Two-mode entanglement in two-component Bose-Einstein condensates

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We study the generation of two-mode entanglement in a two-component Bose-Einstein condensate trapped in a double-well potential. The dynamics of the system is shown to be equivalent to that of two coupled harmonic oscillators in the low excitation regime. Strong entanglement between the two components can be achieved if the interspecies interaction is sufficiently strong. In particular, the condensate constitutes a symmetric Gaussian system and hence its entanglement of formation can be measured directly by fluctuations in the quadratures of the two constituent components.

I. INTRODUCTION

Soon after the experimental realization of Bose-Einstein condensates (BEC's), rich physical phenomena have been observed and predicted as well [1–3]. In particular, there has been a surge of interest in the quantum tunneling dynamics of BEC's trapped in multiple wells, and much attention has been focused on Josephson effect in such systems [2–4]. On the other hand, it was shown that multiparticle entanglement can be generated in BEC's via the coherent interactions between the atoms [5,6], and thereafter BEC's have played a prominent role in the field of quantum information. For example, both multiparticle entanglement (i.e., spin squeezing) and two-mode entanglement can be generated in a spin-1 condensate with three hyperfine sublevels [7,8]. In this case, two-mode entanglement, describing the inseparability between the two modes (respectively with spin projection $m = \pm 1$), can be used as quantum information protocols to facilitate quantum teleportation of continuous variables [9,10].

On the other hand, in addition to BEC's with spin 0 and 1, BEC's with two internal degrees of freedom, e.g., the $|F=1, m=-1\rangle$ and $|F=2, m=2\rangle$ sublevels of ^{87}Rb, are also achievable experimentally and are often termed as two-component condensates [11,12], which give rise to interesting features such as phase separation [13] and the cancellation of mean-field energy shift [14]. Such condensates can be viewed as collections of interacting spin-$\frac{1}{2}$ particles and can consequently display multiparticle entanglement through the emergence of spin squeezing [15]. Likewise, two different kinds of atoms (e.g., ^{41}K and ^{87}Rb) can also form stable two-component BEC's [16]. More interestingly, the interspecies interaction of these two-component BEC's can be varied by the application of magnetic control [17], thus hastening various experimental and theoretical investigations in this field.

Recently, there have been several discussions on the dynamics of a two-component condensate trapped in a double-well potential [18,19]. The behaviors of such systems, which can be realized experimentally with the current technology, are arguably much richer than those of single-component condensates because of the interspecies interaction. For example, in a recent paper [19] we studied the tunneling dynamics of a two-component condensate whose two components are initially separated by the potential barrier between a double well. We found that in the strong scattering regime atoms in the two components can tunnel through the barrier in a correlated manner [19]. As both the scattering strength and the tunneling strength between the wells can be tuned independently with various experimental techniques [17,20], it is expected that such phenomena will become observable in the near future.

In this paper, we consider a two-component condensate trapped in a double-well potential. Both components of the condensate are initially prepared in the ground state of the double well and hence are separable at time $t=0$. The objective of the present paper is to study and quantify the generation of two-mode entanglement in such a condensate. Two-mode entanglement is commonly attributed to the inseparability of the density matrix describing two systems, which can be found in various experimental situations and is very useful in the applications of quantum measurement and quantum information [21–23]. In fact, entanglement between two ensembles of atoms with spin has recently been achieved experimentally via interaction with polarized light [23], and the concept of two-mode entanglement has been generalized to describe such spin systems [7]. Since each component of a two-component condensate in a double well can be described as a collection of spin-$\frac{1}{2}$ particles, with the two spatially localized modes in the two wells playing respectively the roles of spin-up and spin-down states, the condensate is in fact equivalent to two interacting gigantic spins that can be described in terms of continuous variables [7]. While it has been shown that the intraspecies interaction is able to create spin squeezing for a single-component condensate trapped a double-well potential [15], in this paper we will demonstrate that the interspecies interaction is responsible for the generation of two-mode entanglement. It is the intriguing interplay of the interspecies and the intraspecies interactions that sparks our investigation in such systems. By applying the Holstein-Primakoff transformation (HPT) [24], which reduces this two-spin system to two coupled oscillators, we show that our system is exactly solvable as long as the number of excitations due to atom-atom interactions remains low.

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More remarkably, the condensate in fact forms a symmetric Gaussian system. Therefore we can directly evaluate the entanglement of formation (i.e., the von Neumann entropy for a pure state) of the system from the fluctuations in the quadratures of the two constituent components [7,21,22,25]. In accordance with the schemes proposed recently by Duan *et al.* [21,22] and Giedke *et al.* [25], we define a two-mode entanglement parameter that measures the fluctuations in the quadratures and in turn analyze the degree of entanglement between the two components of the condensate in the present paper. Our discovery is that strong interspecies interaction can lead to significant two-mode squeezing in the condensate and hence strong entanglement between the two components.

The structure of our paper is as follows. In Sec. II, we introduce the Hamiltonian of our system and classify it into symmetric and asymmetric cases according to the properties of the constituent condensates. In Sec. III, we consider specifically a system with a large number of atoms. An effective Hamiltonian, which is exactly solvable, is then derived from the HPT. In Sec. IV, we introduce the two-mode entanglement parameter [7,21,22,25] to describe and quantify the entanglement in our system. In Sec. V, we study in detail the two-mode entanglement parameter for several typical cases. Finally, we discuss the physical meaning of the two-mode entanglement parameter in Sec. VI and consider generalization our approach to mixed states.

II. TUNNELING TWO-COMPONENT BEC

We first consider the tunneling dynamics of a two-component BEC trapped in a symmetric double-well potential. The total number of atoms in components A and B of the condensate are N_a and N_b, respectively. We will further assume that the interaction between the atoms is sufficiently weak and adopt the two-mode approximation to describe the tunneling process. Under such approximation, the condensate dwelling in each potential minimum is adequately described by a single localized mode function [26–28]. Considering the effect of quantum tunneling and the conservation of the particle number of each component, we obtain the Hamiltonian of the system:

$$H = \frac{\Omega_a}{2}(\hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L) + \frac{\Omega_b}{2}(\hat{b}_L^\dagger \hat{b}_R + \hat{b}_R^\dagger \hat{b}_L) + \kappa(\hat{a}_L^\dagger \hat{a}_L \hat{b}_L^\dagger \hat{b}_L$$
$$+ \hat{a}_R^\dagger \hat{a}_R \hat{b}_R^\dagger \hat{b}_R) + \frac{\kappa_a}{2}[(\hat{a}_L^\dagger \hat{a}_L)^2 + (\hat{a}_R^\dagger \hat{a}_R)^2] + \frac{\kappa_b}{2}[(\hat{b}_L^\dagger \hat{b}_L)^2$$
$$+ (\hat{b}_R^\dagger \hat{b}_R)^2]. \quad (2.1)$$

Here $\hat{a}_j^\dagger$ ($\hat{a}_j$) and $\hat{b}_j^\dagger$ ($\hat{b}_j$) are, respectively, the creation (annihilation) operators of components A and B residing in the jth well, $j=L,R$. Since there are two spatial modes (the L and R modes) available for each component, the Hamiltonian above in fact consists of four Bosonic operators. Besides, the parameters $\Omega_a(\Omega_b)$, $\kappa_a(\kappa_b)$, and κ are the tunneling, intraspecies interaction strength of component $A(B)$, and the interspecies interaction strength, respectively.

