8}$$

where $\gamma_A(t)$ and A are the correlation function and the mean value of A, respectively.

It is clear from this equation that it is always possible to modulate the density of the process by varying α without changing the shape of its coincidence function. This is called an invariance by modulation. This invariance by modulation is in fact a test for classical fields [19,20]. It is especially important for applying the first-order approximation introduced above. Indeed, as the density λ is αA, it is always possible to reach the condition $\lambda t_0 \ll 1$ by using a sufficiently small α. In this operation the shape of $c(t)$ given by Eq. (3.8) does not change. However, this property has no reason to be general. In particular, since a CPP yields necessarily a bunching effect, any PP with antibunching effect cannot be a CPP one, and there is no reason to assume that the first approximation can be valid.

Finally, we can state the question whether or not this invariance by modulation is characteristic of a CPP. The answer is certainly no, because we have only a second-order property that is insufficient to define a PP. However, if this invariance property is valid for any coincidence function defined as Eq. (2.3) but with n arbitrary time instants $\{t_i\}$ instead of 2, then the process is necessarily a CPP. We do not present here the proof of this result.

Starting from the PP generated by photodetection of an optical field, two questions can be asked: (a) Is the field classical or not? (b) If not, how does one determine some properties of the corresponding nonclassical PP and, especially, its bunching or coincidence function? The answers to these question can be obtained from counting or from lifetimes measurements or from a combination of the two.

C. Counting experiments

For special states of light—e.g., squeezed states—several photoncounting experiments have been performed to prove the nonclassical character of these states (see, for example, references in [13]).

From these experiments, it can be shown that the condition (3.7) is not satisfied and the conclusion is that the PP

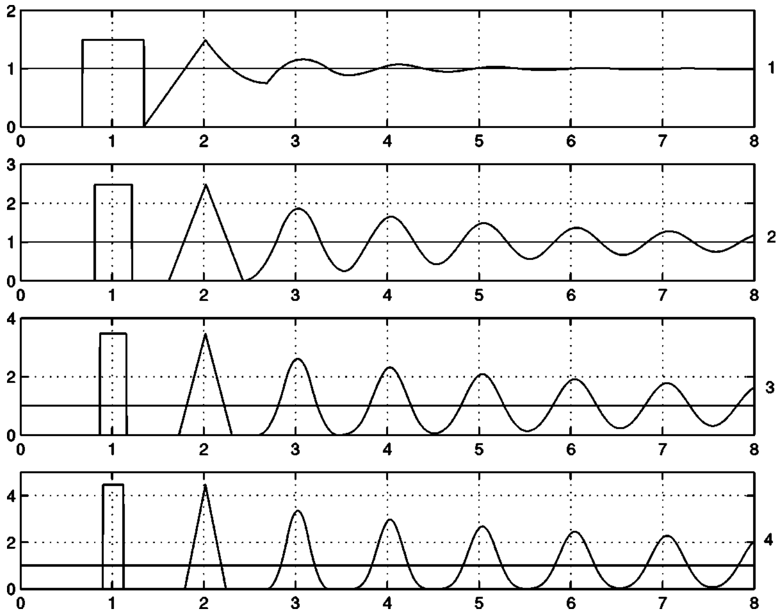

FIG. 1. Bunching functions versus time in dimensionless units for renewal process with uniform lifetime PDF. Curves 1, 2, 3, and 4 are plotted for values of a equal to $1/3$, $1/5$, $1/7$, and $1/9$, respectively.

cannot be classical. However, from this result only, it is impossible to get some properties of this nonclassical PP. The minimum work would be to present an analysis of $\sigma_N^2(T)$ in terms of T and use Eqs. (2.16) and (2.17) to obtain some information about the coincidence function $c(t)$ which describes the second-order properties of the PP.

D. Lifetime experiments

The principle of the lifetime experiments is based on the utilization of TAC devices. The output of these devices delivers the time distances between successive points. These time distances are then treated by a multichannel analyzer, for example, or processed with a computer. Finally, the result is an estimation of $f_1(t)$, the PDF of the lifetime of order 1 introduced above. The experiments show that this function is not maximum at the origin of time axis. Using the first-order approximation introduced above, many authors conclude that $f_1(t)$ is an approximation of the coincidence function called correlation function in the discussion. Thus it seems that the results of the experiments yield answers to the two questions stated above: (a) The PP is not classical. (b) Its coincidence function is measured and exhibits an antibunching effect.

These two conclusions must be analyzed with some care.

(i) The PP is certainly not classical, but this conclusion can be obtained only by contradiction or by violation of at least one of the properties of classical fields indicated above. For classical fields the first-order approximation is valid and the PDF of the first-order lifetime measured with a TAC tends to the coincidence function which is a correlation function with a maximum at the origin. As this is not the case, the PP is not classical.

(ii) On the other hand, the conclusion concerning the antibunching effect, as indicated in the title of the paper, cannot be deduced from these experiments. Indeed, as the PP is not classical [see (i)] the first-order approximation has no reason to be valid and there is no reason to deduce the coincidence function (and thus the antibunching effect) only from knowledge of the PDF $f_1(t)$ obtained by using a TAC.

Let us examine this point by analyzing the case of a renewal PP with a uniform distribution. Such a process is characterized by the fact that the distances $L_1(i)$ between successive points are independent and identical distributed (IID) RV's. Thus their common PDF function $f_1(t)$ defines completely the PP.

