Fault-tolerant quantum computation for local non-Markovian noise

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We derive a threshold result for fault-tolerant quantum computation for local non-Markovian noise models. The role of error amplitude in our analysis is played by the product of the elementary gate time t_0 and the spectral width of the interaction Hamiltonian between system and bath. We discuss extensions of our model and the applicability of our analysis.

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I. INTRODUCTION

Whether or not quantum computing will become reality will at some point depend on whether we can implement quantum computation fault-tolerantly. This would imply that even though the quantum circuitry and storage are faulty, it is possible by error-correction to perform error-free quantum computation for an unlimited amount of time while incurring an overhead that is polylogarithmic in time and space, see [1–7]. For this "software" solution that uses concatenated coding techniques, an error probability threshold of the order of 10^{-4}–10^{-6} per qubit per clock-cycle has been given for the simplest error models, meaning that for an error probability below this threshold fault-tolerant quantum computation is possible. These estimates heavily depend on error modeling, the efficiency of the error-correcting circuits, and the codes that are used. Different and potentially better estimates are possible, see, for example [8–10]. Another solution to the fault-tolerance problem proposed by Kitaev is to make the hardware intrinsically fault-tolerant by using topological degrees of freedom such as anyonic excitations as qubits [11].

In Refs. [3,4] the threshold result for fault-tolerance is derived for various error models, including ones with exponentially decaying correlations. However, this general model of exponentially decaying correlations does not make direct contact with a detailed physical model of decoherence. Such a physical model of decoherence starts from a Hamiltonian description involving the environmental degrees of freedom and the computer "system" degrees of freedom.

Starting from such a Hamiltonian picture it was argued in a paper by Alicki *et al.* [12] that fault-tolerant quantum computation may not be possible when the environment of the quantum computer has a long-time memory.

In this paper we carry out a detailed threshold analysis for some non-Markovian error models. Our findings are not in agreement with the views put forward in the paper by Alicki *et al.*, that is, we can derive a threshold result in the non-Markovian regime if we make certain reasonable assumptions about the spatial structure and interaction amongst the environments of the qubits. The results of our paper and the previous results in the literature are summarized in Sec. IV of this paper. In Sec. I A we introduce our notation and our assumptions on the decoherence model. In Sec. I B we introduce our measure of error or decoherence strength which we motivate with a small example. Then in Sec. I C we prove some simple lemmas that will be used in the fault-tolerance analysis and in Sec. I D we discuss the overall picture of a fault-tolerance derivation, in particular the parts of this derivation that do not depend on the decoherence model. Then in Sec. II we fill in the technical details to obtain the threshold result expressed in Theorem 1. In Sec. III we generalize our decoherence model to incorporate more relaxed conditions on the spatial structure of the bath and we discuss further possible extensions. In Sec. IV we give an overview of all known fault-tolerance results including ours and in Sec. V we discuss several physical systems in which our analysis may be applicable.

A. Notation and explanation of the decoherence model

We use the following operator norm: $\|A\| = \max_{\|\psi\|=1}\|A|\psi\rangle\|$ where $\||\psi\rangle\| \equiv \|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. The following properties will be used: $\|A+B\| \leqslant \|A\|+\|B\|$, $\|U\|=1$ if U is unitary, and $\|AB\| \leqslant \|A\|\|B\|$. An operator H that acts on system qubit i or qubits i and j (and potentially another quantum system) is denoted as $H[\mathrm{q}_i]$ or $H[\mathrm{q}_i,\mathrm{q}_j]$. A unitary evolution for the time-interval t to $t+t_0$ is denoted as $U(t+t_0,t)$. t_0 is the time it takes to do an elementary (one or two qubit) gate. The identity operator is denoted as $\mathbf{I}$ and $\mathbf{e}$ denotes the base of the natural logarithm. We will also use the trace-norm denoted by $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ and the classical variation distance between probability distributions $\mathbb{P}$ and $\mathbb{Q}$: $\|\mathbb{P}-\mathbb{Q}\|_1 = \Sigma_i|\mathbb{P}(i)-\mathbb{Q}(i)|$.

The following assumptions have been shown to be necessary for fault-tolerance and thus we keep these assumptions in our analysis.

(1) It is possible to operate gates on different qubits in parallel.

(2) We have fresh ancilla qubits at our disposal. These ancilla qubits are prepared off-line in the exact computational state $|00\cdots 0\rangle$ and they can be used in the circuit when necessary. They function as a heat-sink which removes entropy from the computation.

In Fig. 1(a) three types of quantum systems are sketched that differ in function and in the amount of control that we can exert over them. First, there is R, for quantum registers, that we can control and use for our computation. Second, there is A, for ancillas, which are used for error-correction and fault-tolerant gate construction during the computation.