For the convenience of the subsequent discussion of two-mode entanglement, it is instructive to represent this Hamil-

tonian in terms of angular momentum operators by following through the standard Schwinger oscillator model to construct a set of spin operators for each component [29]:

$$\hat{J}_{\alpha x} = \frac{1}{2}(\hat{\alpha}_L^\dagger \hat{\alpha}_L - \hat{\alpha}_R^\dagger \hat{\alpha}_R),$$

$$\hat{J}_{\alpha y} = \frac{1}{2i}(\hat{\alpha}_L^\dagger \hat{\alpha}_R - \hat{\alpha}_R^\dagger \hat{\alpha}_L), \quad (2.2)$$

$$\hat{J}_{\alpha z} = \frac{1}{2}(\hat{\alpha}_L^\dagger \hat{\alpha}_R + \hat{\alpha}_R^\dagger \hat{\alpha}_L),$$

where $\alpha = a, b$. Here $\hat{\mathbf{J}}_\alpha = (\hat{J}_{\alpha x}, \hat{J}_{\alpha y}, \hat{J}_{\alpha z})$ obey the usual angular momentum commutation relations. In the following we will denote the eigenstates of $\hat{J}_\alpha^2 \equiv \hat{J}_{\alpha x}^2 + \hat{J}_{\alpha y}^2 + \hat{J}_{\alpha z}^2$ and $\hat{J}_{\alpha z}$ with $|j_\alpha, m_\alpha\rangle$ such that $\hat{J}_\alpha^2 |j_\alpha, m_\alpha\rangle = j_\alpha(j_\alpha+1)|j_\alpha, m_\alpha\rangle$, and $\hat{J}_{\alpha z}|j_\alpha, m_\alpha\rangle = m_\alpha |j_\alpha, m_\alpha\rangle$, where $j_\alpha = N_\alpha/2$. After discarding some constant terms, we rewrite the Hamiltonian (2.1) in terms of $\hat{\mathbf{J}}_a$ and $\hat{\mathbf{J}}_b$:

$$H = \Omega_a \hat{J}_{az} + \Omega_b \hat{J}_{bz} + \kappa_a \hat{J}_{ax}^2 + \kappa_b \hat{J}_{bx}^2 + 2\kappa \hat{J}_{ax}\hat{J}_{bx}. \quad (2.3)$$

Despite the signs of the tunneling coefficients Ω_a and Ω_b are unimportant, we assume, for concreteness, $\Omega_a > 0$ and $\Omega_b > 0$ in the following discussion.

In the absence of atom-atom interactions, the ground state of the system is obviously given by the product state $|j_a, -j_a\rangle |j_b, -j_b\rangle$ and the two components are not entangled. In the following discussion we consider how the atom-atom interactions affect the evolution of the initial state,

$$|\Psi(t=0)\rangle = |j_a, -j_a\rangle |j_b, -j_b\rangle, \quad (2.4)$$

and show that the two components will get entangled through the inter- and intraspecies interactions.

To facilitate later discussion on the phenomenon of entanglement, we further classify two-component condensates according to the symmetry properties of the two components constituting the condensate. In a *symmetric* two-component BEC, the parameters of component A and component B are equal to one another, namely $\Omega_a = \Omega_b$ and $\kappa_a = \kappa_b$. These conditions hold approximately for condensates consisting of the hyperfine states $|F=1, m=-1\rangle$ and $|F=2, m=1\rangle$ of ^{87}Rb [12]. The two components of such condensates have essentially same masses and magnetic moments and hence $\Omega_a = \Omega_b$. Their intraspecies scattering lengths are quite close and, in addition, $\kappa_a \approx \kappa_b \approx \kappa$ [11]. In a *quasi-identical* two-component BEC where $\Omega_a = \Omega_b = \Omega$ and $\kappa_a = \kappa_b = \kappa$, the Hamiltonian (2.3) reduces to

$$H_1 = \Omega \hat{J}_z + \kappa \hat{J}_x^2, \quad (2.5)$$

where $\hat{\mathbf{J}} = \hat{\mathbf{J}}_a + \hat{\mathbf{J}}_b$ is the total angular momentum of the system. Despite that the Hamiltonian of such a two-component condensate is identical to that of a single-component one [15], the distinguishability of the two species entails the study of two-mode entanglement.

Meanwhile, for *asymmetric* two-component BEC's relevant physical parameters of the two components are gener-

ally different. For example, it has recently been observed in the experiment that the condensates of potassium and rubidium (Rb—K), which have different scattering lengths and masses, can form stable two-component condensates [16]. Therefore it is deemed appropriate to develop a generic analytical scheme to study such condensates. In the following discussion, we will make use of the HPT [24] to carry out a thorough analytical investigation on the entanglement between the two constituent components.

III. BOSONIC OPERATOR APPROXIMATION

In this section, we consider the evolution of a condensate with large numbers of atoms and sufficiently weak scattering strengths, namely $N_{a(b)} \gg 1$ and $\Omega_{a(b)} \gg \kappa_a, \kappa_b, \kappa$. As the initial state, given by Eq. (2.4), is the ground state of a noninteracting condensate and the scattering strengths are weak, only the low-lying states will be excited in the evolution and the coherence of tunneling process can be maintained. The current situation is in contradistinction to our previous study [19] that discovered correlated tunneling of the two components in the strong scattering regime. However, we will show that the interspecies interaction does lead to nontrivial entanglement of the two components.