Because of the *independence* of the successive $L_1(i)$, their sum, which yields higher-order lifetimes, is given by the convolution (denoted by asterisks)

$$f_k(t) = \underbrace{f(t) * f(t) * \dots * f(t)}_{k} \triangleq f^{*k}(t). \tag{3.9}$$

Note that in case that $f(t)$ is exponential, the renewal process is a Poisson process.

Suppose now that L_1 (lifetime of order 1) is uniformly distributed in a given interval. In this case the explicit expression of $f_k(t)$ is complicated, except for $k=2$ and $k=3$, but these convolutions (3.9) can easy be calculated numerically.

More precisely, suppose that the PDF of L_1 is equal to $1/2a$ in the interval $[1-a, 1+a]$ and zero elsewhere. As a consequence its mean value is 1 for any a and the density satisfies $\lambda=1$. For $a=1$ we obtain a uniform distribution in the interval $[0,2]$. On the contrary when $a \ll 1$ the RV L_1 is almost equal to 1, and this occurs for example in the jitter effect in communication systems.

Furthermore, it is easy to verify that the PDF $f_k(t)$ is symmetric with respect to $t=k$ and equal to zero outside the interval $[k(1-a), k(1+a)]$. As a consequence there is no overlapping between $f_{k-1}(t)$ and $f_k(t)$ if $(n-1)(1+a) < n(1-a)$ or $a < 1/(2n-1)$. Finally, if a satisfies this inequality, the PDF's $f_\ell(t)$ do not overlap for $\ell \leqslant k$ and there is an overlapping effect if $\ell > k$.

These properties appear clearly in Fig. 1. Figure 1.1 is drawn for $a=1/3$. There is no overlapping between f_1 and f_2 and the four discontinuities appear clearly. Figures 1.2, 1.3, and 1.4 are drawn for $a=1/5$, $a=1/7$, and $a=1/9$, respectively. In Fig. 1.4 there is no overlapping between f_1, f_2, f_3, f_4, and f_5 and there are five points of discontinuity.

Note also that all these curves tend to the asymptotic value $\lambda=1$ but the speed of convergence decrease with a. For small values of a there are some oscillations that disappear when t increases and at the limit of $a=0$ the RV's become deterministic and $c(t)$ is a set of Dirac functions at the time instants k.

Let us now present some properties of $b(t)$. It is possible to show that the derivative of $b(t)$ is continuous with respect to time, except at points of the abscissa, $1-a, 1+a, 2-2a, 2, 2+2a$. For the particular value $a=1/3$, obtained from $1+a=2-2a$, there remain only four abscissa of discontinuity. There is of course an antibunching effect for small values of t. However, it is not permanent since there are values of t where $b(t)>\lambda=1$.

Let us come back to the question stated at the beginning of this section. It is clear that the bunching function appearing in Fig. 1 cannot be approximated by the first-order PDF which is the rectangular function centered at 1 and appearing also in the figure.

Similarly the conclusion on a bunching or antibunching effect cannot be deduced from the analysis of $f_1(t)$ only, and for enlightening this point we shall now present some theoretical results and computer experiments on some nonclassical PP's. We do not claim that these PP's correspond to specific optical fields, which is a question outside the scope of this paper, and this is the reason to use the expressions of classical or nonclassical PP's.

IV. CORRELATED POINT PROCESSES

Let $\{t_i\}$ be the time instants (or points) of a stationary PP and $x_i=t_i-t_{i-1}$ the distance between successive points, or lifetime, also noted previously as $L_1(i)$. The quantities x_i are the values of a stationary discrete time (DT) positive random signal. The relation $x_i=L_1(i)$ means that the set of all stationary PP's is equal to the set of all stationary positive random signals, and this remark is the starting point of this section.

If the RV's x_i are independent, which means that the signal x_i is a positive white noise, the corresponding PP is a renewal PP. If, moreover, the PDF of the RV's x_i is exponential, the PP is a pure Poisson process.

Our purpose is to delete the *assumption of independence* or to study some PP's with correlated lifetimes. However, as the property of noncorrelation is quite insufficient to define a random signal, we shall introduce a specific model of correlated PP's including as a particular case the renewal processes and also the Poisson processes.

This model is defined by two parameters. The first one p specifies the correlation function between the successive intervals of the PP (lifetime of order 1). The latter r specifies the marginal PDF of these intervals.

By using this model, it is possible to obtain some analytical expressions concerning the PDF's appearing in Eq. (2.18). In this case we shall compare the results of computer experiments with the relevant theoretical calculations. In other cases calculations are almost impossible and we shall limit our aim to present only results of computer experiments.

A. Definition of the model

Let u_i be a sequence of IID positive random variables characterized by the PDF $f(t)$. This defines a renewal process P. Let also v_i be sequence of IID random variables independent of the u_i's, and taking only the values 1 or 0 with the probabilities p and $q=1-p$, respectively.

Consider the signal x_i defined by the recursion

$$x_{i+1} = v_i x_i + (1-v_i)u_i = v_i x_i + \overline{v_i} u_i. \tag{4.1}$$

It is clear from this equation that the marginal PDF is common to all x_i and is still $f(t)$.