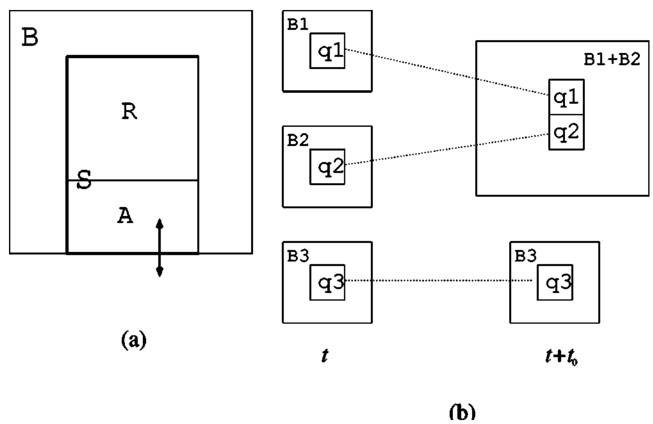

FIG. 1. Schematic representation of the model. (a) The system S consists of a register R of qubits plus ancillas A that can be reset during the computation. The system S is coupled to the environment, or bath, B. (b) The decoherence model. Each qubit q_i is coupled to an individual bath B_i. When two qubits interact, they may interact with one common bath.

The systems R and A taken together are denoted as S for system of which single qubits are denoted by the letter q. Clean ancilla registers set to $|00\cdots 0\rangle$ are added during the computation and can be removed after having interacted with (1) other parts of the system S by error-correcting procedures and (2) the bath B according to some fixed interaction Hamiltonian.

We will assume that the third system, the bath B, which interacts with the system and ancillas has a local structure, illustrated in Fig. 1(b). We will generalize this model in Sec. III. Every qubit $(q_1, q_2, q_3, \ldots)$. of the system has its own bath $(B_1, B_2, B_3, \ldots)$. Only during the time when two qubits interact their baths (B_1 and B_2 in the figure) can interact. The idea behind this modeling is that the bath is localized in space, i.e., is associated with the place where the qubit is stored. But when qubits interact, they need to be brought together and so they may share a common bath. In the picture B_1+B_2 at time $t+t_0$ are suggested to be the same baths that qubits q_1 and q_2 interacted with at time t, but in general they may also be different baths. For example, when qubits q_1 and q_2 have to be moved in order to interact, they may see a partially new environment at time $t+t_0$. This distinction will not be important in our analysis.

Most importantly, in this model, each bath can have an arbitrarily long memory; at no point in our derivation will we make a Markovian assumption. This implies that, for example, the bath B_1 may contain information about qubit q_1 at time t, then interact with bath B_2 at time $t+t_0$ and pass this information on to bath B_2, etc. The interaction Hamiltonian of a single qubit q_i of the system (R or A) with the bath is given by

$$H_{SB}[q_i] = \sum_k \sigma_k[q_i] \otimes A_k, \tag{1}$$

with the Pauli-matrices σ_k acting on qubit q_i and A_k is some Hermitian operator on the bath of the qubit q_i which is not equal to the identity $\mathbf{I}$. During a two qubit gate both qubits may interact with both baths. For simplicity (see [13]) we assume that the interaction is of the form

$$H_{SB}[q_i, q_j] = H_{SB}[q_i] + H_{SB}[q_j], \tag{2}$$

where the bath part of each $H_{SB}[q_i]$ is an operator on the joint bath of qubits q_i and q_j. We do not care about the time-evolution of the baths except that it has to obey the "local bath assumption," i.e., noninteracting qubits have noninteracting baths. The system (register and ancilla) evolution $H_{RA}(t)$ is time-dependent and represents the fault-tolerant quantum circuit that we want to implement. This evolution is built from a sequence of one and two qubit gates and, as was said before, t_0 is the time it takes to perform any such gate.

B. Measure of decoherence strength

Our results will depend on the strength of the coupling Hamiltonian $H_{SB}[q_i]$. There is an additional freedom in determining $H_{SB}[q_i]$, namely we can always add a term $\alpha \mathbf{I}_S[q_i] \otimes \mathbf{I}_B$ where α is an arbitrary real constant and $\mathbf{I}$ is the identity operator. This is possible since it merely shifts the spectrum (see [14]). Let μ_i be the eigenvalues of H_{SB}. With this freedom we see that

$$\min_{\alpha} \| H_{SB}[q_i] + \alpha \mathbf{I}_S[q_i] \otimes \mathbf{I}_B \| = (\mu_{max} - \mu_{min})/2 \equiv \Delta_{SB}[q_i], \tag{3}$$

the spectral width of the interaction Hamiltonian (divided by 2). Our analysis will apply to physical systems where one can bound

$$\forall\ q_i \in S, \quad \Delta_{SB}[q_i] \leq \lambda_0, \tag{4}$$

where λ_0 is a small constant which will enter the threshold result, Theorem 1, together with t_0, the fundamental gate time. In what follows we will denote $\Delta_{SB}[q_i]$ as Δ_{SB} or Δ assuming that the spectral width is the same for each qubit in S.