To proceed, we apply the HPT to map angular momentum operators into Bosonic operators [24,30] and show that under the HPT our system is in fact equivalent to two coupled harmonic oscillators. In HPT, the angular momentum operators

$$\hat{J}_{\alpha\pm} = \hat{J}_{\alpha x} \pm i\hat{J}_{\alpha y}, \quad \alpha = a,b, \tag{3.1}$$

and $\hat{J}_{\alpha z}$ are expressed in terms of Bosonic operators $\hat{\alpha}$, $\hat{\alpha}^\dagger$:

$$\hat{J}_{\alpha+} = \hat{\alpha}^\dagger \sqrt{2j_\alpha - \hat{\alpha}^\dagger \hat{\alpha}}, \tag{3.2}$$

$$\hat{J}_{\alpha-} = (\sqrt{2j_\alpha - \hat{\alpha}^\dagger \hat{\alpha}})\hat{\alpha}, \tag{3.3}$$

$$\hat{J}_{\alpha z} = (\hat{\alpha}^\dagger \hat{\alpha} - j_\alpha). \tag{3.4}$$

Here $\hat{\alpha}^\dagger$ and $\hat{\alpha}$ are standard Bosonic operators satisfying $[\hat{\alpha}, \hat{\alpha}^\dagger] = 1$. Hence the Hamiltonian (2.3) can be written as

$$H = \sum_{\alpha=a,b} \left[\Omega_\alpha(\hat{\alpha}^\dagger \hat{\alpha} - j_\alpha) + \frac{\kappa_\alpha j_\alpha}{2} \left(\hat{\alpha}^\dagger \sqrt{1 - \frac{\hat{\alpha}^\dagger \hat{\alpha}}{2j_\alpha}} \right. \right.$$
$$+ \left. \sqrt{1 - \frac{\hat{\alpha}^\dagger \hat{\alpha}}{2j_\alpha}} \hat{\alpha} \right)^2 \right] + \kappa \sqrt{j_a j_b} \left(\hat{a}^\dagger \sqrt{1 - \frac{\hat{a}^\dagger \hat{a}}{2j_a}} \right.$$
$$+ \left. \sqrt{1 - \frac{\hat{a}^\dagger \hat{a}}{2j_a}} \hat{a} \right) \left(\hat{b}^\dagger \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2j_b}} + \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2j_b}} \hat{b} \right). \tag{3.5}$$

Since $\Omega_{a(b)} \gg \kappa, \kappa_a, \kappa_b$, it is arguable that

$$\frac{\langle \hat{\alpha}^\dagger \hat{\alpha} \rangle}{2j_\alpha} \ll 1, \tag{3.6}$$

leading to an approximate effective Hamiltonian,

$$H_{\text{eff}} = \Omega_a \hat{a}^\dagger \hat{a} + \Omega_b \hat{b}^\dagger \hat{b} + \frac{1}{2}[\kappa_a j_a(\hat{a}^\dagger + \hat{a})^2 + \kappa_b j_b(\hat{b}^\dagger + \hat{b})^2$$
$$+ 2\kappa\sqrt{j_a j_b}(\hat{a}^\dagger + \hat{a})(\hat{b}^\dagger + \hat{b})]. \tag{3.7}$$

This Hamiltonian is analogous to that of two coupled oscillators and completely captures the essence of the dynamics of the two interacting BEC's. Correspondingly, the initial state (2.4) is given by the vacuum state $|0_a, 0_b\rangle$ of the two decoupled oscillators described by the first two terms in H_{eff}, where $|n_a, n_b\rangle$ represents the Fock state of the oscillators.

To study the two-mode entanglement in the condensate, it is advantageous to make use of the position and the momentum operators:

$$\hat{q}_a = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p}_a = i\frac{1}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{3.8}$$

$$\hat{q}_b = \frac{1}{\sqrt{2}}(\hat{b}^\dagger + \hat{b}), \quad \hat{p}_b = i\frac{1}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}), \tag{3.9}$$

and to rewrite the effective Hamiltonian as

$$H_{\text{eff}} = \frac{\Omega_a}{2}(\hat{q}_a^2 + \hat{p}_a^2) + \frac{\Omega_b}{2}(\hat{q}_b^2 + \hat{p}_b^2) + \kappa_a j_a \hat{q}_a^2 + \kappa_b j_b \hat{q}_b^2$$
$$+ 2\kappa\sqrt{j_a j_b}\hat{q}_a\hat{q}_b. \tag{3.10}$$

It is then straightforward to solve the resulting equations of motion of $\hat{q}_a(t)$, $\hat{p}_a(t)$, $\hat{q}_b(t)$, and $\hat{p}_b(t)$. For convenience, we express the solutions in matrix form, which reads

$$X(t) \equiv (\hat{q}_a(t), \hat{p}_a(t), \hat{q}_b(t), \hat{p}_b(t))^T = U(t)X(t=0). \tag{3.11}$$

Here $U(t)$ is a real 4×4 matrix representing the evolution operator and can be written as

$$U = \begin{pmatrix} C & E_1 \\ E_2 & D \end{pmatrix},$$

with the 2×2 matrices C, D, and E being explicitly given by

$$C = \frac{1}{(\mu_2 - \mu_1)} \begin{pmatrix} \mu_2 \cos \omega_1 t - \mu_1 \cos \omega_2 t & \Omega_a(\mu_2 \sin \omega_1 t/\omega_1 - \mu_1 \sin \omega_2 t/\omega_2) \\ -(\omega_1\mu_2 \sin \omega_1 t - \omega_2\mu_1 \sin \omega_2 t)/\Omega_a & \mu_2 \cos \omega_1 t - \mu_1 \cos \omega_2 t \end{pmatrix},$$

$$D = \frac{1}{(\mu_2 - \mu_1)} \begin{pmatrix} -\mu_1 \cos \omega_1 t + \mu_2 \cos \omega_2 t & -\Omega_b(\mu_1 \sin \omega_1 t/\omega_1 - \mu_2 \sin \omega_2 t/\omega_2) \\ (\mu_1\omega_1 \sin \omega_1 t - \mu_2\omega_2 \sin \omega_2 t)/\Omega_b & -\mu_1 \cos \omega_1 t + \mu_2 \cos \omega_2 t \end{pmatrix},$$

$$E_1 = \frac{1}{(\mu_2 - \mu_1)} \begin{pmatrix} -(\cos \omega_1 t - \cos \omega_2 t) & -\Omega_b(\sin \omega_1 t/\omega_1 - \sin \omega_2 t/\omega_2) \\ (\omega_1 \sin \omega_1 t - \omega_2 \sin \omega_2 t)/\Omega_a & -\Omega_b(\cos \omega_1 t - \cos \omega_2 t)/\Omega_a \end{pmatrix},$$

and

$$E_2 = \frac{1}{(\mu_2 - \mu_1)} \begin{pmatrix} -\Omega_b(\cos \omega_1 t - \cos \omega_2 t)/\Omega_a & -\Omega_b(\sin \omega_1 t/\omega_1 - \sin \omega_2 t/\omega_2) \\ (\omega_1 \sin \omega_1 t - \omega_2 \sin \omega_2 t)/\Omega_a & -(\cos \omega_1 t - \cos \omega_2 t) \end{pmatrix},$$

where

$$\mu_{1(2)} = \frac{\omega_{1(2)}^2 - \Omega_a^2 - 2\kappa_a j_a \Omega_a}{2\kappa \Omega_a \sqrt{j_a j_b}}, \tag{3.12}$$

and the normal-mode frequencies of the coupled oscillation are

$$\omega_{1(2)} = \left\{ \left(\frac{\Omega_a^2 + \Omega_b^2}{2} + \kappa_a j_a \Omega_a + \kappa_b j_b \Omega_b \right) \pm \left[\left(\frac{\Omega_a^2 - \Omega_b^2}{2} \right. \right. \right.$$
$$\left. \left. \left. + \kappa_a j_a \Omega_a - \kappa_b j_b \Omega_b \right)^2 + 4\kappa^2 j_a j_b \Omega_a \Omega_b \right]^{1/2} \right\}^{1/2}. \tag{3.13}$$