If $p=0$, $x_{i+1}=u_i$, and these RV's describe the renewal PP defined by u_i. If $p\neq 0$, there exists a correlation between the x_i's and the corresponding correlation function can be calculated. Let m and σ^2 be the mean value and the variance associated with $f(t)$, respectively. This means that the RV's x_i are second order. The case where σ^2 is not finite requires a specific treatment not presented here. It results from Eq. (4.1) that

$$\begin{aligned} \mathsf{E}[x_0 x_k] &= \mathsf{E}[x_0(v_{k-1}x_{k-1} + \overline{v}_{k-1}u_{k-1})] \\ &= p\mathsf{E}[x_0 x_{k-1}] + (1-p)m^2. \end{aligned} \tag{4.2}$$

Introducing the correlation function defined by $\gamma_k = \mathsf{E}[x_0 x_{k-1}] - m^2$ yields

$$\gamma_{k+1} = p\gamma_k = \sigma^2 p^{k+1} = \sigma^2 p^{|k+1|}, \tag{4.3}$$

because any correlation function is an even function. This means that Eq. (4.1) introduces an exponential correlation function whatever the PDF $f(t)$. Note that the correlated PP is entirely defined by $f(t)$ and p. In other words p introduces a correlation in the starting renewal process P defined by u_i without changing the marginal distribution of the lifetime.

B. Distribution of the lifetime of order 1

As we are interested in the discussion of bunching and antibunching effects in PP's, we shall introduce a class of PDF's depending on one parameter r and including as a particular case the exponential distribution corresponding to pure Poisson processes.

The family of PDF's used below is introduced by the following argument. Let $F(t)$ be an arbitrary distribution function (DF). It is a nondecreasing function varying from 0 to 1. The same property is valid for $F^{[r]}(t)=F^r(t)$, which ensures that it is still a DF. But in this transformation there is a shift of the possible values to the right. As an example, if $F(\lambda)=1$ or if the RV X defined by $F(\cdot)$ satisfies $X\leq\lambda$, we obtain in the limit $r\to\infty$ the DF $F^{[\infty]}(t)=u(t-\lambda)$, which means that the RV X is almost surely equal to λ.

We shall apply this idea to the exponential distribution characterizing a pure Poisson process when the parameter p of the model (4.1) is zero. The exponential DF of mean value $1/\lambda$ is $F(t)=1-\exp(-\lambda t)$. The PDF associated with $F^{[r]}(t)$ is

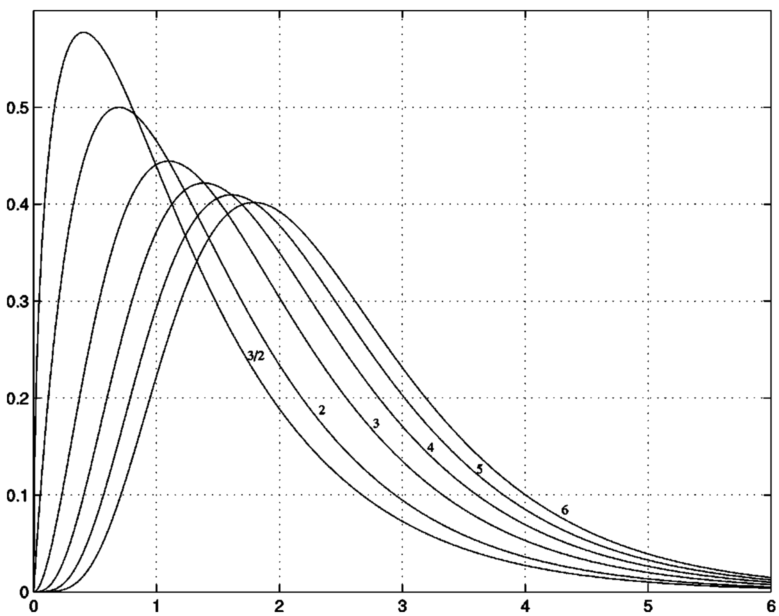

FIG. 2. Shifted exponential PDF's versus time in dimensionless units. The curves are plotted for $\lambda = 1$ and indexed with the values of $r = 3/2, 2, 3, 4, 5, 6$.

obtained by calculating its derivative, which yields the value of the PDF of the first-order lifetime by

$$f_1^{[r]}(t) = r[1 - \exp(-\lambda t)]^{r-1}\lambda\exp(-\lambda t). \tag{4.4}$$

These functions are displayed in Fig. 2 for $\lambda = 1$ and the following values of r: 1.5, 2, 3, 4, 5, 6. The exponential distribution obtained for $r = 1$ is not represented in this figure. The effect on the small values of t is evident and the maximum is shifted to the right when r increases. It is obvious that the mean value, which has no simple analytical expression, is an increasing function of r.

The last point is to generate a white noise u_i with the PDF (4.4). For this there is a classical procedure which makes use of the inverse function $G^{[r]}$ of the DF function $F^{[r]}(t)$. An elementary calculation yields

$$G^{[r]}(x) = -(1/\lambda)\ln(1 - x^{1/r}). \tag{4.5}$$

It is known that if w_i is a white noise uniformly distributed in [0, 1], the signal defined by $u_i = G^{[r]}(w_i)$ is a white noise with the PDF (4.4).

In conclusion, it is possible to construct a PP defined by the x_i with exponential correlated lifetimes of order 1 defined by p and of PDF (4.4) defined by r and represented in Fig. 2.

At this step it is worth pointing out that we have here a good example of PP where the first-order approximation discussed in Sec. II E *cannot be applied*. Indeed by varying p and r without changing $f(t)$ we obtain different PP's with the same PDF of the lifetime or order 1. As a consequence all these processes cannot be distinguished by using only this PDF, even by varying the density, which is the main idea of a first-order approximation.

C. Calculations

In order to calculate the bunching function $b(t)$ defined by Eq. (2.18), it would be necessary to calculate all the PDF's $f_k(t)$ appearing in this expression for the PP defined by Eq. (4.1). It is in general an almost impossible task.

Let us first indicate the general procedure of this calculation. Let t_0 be an arbitrary point of the PP. The RV x_0 is the distance L_1 between t_0 and the first point t_1 of the PP posterior to t_0. Similarly the distance L_k between t_0 and the kth point of the PP posterior to t_0 is $L_k = x_0 + x_1 + \cdots + x_{k-1}$. The function $f_k(t)$ is the PDF of L_k.

It results from Eq. (4.1) that x_1 takes two values x_0 and u_0 with probabilities p and q, respectively. To each of these values we can associate by the same procedure two values of x_2. By repeating the procedure it is clear that $x_k = L_k$ takes 2^k possible values which are sum of independent RV's. As a consequence the PDF of L_k is a sum of convolutions. The analytical expression of these convolutions is difficult to be obtained except in the case of exponential distribution

In this last case, the principle of the calculation is presented in the Appendix. Let us give the results of the calculation summarized by the expressions of the first five values of $f_k(t)$. By definition we have $f_1(t) = \lambda\exp(-\lambda t)$. In order to simplify the presentation of the results, we introduce the functions $\hat{f}_k(x) = (1/\lambda)f_k(x)$ where $x = \lambda t$ and $e_k = e_k(x) = \exp(-x/k)$, k integer. We have $\hat{f}_1(x) = e_1$ and $\hat{f}_2(x) = (1/2)pe_2 + qxe_1$. The other functions $\hat{f}_k(x)$ are

$$\hat{f}_3(x) = (1/3)p^2e_3 + 2pq(e_2 - e_1) + (1/2)q^2x^2e_1, \tag{4.6}$$

$$\hat{f}_4(x) = p^3(1/4)e_4 + p^2q[e_3 - e_1 + (1/4)xe_2] + 3pq^2[2e_2 - (x+2)e_1] + (1/6)q^3x^3e_1, \tag{4.7}$$

and finally

$$\hat{f}_5(x) = (1/5)p^4e_5 + p^3q[(2/3)(e_4 - e_1) + 2(e_3 - e_2)] + (3/4)p^2q^2[(1 - 2x)e_1 + (2x - 4)e_2 + 3e_3] + 2pq^3[-(x^2 + 4x + 8)e_1 + 8e_2] + (1/24)q^4x^4e_1. \tag{4.8}$$

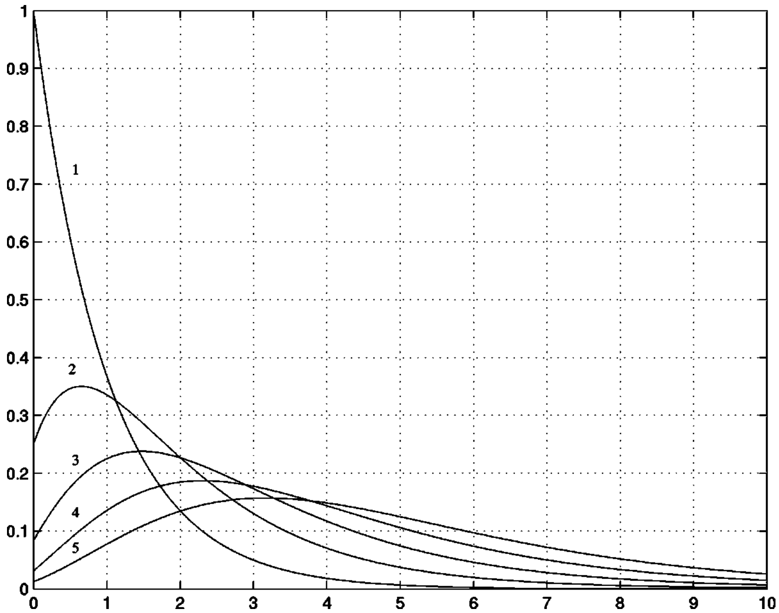

FIG. 3. Lifetime distributions $\hat{f}_k(t)$ versus t, time in dimensionless units. The curves are plotted for $p=0.5$ and indexed with the values of $k=1,2,3,4,5$.

These functions are presented in Fig. 3 for $p=0.5$. The function $\hat{f}_5(x)$ is presented in Fig. 4 for various values of p.

It is clear that $\hat{f}_k(0)=p^{k-1}/k$, and it results from the property (ii) of Sec. III B 1 that the PP analyzed in this section cannot be a CPP except when $p=0$, where it is a pure Poisson process.