This approximate solution, which is based on HPT, is valid as long as the condition (3.6) holds. However, if the normal frequency is complex, the system will become unstable. We can therefore determine the stability condition from Eq. (3.13):

$$|\kappa| < \frac{1}{2} \sqrt{\left(\frac{\Omega_a}{j_a} + 2\kappa_a \right)\left(\frac{\Omega_b}{j_b} + 2\kappa_b \right)} \equiv \kappa_c. \tag{3.14}$$

The HPT fails to yield a self-consistent solution for systems violating the inequality. It is interesting to note that in the limit where $j_a, j_b \rightarrow \infty$, the condition for stability reduces to

$$|\kappa| < \sqrt{\kappa_a \kappa_b} \equiv \kappa_e, \tag{3.15}$$

which is a well-known result for BEC's in extended space, and violation of Eq. (3.15) will lead to the phase separation of two-component BEC's [13,31]. It is also worthwhile to note that the stability criterion (3.14) depends on the numbers of atoms in the double well and similar dependence has previously been found for two-component BEC's in a single well [31]. We will, however, assume the stability criterion (3.14) is satisfied throughout the present study and obtain analytically the two-mode entanglement parameter for the condensate, which will be defined in the following section. We will see that in addition to yielding the analytic solution to the tunneling dynamics, the HPT performed here also facilitates our study on two-mode entanglement.

IV. THEORY OF TWO-MODE ENTANGLEMENT

Entanglement between two systems that are described in terms of continuous variables is usually indicated by an inequality in its Einstein-Podolsky-Rosen (EPR) uncertainty [7,21,22]:

$$\frac{1}{2}\{\langle[\Delta(\hat{q}_a + \hat{q}_b)]^2\rangle + \langle[\Delta(\hat{p}_a - \hat{p}_b)]^2\rangle\} < 1, \tag{4.1}$$

where $[\hat{q}_m, \hat{p}_n] = i\delta_{mn}$ for $m, n = a, b$. $\hat{q}_{a(b)}$ and $\hat{p}_{a(b)}$ are, respectively, the position and momentum operators (or any pairs of quadratures) of system $a(b)$, and the above inequality simply implies that the positions (momenta) of the particles are strongly anticorrelated (correlated). In general, condition (4.1) is only a sufficient condition for entanglement and does not imply separability of the two systems even if it is violated [7,21]. However, it has recently been shown that a necessary and sufficient condition for entanglement, which is analogous to Eq. (4.1), can be established if the combined system is a Gaussian one in the sense that if its Wigner characteristic function, defined by

$$\chi^{(w)}(\lambda_a, \lambda_b) = \text{tr}[\rho \exp(\lambda_a \hat{a} - \lambda_a^* \hat{a}^\dagger + \lambda_b \hat{b} - \lambda_b^* \hat{b}^\dagger)], \tag{4.2}$$

is a Gaussian function of λ_a and λ_b [21,25]. Without loss of generality, one can assume that the expectation values of all quadratures vanish and hence the Wigner characteristic function of a Gaussian system is expressible as

$$\chi^{(w)}(\lambda_a, \lambda_b) = \exp\left[-\frac{1}{2}\Lambda^T M \Lambda \right], \tag{4.3}$$

where M is a 4×4 real symmetric matrix and the matrix Λ is defined by $\Lambda \equiv (\lambda_a^I, \lambda_a^R, \lambda_b^I, \lambda_b^R)^T$. As the characteristic function can also be written as

$$\chi^{(w)}(\lambda_a, \lambda_b) = \text{tr}[\rho \exp(i\sqrt{2}\Lambda^T X)], \tag{4.4}$$

it is obvious that the matrix elements of M are the correlation functions of the quadrature variables $X = (\hat{q}_a, \hat{p}_a, \hat{q}_b, \hat{p}_b)^T$. In fact, $M_{ij} = \langle(X_i X_j + X_j X_i)\rangle$ and therefore M is termed the covariance matrix.

As the amount of entanglement between the two systems is unaffected by local unitary operations, say local rotations in the q-p plane and local squeezing operations, the matrix M can be transformed into a standard form M_s by several local operations [21]:

$$M_s = \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{pmatrix},$$

where n_1, n_2, m_1, m_2 are positive numbers, and

$$\frac{n_1 - 1}{m_1 - 1} = \frac{n_2 - 1}{m_2 - 1}, \tag{4.5}$$

$$|c_1| - |c_2| = \sqrt{(n_1 - 1)(m_1 - 1)} - \sqrt{(n_2 - 1)(m_2 - 1)}. \tag{4.6}$$

It has recently been shown by Duan *et al.* that a Gaussian system is entangled if and only if the following inequality is satisfied [21,25]:

$$a_0^2(n_1 + n_2) - 2(|c_1| + |c_2|) + (m_1 + m_2)/a_0^2 < 2a_0^2 + 2/a_0^2, \tag{4.7}$$

where

$$a_0^2 = \sqrt{\frac{m_1 - 1}{n_1 - 1}} = \sqrt{\frac{m_2 - 1}{n_2 - 1}}. \tag{4.8}$$

We therefore accordingly construct a two-mode entanglement parameter ξ_t [21,25]:

$$\xi_t \equiv \frac{a_0^2(n_1 + n_2) - 2(|c_1| + |c_2|) + (m_1 + m_2)/a_0^2}{2a_0^2 + 2/a_0^2}. \tag{4.9}$$