Note also that the first-order approximation discussed previously cannot be applied. Indeed, as all the $f_t(t)$ satisfy $\hat{f}_k(x)=(1/\lambda)f_k(x)$, the bunching function defined by Eq. (2.18) has the same property and can be written as $\hat{b}(x)=(1/\lambda)b(x)$. Therefore a variation of the density λ of the process does not change $\hat{b}(x)$ which can never be approximated by $\hat{f}_1(x)$, even when $\lambda \to 0$.

D. Computer experiments

1. Principles of realizations and processing

As noticed above, the model introduces a correlated PP defined by the parameters p and r and giving the Poisson processes in the particular case $p=0$ and $r=1$. However, the calculation of the bunching function and even of the PDF's $f_k(t)$ is not easy, except for $r=1$, as discussed in the Appendix. Then it is necessary to proceed by simulation of by making a computer experiment. It is easy to generate by computer samples of the signal x_i or samples of the associated PP. For this we note that in Eq. (4.1) the signal x_i is generated from two independent white signals u_i and v_i. The first one is deduced from a white noise w_i uniformly distributed in $[0,1]$ by applying the transformation $u_i=G^{[r]}(w_i)$

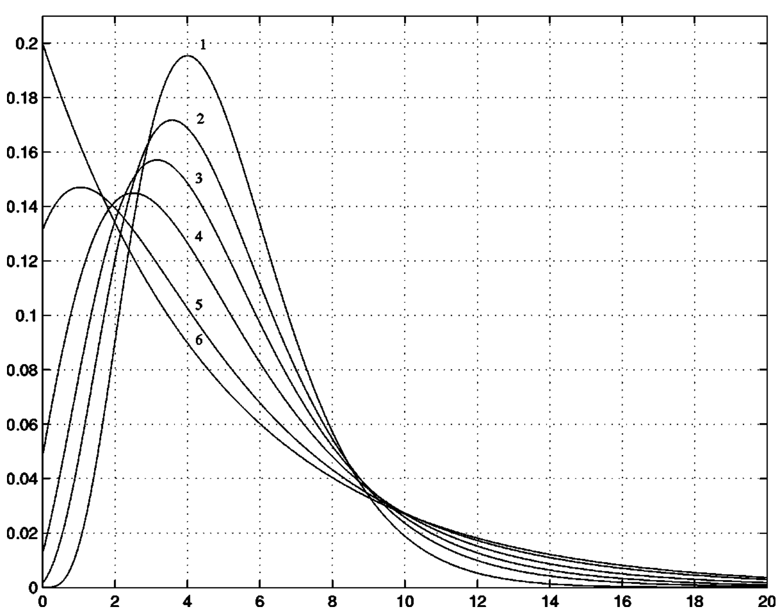

FIG. 4. Lifetime distributions $\hat{f}_5(t)$ versus t, time in dimensionless units. Curves 1, 2, 3, 4, 5, and 6 correspond to $p=0$, 0.3, 0.5, 0.7, 0.9, and 1, respectively.

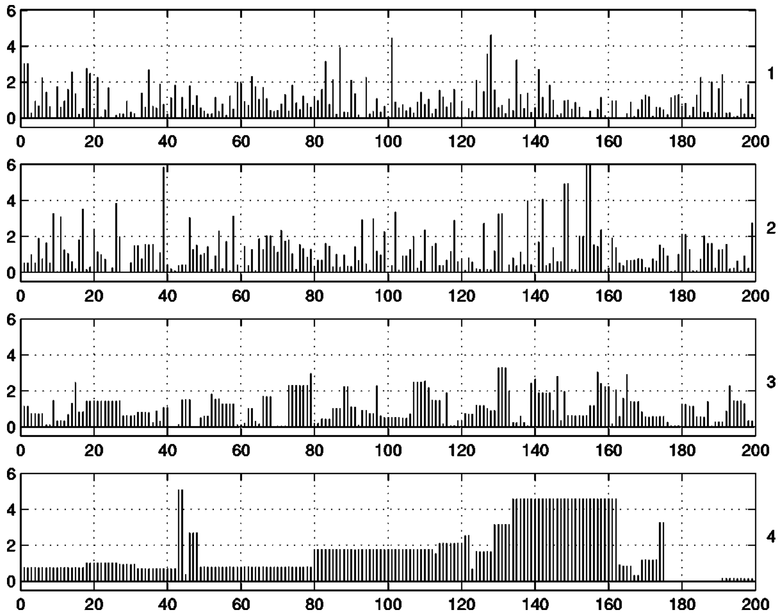

FIG. 5. Computer samples 1–4 of lifetimes simulated with $p=0$, 0.1, 0.5, 0.9, 1.

where $G^{[r]}(x)$ is the function defined by Eq. (4.5). The latter v_i takes only two values 0 or 1 and can be obtained from the white noise $\overline{w_i}$ independent of w_i and also uniformly distributed in [0,1]. For this we introduce $v_i = u(\overline{w_i} - 1 + p)$, where $u(\cdot)$ is the conventional unit step function. It is clear that v_i is still a white noise and that $P(v_i=1)=p$. The two signals u_i and v_i define entirely the signal x_i and therefore the PP.

From samples of x_i, it is possible to evaluate the PDF's $f_k(t)$ and to obtain an approximation of the bunching function by the following procedure.

In order to apply Eq. (2.18) it is necessary to estimate the PDF's $f_k(t)$ of a PP generated by a computer experiment. Let L_k be the RV defined in Sec. II D as the lifetime of order k. In order to estimate experimentally $f_k(t)$ it suffices to use a normalized histogram of the realizations $L_k(i)$ of L_k obtained in a PP generated experimentally by a computer. For this purpose we consider the sequence $L_1(i)$ of distances between *successive points* P_i in a particular realization of the PP. The experimental realizations of $L_k(i)$ are thus

$$L_k(i) = L_1(i+1) + L_1(i+2) + \cdots + L_1(i+k). \quad (4.9)$$

The normalized histogram of $L_k(i)$ yields an estimation of $f_k(t)$. The precision of the method depends first on the total number N of points of the PP analyzed in the experiment. In the following this number is usually of the order of 10^6. The other parameters involved in the precision of the method are the number and width of windows used for calculating the histogram. This is a classical topic in all the procedures of PDF estimation from histograms and standard methods are available.

The last problem is to find the value of the K, the number of terms of Eq. (2.18) necessary to obtain a good approximation of Eq. (2.18). There is no simple and general method for this problem. However, by examining the case of pure

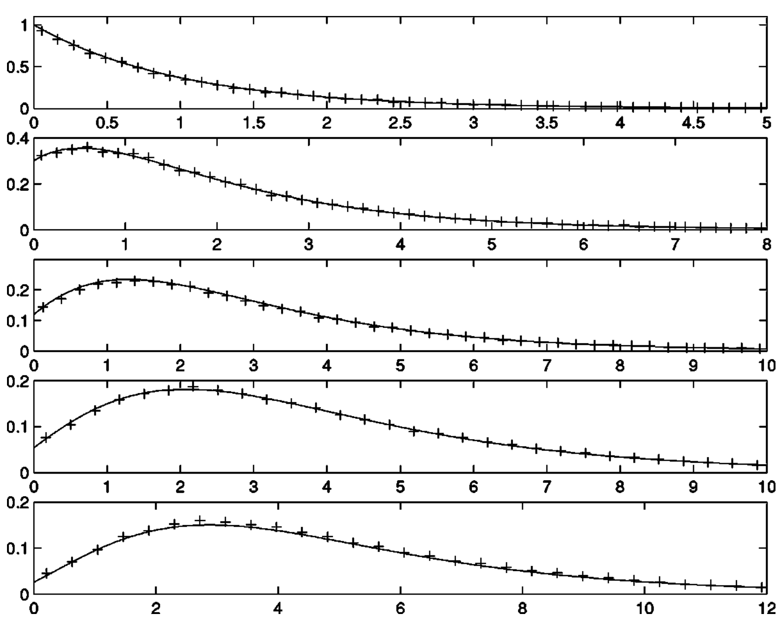

FIG. 6. Computer simulations and theoretical results of various lifetime distributions $f_k(t)$ for $k=1,2,3,4,5$ versus t, time in dimensionless units.

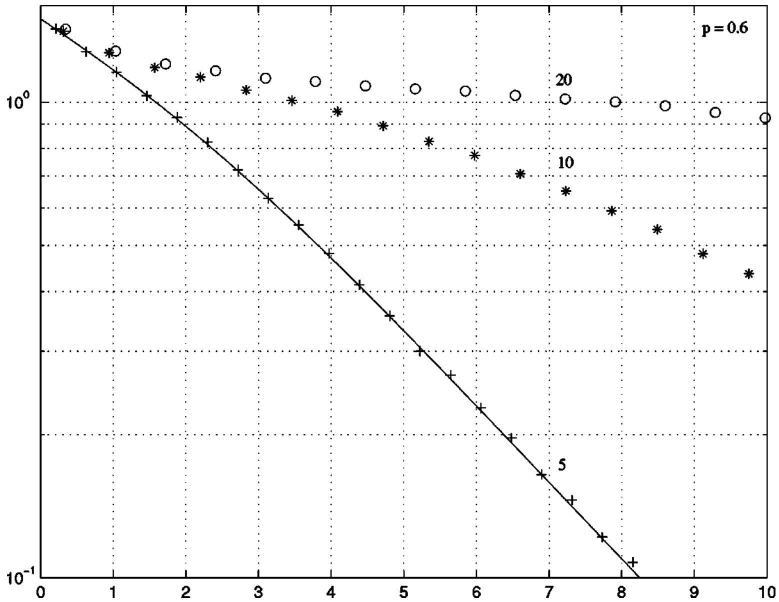

FIG. 7. Approximated bunching functions $b(t)$ versus t, time in dimensionless units in cases of correlated PP's ($p=0.6$) for three values of $K=5,10,20$. The curve in solid line is the exact function calculated for $K=5$ (see the Appendix).

Poisson processes for which the analytical expressions of all the PDF's $f_k(t)$ are known, it is simple to derive it. In fact, it is easily shown by computer calculations that for a Poisson process of density $\lambda=1$, the value $K=10$ yields an excellent approximation of $b(t)=1$ of Eq. (2.18) for $t<5$. For $K=50$, the approximation is excellent for $t<30$. But instead of estimating the K PDF's $f_k(t)$ appearing in Eq. (2.18), it is simpler to make a direct estimation of this function for a given K. This is simply obtained by using in the same histograms all values $L_k(i)$ for $1\leqslant i\leqslant N$ and $1\leqslant k\leqslant K$. This procedure yields directly an estimation of $b(t)$ without calculating the PDF's $f_k(t)$.