Physically speaking, $\xi_t > 0$ is merely a suitably weighted EPR-type uncertainty in the squeezed quadratures of the system. The sufficient and necessary condition for entanglement mentioned above can then be expressed in terms of an inequality involving this parameter:

$$\xi_t < 1. \tag{4.10}$$

If, in addition, the system is a symmetric one such that $n_1 = n_2 = m_1 = m_2$, Giedke *et al.* [25] have recently shown that the parameter ξ_t can also yield the entanglement of formation (EOF) of the system, E_F, [32]:

$$E_F(\xi_t) = c_+(\xi_t)\ln[c_+(\xi_t)] - c_-(\xi_t)\ln[c_-(\xi_t)] \text{ if } 0 < \xi_t < 1,$$

$$= 0 \text{ otherwise}, \tag{4.11}$$

where the functions $c_\pm(z)$ are defined by $c_\pm(z) = [(z)^{-1/2} \pm (z)^{1/2}]^2/4$. It is noteworthy that EOF is a proper measure of the degree of entanglement between two systems and is equal to the von Neumann entropy if the compound system remains in a pure state. However, unlike the von Neumann entropy, EOF still works for mixed states. In fact, Josse *et al.* have recently measured the EOF of a pair of non separable light beams with this scheme [33]. In the following

section, we shall make use of the two-mode entanglement parameter ξ_t to study how the two components of a BEC condensate trapped in double well are entangled.

V. TWO-MODE ENTANGLEMENT IN BEC'S

To study two-mode entanglement in BEC's, we first show that a two-component BEC indeed forms a Gaussian system. Since

$$\chi^{(w)}(\lambda_a, \lambda_b, t) = \text{tr}\{\rho \exp[i\sqrt{2}\Lambda^T X(t)]\}$$
$$= \text{tr}\{\rho \exp[i\sqrt{2}(U(t)^T\Lambda)^T X(t=0)]\}, \tag{5.1}$$

$\chi^{(w)}(\lambda_a, \lambda_b, t)$ can be obtained from $\chi^{(w)}(\lambda_a, \lambda_b, t=0)$ and we therefore consider the characteristic function at $t=0$. For the initial state $|0_a, 0_b\rangle$, it is readily shown that $\chi^{(w)}(\lambda_a, \lambda_b, t=0)$ is a Gaussian function and $M(t=0)$ is an identity matrix. $\chi^{(w)}(\lambda_a, \lambda_b, t)$ at other times can simply be obtained by the replacement $\Lambda \to U(t)\Lambda$. Consequently, we show that $\chi^{(w)}(\lambda_a, \lambda_b, t)$ is a Gaussian function of Λ and

$$\chi^{(w)}(\lambda_a, \lambda_b, t) = \exp\left\{-\frac{1}{2}[U^T(t)\Lambda]^T M(0) U^T(t)\Lambda\right\}, \tag{5.2}$$

which directly yields the covariance matrix $M(t)$:

$$M(t) = U(t)M(0)U^T(t) = U(t)U^T(t). \tag{5.3}$$

Hence the condensate constitutes a Gaussian system to which the two-mode entanglement parameter applies.

After applying several local unitary transformations to $M(t)$, we obtain the matrix $M_s(t)$ of our system, which reads

$$M_s(t) = \begin{pmatrix} n & 0 & c & 0 \\ 0 & n & 0 & -c \\ c & 0 & n & 0 \\ 0 & -c & 0 & n \end{pmatrix}.$$

Moreover, the matrix elements of the matrix M_s are expressible in terms of the variances of the physical quantities as follows:

$$n = 2\{\langle\hat{q}_a^2(t)\rangle\langle\hat{p}_a^2(t)\rangle - [\text{Re}\langle\hat{q}_a(t)\hat{p}_a(t)\rangle]^2\}^{1/2}$$
$$= 2\{\langle\hat{q}_b^2(t)\rangle\langle\hat{p}_b^2(t)\rangle - [\text{Re}\langle\hat{q}_b(t)\hat{p}_b(t)\rangle]^2\}^{1/2}, \tag{5.4}$$

$$c = 2[\text{Re}(\langle\hat{q}_a(t)\hat{p}_b(t)\rangle\langle\hat{q}_b(t)\hat{p}_a(t)\rangle$$
$$- \langle\hat{q}_a(t)\hat{q}_b(t)\rangle\langle\hat{p}_a(t)\hat{p}_b(t)\rangle)]^{1/2}, \tag{5.5}$$

and the variances are given in the Appendix for reference. It is then obvious that our system is a symmetric one with $n_1 = n_2 = m_1 = m_2 = n$ and hence $a_0 = 1$. Therefore one can use the corresponding two-mode entanglement parameter, which is given by

$$\xi_t = n - c, \tag{5.6}$$

to evaluate the EOF of the system [25].

Now we are ready to investigate the entanglement in our system with the help of ξ_t and $E_F(\xi_t)$. We first consider a
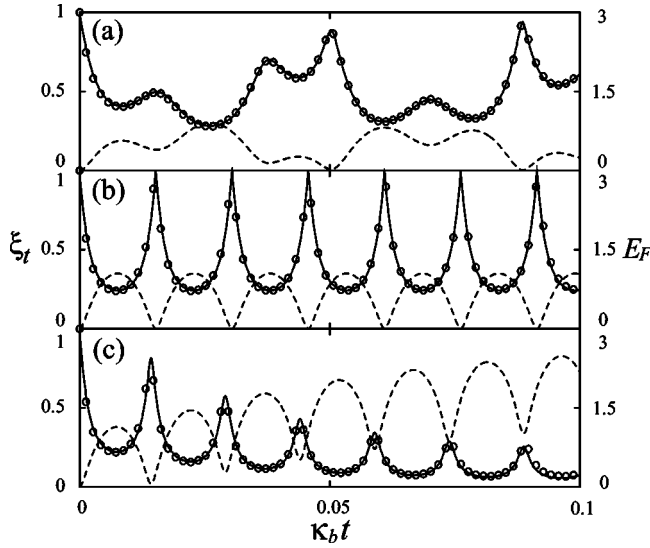
FIG. 1. The time evolution of ξ_t (solid line, left scale) and the EOF E_F (dashed line, right scale) are shown for the symmetric case with $N_a=N_b=400$; $\Omega_a=\Omega_b=50\kappa_b$; and (a) $\kappa/\kappa_b=0.50$; (b) $\kappa/\kappa_b=1.00$; (c) $\kappa/\kappa_b=1.12$. To gauge the accuracy of the HPT, the exact numerical solution of ξ_t is shown by the empty circles.
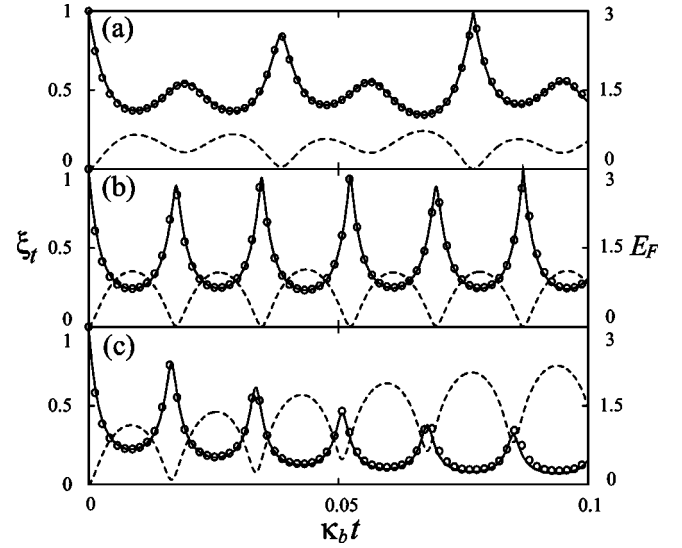