2. Results of computer experiments

a. Samples and PDF's of the lifetimes. Various examples of samples of $L_1(i)=x_i$ are presented in Fig. 5. In this figure the distribution $f(t)$ common to all the cases is exponential. This yields that if x_i's are independent, which is the case for $p=0$, the PP is a Poisson process. This corresponds to the samples presented in the top figure of Fig. 5. The other values of p are 0.1, 0.5, and 0.9. The effect of the correlation between successive lifetimes is especially evident in the last figure.

In Fig. 6 are represented experimental values of the PDF's of lifetimes of order 1–5 calculated with $p=0.6$ and $r=1$. This last value allows us to use results of calculations presented in Sec. VC. The experimental points are indicated by the symbol $(+)$ and the solid curves are calculated from Eqs. (4.6)–(4.8). There is an excellent accordance between experimental and theoretical results.

We emphasize that all the PP's studied here have a lifetime of order 1 identical to the lifetime of order 1 of a Poisson process. However, they are not Poisson because of the correlation of successive lifetimes. They are not either CPP because the PDF's $f_k(t)$ do not satisfy $f_k(0)=0$ for $k>1$.

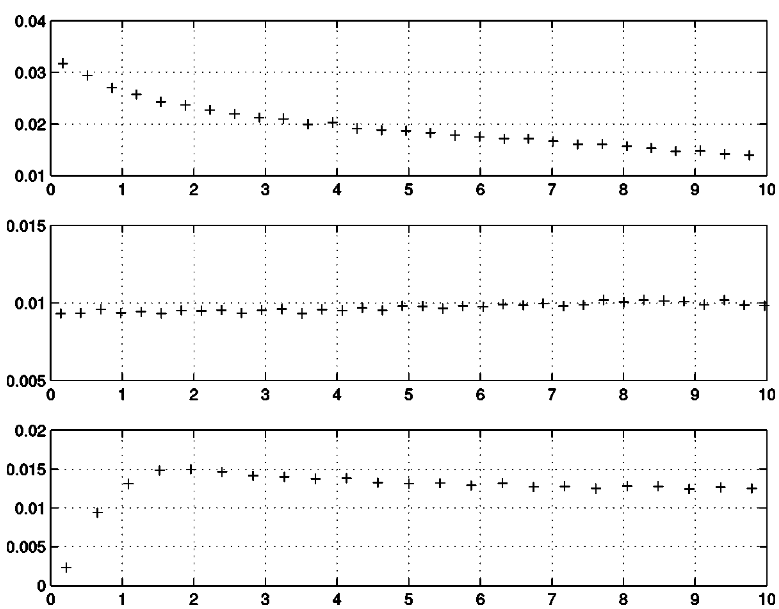

FIG. 8. Approximated bunching functions $b(t)$ versus t, time in dimensionless units for three types of PP's. From the top, the values of the parameters are $K=20$ and $r=1$, $p=0.6$, $r=1$, $p=0$, $r=3$, $p=0.6$, respectively.

b. Evaluation of the parameter K. We have previously mentioned the consequences of the replacement of the series (2.18) by a finite sum in studying a pure Poisson process. We shall analyze the same problem for correlated PP's introduced in this section. We use the same condition as in Fig. 6 that are $p=0.6$, $r=1$, and $f_1(t)$ exponential. In this case we have the explicit expressions of $f_k(t)$ for $1\leq k\leq 5$ by Eqs. (4.6)–(4.8). By using these expressions and limiting the series (2.18) to the first five terms (which means that the value of K introduced above is 5), we can calculate the approximated bunching function and compare the results with those obtained experimentally. This appears in the curve indexed by 5 in Fig. 7. Here also we note an excellent fit between theoretical and experimental results. On the other hand, the two experimental curves obtained with $K=10$ and $K=20$ cannot be verified by the calculation because in these cases the explicit analytical expressions of the PDF's of lifetimes of order higher than 5 is very complicated.

As a conclusion, two comments are of interest

(i) When they are available, the theoretical calculations fit perfectly the computer results.

(ii) In the domain $t\in[0,3]$, the approximations with 10 or 20 terms are quite similar.

c. Bunching Effects. Simulation results concerning bunching effects are displayed in Figs. 7 and 8. The following comments are of some importance.

The two first subfigures of Fig. 8 are calculated for $r=1$. This means that in the two cases the first-order lifetime has an exponential distribution. The second subfigure is calculated for $p=0$, which means that the PP is a Poisson process with a constant bunching function. This appears in the figure and yields a good idea of the quality of the approximation. These two figures clearly show that the bunching effect cannot be deduced from the analysis of the PDF $f_1(t)$ of the first-order lifetime.

Finally, the last subfigure of Fig. 8 shows that, as expected, the use of a displaced exponential distribution defined by Eq. (4.4) yields an antibunching effect. Other results not reported here show an interesting behavior of the bunching function when r increases, displaying some analogy with the results appearing in Fig. 1. This point will be discussed elsewhere.

ACKNOWLEDGMENTS

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APPENDIX: CALCULATIONS OF THE PDF'S OF LIFETIMES

As indicated above, the lifetime of order 1 is the RV L_1 equal to x_0. The lifetime of order 2 is the RV $L_2=x_0+x_1$ where x_1 is deduced from x_0 by Eq. (4.1). As a consequence L_2 is equal to $2x_0$ and x_0+u_1 with the probabilities p and q, respectively. By repeating the procedure the RV L_3 is equal to $x_0+x_1+x_2$ and Table I indicates its value and their corresponding probabilities.