FIG. 2. The time evolution of ξ_t (solid line, left scale) and the EOF E_F (dashed line, right scale) are shown for the asymmetric case with $N_a=N_b=400$; $\kappa_a=0.75\kappa_b$; $\Omega_a=34.5\kappa_b$, $\Omega_b=50\kappa_b$; and (a) $\kappa/\kappa_b=0.50$; (b) $\kappa/\kappa_b=0.875$; (c) $\kappa/\kappa_b=0.965$. To gauge the accuracy of the HPT, the exact numerical solution of ξ_t is shown by the empty circles.

symmetric two-component condensate (e.g., a Rb—Rb condensate), where $\kappa_a=\kappa_b$ and $\Omega_a=\Omega_b$. The time evolution of ξ_t for such a condensate is shown in Fig. 1, where $N_a=N_b=400$, $\Omega_a=\Omega_b=50\kappa_b$, and $\kappa=0.50\kappa_b$ [Fig. 1(a)]; $\kappa=\kappa_b$ [Fig. 1(b)]; $\kappa=1.12\kappa_b$ [Fig. 1(c)]. The solid line and empty circles are, respectively, results obtained from the HPT and numerical diagonalization of the original Hamiltonian, showing that the HPT indeed yields a good approximation in this regime. We also show the time evolution of the EOF, which equals the von Neumann entropy in this case, by the dashed line in Fig. 1. It is obvious that our system is able to generate a substantial amount of two-mode entanglement for most of the time. Besides, one can see that there is a strong anticorrelation between these two curves, which can be understood as $E_F(\xi_t)$ is a monotonically decreasing function of ξ_t for $0 \leq \xi_t \leq 1$.

It is remarkable that the degree of entanglement depends crucially on the interspecies interaction. Of course, it is obvious that the EOF or the entropy is zero when the interspecies interaction κ vanishes. As shown in Figs. 1(a)–1(c), the entanglement parameter ξ_t decreases while the EOF increases with increasing κ. Therefore one can achieve optimal squeezing by properly controlling the interaction parameters. Besides, it is worthy of remark that in Fig. 1(c) $\kappa_c > \kappa > \kappa_e$. Therefore the system is still a stable one and the HPT remains valid. Physically speaking, the availability of the two spatially separated modes in fact stabilizes the condensate despite that $\kappa > \kappa_e$ [31].

We now switch our attention to asymmetric BEC's consisting of two components with different physical characteristics. In fact, stable BEC's of rubidium and potassium have recently been achieved in experiments and it is also possible to control the strength of interspecies interaction between the two components with a magnetic field [17]. It is therefore deemed appropriate to investigate how the entanglement in

such condensates changes with the interspecies interaction strength κ. In the following, we assume that the tunneling strengths and the intraspecies interaction strengths of the two species are in the ratios of 1:1.45 and 1:1.33, respectively, which are reasonable estimates of experimental data for a Rb—K condensate [26].

The two-mode entanglement parameters for three asymmetric cases with $\kappa/\kappa_b=0.5,0.875,0.965$ are, respectively, shown in Figs. 2(a)–2(c). Considering the interspecies interaction κ as an adjustable parameter, we find that a smaller ξ_t (i.e., higher entanglement) can be obtained as the system becomes closer to the point of stability limit given by Eq. (3.14). As shown in Fig. 2(c), if κ is increased and approaches the stability limit given by Eq. (3.14) from below, the two-mode entanglement parameter (the EOF) can attain much smaller (greater) values. Thus the significance of the strength of interspecies interaction in two-mode entanglement generation is clearly demonstrated.

From the results illustrated in Figs. 1 and 2 it is manifest that a substantial increase of entanglement can be achieved in the vicinity of the stability limit (3.14). In fact, this interesting feature can be understood heuristically as follows. In general, position (momentum) squeezed states of a harmonic oscillator can be produced by strengthening (weakening) its spring constant [34]. The two-component condensates can be viewed as a coupled oscillators and at the critical point of stability the eigenfrequenices are $(\Omega_a^2+\Omega_b^2+2\kappa_a j_a \Omega_a +2\kappa_b j_b \Omega_b)^{1/2}$ and zero. As one of these frequencies, $\omega_2=0$, is markedly different from those in the interaction free case, the effect of squeezing in the position space and the momentum space is much pronounced in the vicinity of the critical point.

To further elaborate this issue, we show the minimal value of ξ_t during the evolution of the coupled condensates and the corresponding EOF as functions of the interspecies interac-
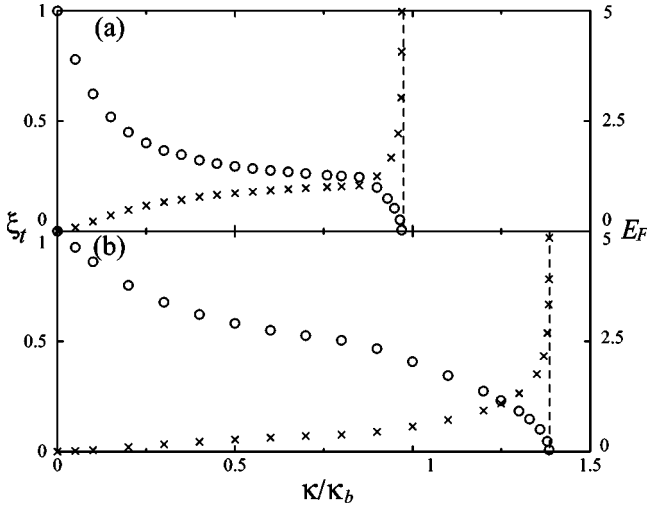
FIG. 3. The maximal achievable values of ξ_t (empty circles, left scale) and the EOF E_F (crosses, right scale) during time evolution are shown for different values of the normalized interspecies interaction strength κ/κ_b. The condensate is an asymmetric one with $N_a=N_b=400$; $\kappa_a=0.75\kappa_b$; and (a) $\Omega_a=34.5\kappa_b$, $\Omega_b=50\kappa_b$, $\kappa_c=0.969\kappa_b$; (b) $\Omega_a=172.5\kappa_b$, $\Omega_b=250\kappa_b$, $\kappa_c=1.385\kappa_b$. The value of κ_c is shown by the vertical dashed line in each diagram.

tion strength κ in Fig. 3, which explicitly confirms that the degree of entanglement increases drastically as the interspecies interaction κ is close to the stability limit given by Eq. (3.14). In fact, both quantities change noticeably once $\kappa > \kappa_e$. It is noteworthy that similar increase in entanglement has previously been found in quantum phase transition of a spin-chain model [35].

VI. DISCUSSION

In the present paper the entanglement between the two components of a BEC condensate trapped in a double well is studied analytically in the low excitation limit with the HPT, and its accuracy is confirmed by comparison with the exact numerical solution. As demonstrated in previous sections, the degree of entanglement, gauged by the two-mode entanglement parameter, depends strongly on the interspecies interaction that can be varied with the current technology [17,20]. We expect that our work can be applied to study entanglement in two-component condensates such as Rb—Rb and Rb—K mixtures. Specifically, our result shows that the two components of the condensate can remain in the tunneling phase and yet get strongly entangled as long as $\kappa_c > \kappa > \kappa_e$. On the other hand, it is worthy of remark that the two-mode approximation adopted here fails to describe the long-term dephasing of Josephson oscillations between two Bose-Einstein condensates [36], which is an important issue for asymmetric condensates. However, the results obtained here still hold before such effect begins to prevail.

To gain more physical insight from our result, we note that $\hat{J}_{\alpha x}$ and $\hat{J}_{\alpha y}$, respectively, represent the population difference (measured by the operator $\delta\hat{n}_\alpha \equiv \hat{\alpha}_R^+\hat{\alpha}_R - \hat{\alpha}_L^+\hat{\alpha}_L$) and the phase difference (measured by the relative phase operator

$\delta\hat{\phi}_\alpha$) of the α-species condensate in the two wells, where $\alpha=a,b$ [3]. Therefore the two-mode entanglement parameter $\xi_t=n-c$, where

$$n = 2\{\langle\delta\hat{n}_a^2(t)\rangle\langle\delta\hat{\phi}_a^2(t)\rangle - [\text{Re}\langle\delta\hat{n}_a(t)\delta\hat{\phi}_a(t)\rangle]^2\}^{1/2}$$

$$= 2\{\langle\delta\hat{n}_b^2(t)\rangle\langle\delta\hat{\phi}_b^2(t)\rangle - [\text{Re}\langle\delta\hat{n}_b(t)\delta\hat{\phi}_b(t)\rangle]^2\}^{1/2}, \quad (6.1)$$

$$c = 2\{\text{Re}[\langle\delta\hat{n}_a(t)\delta\hat{\phi}_b(t)\rangle\langle\delta\hat{n}_b(t)\delta\hat{\phi}_a(t)\rangle$$
$$- \langle\delta\hat{n}_a(t)\delta\hat{n}_b(t)\rangle\langle\delta\hat{\phi}_a(t)\delta\hat{\phi}_b(t)\rangle]\}^{1/2}, \quad (6.2)$$

indeed measure the correlation of $\delta\hat{n}_\alpha$ and $\delta\hat{\phi}_\alpha$. If the two components are entangled and therefore $\xi_t<1$, the fluctuations in the population difference and phase difference of the composite system are squeezed accordingly. Therefore it is likely that the two-mode entanglement parameter between the condensates could be measured directly with the current experimental techniques (see, e.g., Ref. [37]).

On the other hand, in real experiments the temperature of the condensate is not exactly zero [37]. Therefore it is worth studying how the effect of finite temperature might affect the entanglement that could be built up in the system during its evolution when the initial state is a mixed state. Specifically, we consider an initial state that can be written as a product of two thermally equilibrium states maintained at a common temperature T, and the density matrix at $t=0$ is given by

$$\rho(0) = \rho_a \otimes \rho_b, \quad (6.3)$$

where for $\alpha=a,b$,

$$\rho_\alpha = [1 - \exp(-\Omega_\alpha/k_BT)]\exp(-\Omega_\alpha\hat{a}_\alpha^\dagger\hat{a}_\alpha/k_BT). \quad (6.4)$$

It is a mixed stated and its two components are obviously separable at $t=0$. The covariance matrix at $t=0$, $M(0)$, for this initial state is

$$M(0) = \begin{pmatrix} 2\bar{n}_a+1 & 0 & 0 & 0 \\ 0 & 2\bar{n}_a+1 & 0 & 0 \\ 0 & 0 & 2\bar{n}_b+1 & 0 \\ 0 & 0 & 0 & 2\bar{n}_b+1 \end{pmatrix},$$

where, as usual, the mean excitation number $\bar{n}_\alpha$ is $[\exp(\Omega_\alpha/k_BT)-1]^{-1}$. If the initial temperature and the tunneling frequency are of order 10^{-8} K and 1 kHz, respectively, which are typical values in current experiment situations [20,37], the mean excitation number may reach order unity and can give rise to non-negligible effect on the entanglement parameter.

It is well known that the Wigner characteristic function of a harmonic oscillator in a thermal mixed state is still a Gaussian function [38]. So, it is readily shown that the system considered in our paper remains in a Gaussian state and one can use the two-mode entanglement parameter to study the entanglement between the condensates. Following the argument outlined previously, one can show that the covariance matrix $M(t)$ at $t>0$ is given by $U(t)M(0)U^T(t)$, from which the two-mode entanglement parameter ξ_t can be obtained accordingly. In particular, for a symmetric two-component BEC with $\Omega_a=\Omega_b$, the covariance matrix $M(t)$ is
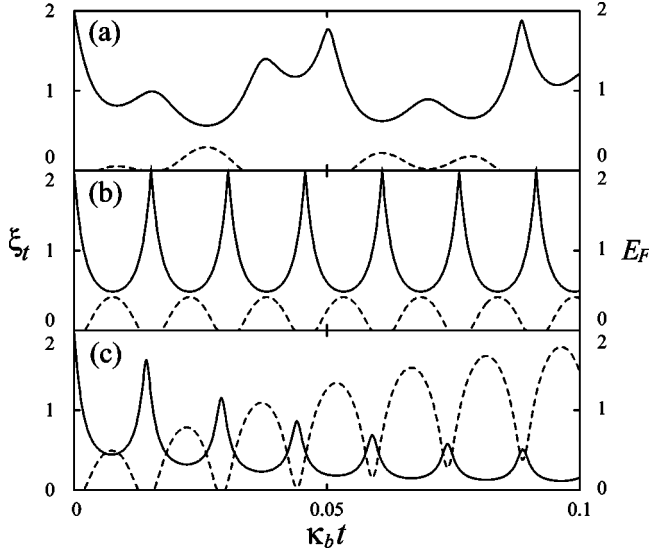
FIG. 4. The time evolution of ξ_t (solid line, left scale) and the EOF E_F (dashed line, right scale) for a symmetric thermal BEC with $N_a = N_b = 400$; $\Omega_a = \Omega_b = 50\kappa_b$; $\bar{n}_a = \bar{n}_b = 0.5$; and (a) $\kappa/\kappa_b = 0.50$; (b) $\kappa/\kappa_b = 1.00$; (c) $\kappa/\kappa_b = 1.12$.