TABLE I. Method for calculation of $\hat{f}_3(x)$.

RV	$3x_0$	$2x_0+u_2$	x_0+2u_1	$x_0+u_1+u_2$
Prob.	p^2	pq	pq	pq
Symbol	[3,0,0]	[2,1,0]	[2,1,0]	[1,1,1]

The last row indicates the possible structure of the sum of independent RV's. For example [2,1,0] means that in the sum of three terms, two are equal. Continuing with this procedure, the lifetime $L_4=x_0+x_1+x_2+x_3$ takes eight possible values indicated below with their corresponding probabilities and symbols (see Table II).

The same procedure can be applied for L_5 which takes 16 possible values easy to express but not reproduced here. All these RV's are sums of independent RV's. Their PDF's can be obtained by convolutions from the PDF $f_1(t)$ of L_1.

The calculation of these convolutions is tedious, except when the PDF common to all the RV's x_i is exponential or when $f_1(t)=\lambda\exp(-\lambda t)$.

Before continuing, let us point out the following property which simplifies the calculation. If $a_1(t)$ and $a_2(t)$ are two PDF's satisfying $a_i(t)=\lambda\hat{a}_i(x)$, with $x=\lambda t$, their convolution $c(t)$ is a PDF with the same structure—i.e., $c(t)=\lambda\hat{c}(x)$. Its proof is obvious and results immediately from the calculation of the convolution. As a consequence we can assume that $\lambda=1$ and calculate only the functions $\hat{c}(x)$.

The remaining task is now to calculate all the convolutions yielding the PDF's of the RV's appearing in the tables.

Order 2. The possible RV's are $2x_0$ and x_0+u_1. As a result,

$$\hat{f}_1(x)=(1/2)pe_2+qxe_1, \tag{A1}$$

where the functions $e_i=e_i(x)$ are those defined previously.

Order 3. It appears in Table I that there are only three distinct convolutions $h_{3i}(x)$ to calculate. The first noted f_{31} is the PDF of $3x_0$ or $h_{31}(x)=(1/3)e_3$. The second is $h_{32}(x)=[(1/2)e_2*e_1](x)$, and one obtains easily $h_{32}(x)=e_2-e_1$. The last one is $h_{33}(x)=e_1*e_1*e_1$ which is $h_{33}(x)=(1/2)x^2e_1$. Using the relation $\hat{f}_3(x)=p^2h_{31}+2pqh_{32}+q^2h_{33}$ yields Eq. (4.6).

Order 4. Among the eight terms appearing in the table, there are only five different structures giving five functions $h_{4i}(x)$. The function coming from [4,0,0,0] is clearly $h_{41}(x)=(1/4)e_4$. Similarly $h_{42}(x)$ coming from [3,1,0,0] is

TABLE II. Method for calculation of $\hat{f}_4(x)$.

RV	$4x_0$	$3x_0+u_3$	$2x_0+2u_2$	$2x_0+u_2+u_3$	x_0+3u_1	$x_0+2u_1+u_3$	$x_0+u_1+2u_2$	$x_0+u_1+u_2+u_3$
Prob.	p^3	p^2q	p^2q	pq^2	p^2q	pq^2	pq^2	q^3
Symbol	[4,0,0,0]	[3,1,0,0]	[2,2,0,0]	[2,1,1,0]	[3,1,0,0]	[2,1,1,0]	[2,1,1,0]	[1,1,1,1]

$[(1/3)e_3 * e_1](x)$, and the calculation yields $h_{42}(x)=(1/2)(e_3 - e_1)$. The term associated with $[2,2,0,0]$ is $h_{43}(x) = (1/4)[e_2 * e_2](x) = (1/4)xe_2$. For $[2,1,1,0]$ we obtain $h_{44}(x) = [(1/2)e_2 * e_1 * e_1](x)(x) = 2e_2 - (x+2)e_1$. Finally the structure $[1,1,1,1]$ yields $h_{45}(x) = [e_1 * e_1 * e_1 * e_1](x) = (1/6)x^2 e_1$. Grouping all these terms with their corresponding probabilities yields Eq. (4.7).

Order 5. The general procedure is the same and we give only the seven different functions appearing in the calculation of the 16 terms of $\hat{f}_5(x)$:

$$[5,0,0,0,0] \quad h_{51} = (1/5)e_5,$$

$$[4,1,0,0,0] \quad h_{52} = (1/3)(e_4 - e_1),$$

$$[3,2,0,0,0] \quad h_{53} = e_3 - e_2,$$

$$[3,1,1,0,0] \quad h_{54} = (1/4)[3e_3 - (2x+3)e_1],$$

$$[2,2,1,0,0] \quad h_{55} = (1/2)[(x-2)e_2 + 2e_1],$$

$$[2,1,1,1,0] \quad h_{56} = (1/2)[-(x^2+4x+8)e_1 + 8e_2],$$

$$[1,1,1,1,1] \quad h_{57} = (1/24)x^4 e_1. \tag{A2}$$

Using these functions for the calculation of all terms of $\hat{f}_3(x)$ yields Eq. (4.8).

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