just $U(t)U^T(t)$ multiplied by $2\bar{n}_a + 1$ (or equivalently $2\bar{n}_b + 1$) and the system is again symmetric. The two-mode entanglement parameter ξ_t can then be used to determine whether the system is entangled or not and to evaluate the entanglement of formation $E_F(\xi_t)$ as well. In Fig. 4 we show the two-mode entanglement parameter ξ_t and the entanglement of formation E_F as functions of time for a symmetric two-component BEC with the mean excitation number $\bar{n}_\alpha = 0.5$. It is clearly manifested that a substantial amount of entanglement can still be achievable when κ is sufficiently strong. Therefore the existence of finite thermal effects does not readily preclude the occurrence of entanglement.

On the other hand, for an asymmetric two-component BEC with $\Omega_a \neq \Omega_b$, the covariance matrix $M(t)$ is no longer symmetric with respect to the two interacting components. Therefore, despite that one can still make use of the parameter ξ_t to determine whether the system is entangled or not, it is no longer possible to apply the method developed here to obtain E_F even if $\xi_t < 1$ and other numerical schemes have to be sought [32].

In summary, although there might be some complications in quantifying the degree of entanglement if the initial state of the condensate is a mixed state, the two-mode entanglement parameter studied here can still give a necessary and sufficient condition of the separability of the two condensates.

ACKNOWLEDGMENTS

APPENDIX: EXPLICIT FORM OF THE VARIANCES

The explicit expressions of the variances are given here for reference:

$$\langle \hat{q}_a^2 \rangle = \sum_{i=1,2} \{ F_i \cos 2\omega_i t + G_i \cos[\omega_1 + (-1)^{i+1}\omega_2]t + H_i \},$$
(A1)

$$\langle \hat{q}_b^2 \rangle = \frac{1}{\Omega_a} \sum_{i=1,2} \{ \mu_i^2 \Omega_a F_i \cos 2\omega_i t - \Omega_b G_i \cos[\omega_1 + (-1)^{i+1}\omega_2]t + \mu_i^2 \Omega_a H_i \},$$
(A2)

$$\langle \hat{p}_a^2 \rangle = \frac{-1}{\Omega_a^2} \sum_{i=1,2} \{ \omega_i^2 F_i \cos 2\omega_i t + (-1)^{i+1}\omega_1\omega_2 G_i \times \cos[\omega_1 + (-1)^{i+1}\omega_2]t + \omega_i^2 H_i \},$$
(A3)

$$\langle \hat{p}_b^2 \rangle = \frac{-1}{\Omega_a \Omega_b^2} \sum_{i=1,2} \{ \mu_i^2 \omega_i^2 \Omega_a F_i \cos 2\omega_i t + (-1)^i \omega_1\omega_2 \Omega_b G_i \times \cos[\omega_1 + (-1)^{i+1}\omega_2]t + \mu_i^2 \omega_i^2 \Omega_a H_i \},$$
(A4)

$$\mathrm{Re}(\langle \hat{q}_a \hat{q}_b \rangle) = \frac{1}{2} \sum_{i=1,2} \{ 2\mu_i F_i \cos 2\omega_i t + (\mu_1 + \mu_2) G_i \times \cos[\omega_1 + (-1)^{i+1}\omega_2]t + 2\mu_i H_i \},$$
(A5)

$$\mathrm{Re}(\langle \hat{p}_a \hat{p}_b \rangle) = \frac{-1}{2\Omega_a \Omega_b} \sum_{i=1,2} \{ 2\mu_i \omega_i^2 F_i \cos 2\omega_i t + (-1)^{i+1}\omega_1\omega_2 \times (\mu_1 + \mu_2) G_i \cos[\omega_1 + (-1)^{i+1}\omega_2]t + 2\mu_i \omega_i^2 H_i \},$$
(A6)

$$\mathrm{Re}(\langle \hat{q}_a \hat{p}_b \rangle) = \frac{-1}{2\Omega_b} \sum_{i=1,2} \{ 2\mu_i \omega_i F_i \sin 2\omega_i t + [\mu_1\omega_1 + (-1)^{i+1}\mu_2\omega_2] G_i \sin[\omega_1 + (-1)^{i+1}\omega_2]t \},$$
(A7)

$$\mathrm{Re}(\langle \hat{q}_b \hat{p}_a \rangle) = \frac{-1}{2\Omega_a} \sum_{i=1,2} \{ 2\mu_i \omega_i F_i \sin 2\omega_i t + [\mu_2\omega_1 + (-1)^{i+1}\mu_1\omega_2] G_i \sin[\omega_1 + (-1)^{i+1}\omega_2]t \},$$
(A8)

where

$$F_i = \frac{\tilde{\mu}_i^2}{4(\mu_1 - \mu_2)^2}\left[1 + \frac{1}{\tilde{\mu}_i^2} - \frac{\Omega_a^2}{\omega_i^2} - \frac{\Omega_b^2}{\tilde{\mu}_i^2\omega_i^2}\right], \qquad (A9)$$

$$H_i = \frac{\tilde{\mu}_i^2}{4(\mu_1 - \mu_2)^2}\left[1 + \frac{1}{\tilde{\mu}_i^2} + \frac{\Omega_a}{\omega_i^2} + \frac{\Omega_b^2}{\tilde{\mu}_i^2\omega_i^2}\right], \quad (A11)$$

$$G_i = \frac{1}{2(\mu_1 - \mu_2)^2}\left[\frac{\Omega_b}{\Omega_a} - 1 + (-1)^i\frac{\Omega_a\Omega_b}{\omega_1\omega_2} + (-1)^{i+1}\frac{\Omega_b^2}{\omega_1\omega_2}\right], \qquad (A10)$$

$$\tilde{\mu}_j = \mu_1(1 - \delta_{j1}) + \mu_2(1 - \delta_{j2}). \qquad (A12)$$

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