We justify the use of this norm in the following way. Consider a single qubit coupled to a bath such that both bath and system Hamiltonians are zero but there exists nonzero coupling. To what extent will an arbitrary initial state of qubit and bath change under this interaction? We can consider the minimum fidelity of an initial state $\psi_{SB}(0)$ with the evolved state at time t:

$$F_{min}(t) = \min_{\psi(0)} |\langle \psi(t) | \psi(0) \rangle|. \tag{5}$$

For small times t such that $\Delta_{SB} t \leq \pi/2$ the minimum fidelity can be achieved by taking $|\psi(0)\rangle = 1/\sqrt{2}(|\psi_{max}\rangle + |\psi_{min}\rangle)$ where $|\psi_{max/min}\rangle$ are the eigenvectors of H_{SB} with largest and smallest eigenvalues. Then we have

$$F_{min}(t) = \cos(\Delta t) \approx 1 - \Delta^2 t^2/2 + O((\Delta t)^4). \tag{6}$$

Note that this fidelity decay *includes* the effects on the bath. For this reason this fidelity decay overestimates the effects of decoherence, in other words $F(\rho_S(t), \rho_S(0)) \geq F_{min}$.

One may compare this fidelity decay with that of other decoherence processes, for example, the depolarizing channel $\mathcal{E}$ with depolarizing probability p. For such a channel we have $F(|\psi\rangle_S, \mathcal{E}(|\psi\rangle\langle\psi|)_S) = \sqrt{1-p/2}$ [15]. Thus, loosely speaking, Δt could be interpreted as an error amplitude whose square is an error probability.

Thus this brief analysis shows that for some initial states $\psi_{SB}(0)$ the norm of the interaction Hamiltonian measures exactly how the state changes due to the interaction. Since our environment is non-Markovian we cannot exclude such bad initial states, in other words we cannot assume that the decoherence is just due to the interactive evolution of an initially unentangled bath and system.

C. Error modeling tools

The following simple lemma will be used repeatedly in this paper:

Lemma 1. Let a unitary transformation $\mathbf{U} = U_n \cdots U_1$ where $U_i = G_i + B_i$ and the operator G_i and B_i are not necessarily unitary. Let $\mathbf{U} = \mathbf{B} + \mathbf{G}$ where we define $\mathbf{B}$ to be the sum of terms which contains at least k factors B_i. Let $\|B_i\| \leq \epsilon$ and thus $\|G_i\| \leq 1+\epsilon$. We have

$$\|\mathbf{B}\| \leq \binom{n}{k} \epsilon^k (1+\epsilon)^{n-k}. \tag{7}$$

If G_i is unitary, we have

$$\|\mathbf{B}\| \leq \binom{n}{k} \epsilon^k. \tag{8}$$

Proof. We can think about $\mathbf{U}$ as a binary tree of depth n such that the children of each node are labeled with G_i or B_i at depth i. We prune the tree in the following way; when a branch has k factors B_i in its path, we terminate this whole branch with the remaining $U_n \cdots U_m$. The sum of these terminated branches is $\mathbf{B}$. $\mathbf{B}$ can be bounded by observing that there are $\binom{n}{k}$ terminated branches each of which have norm at most $\|B_i\|^k \|G_i\|^{n-k}$ (since each branch is a sequence of G_i transformations interspersed with k B_i transformations followed by unitary transformations. ■

It is easy to prove the following (see also Ref. [16]).

Lemma 2. Consider a time-interval $[t, t+t_0]$ and a single qubit $q \in S$ which does not interact with any other qubit in S at that time. The time-evolution for this qubit is given by some unitary evolution $U[q]$ involving its bath B. Let $U_0[q] = U_S[q] \otimes U_B$ be the free uncoupled evolution for this qubit. We can write

$$U[q] = U_0[q] + E[q], \tag{9}$$

where $E[q]$ is a fault-operator with norm

$$\|E[q]\| \leq t_0 \|H_{SB}[q]\| = t_0 \Delta_{SB}[q] \leq t_0 \lambda_0. \tag{10}$$

Proof. We drop writing the dependence on qubit q for the proof. For the qubit evolution in the interval, using the Trotter expansion we can write

$$U = \lim_{n\to\infty} \Pi_{m=1}^{n} (U_S^{t_m} U_{SB}^{t_m} U_B^{t_m}), \tag{11}$$

where $U_K^{t_m}$ is the time-evolution for K=S, B or coupling SB during the time-interval t_m of length t_0/n. Now in this expansion we may write $U_{SB}^{t_m} = \mathbf{I} - iH_{SB}t_0/n + O(t_0^2/n^2)$ and omit these higher order terms. Let us call $G_m = U_S^{t_m} U_B^{t_m}$ and $B_m = -i(t_0/n) U_S^{t_m} H_{SB} U_B^{t_m}$ as in Lemma 1. We thus have $\|B_m\| \leq t_0 \|H_{SB}\|/n$. Note that G_m is unitary and we have a binary tree of depth $n \to \infty$ and can use Lemma 8 with $k=1$. This gives

$$\|E\| = \|\mathbf{B}\| \leq t_0 \|H_{SB}\|. \tag{12}$$

■

A similar statement holds when we consider the evolution of two interacting qubits. We have that

$$U_{SB}[q_i, q_j] = U_0[q_i, q_j] + E[q_i, q_j], \tag{13}$$

where $\|E[q_i, q_j]\| \leq 2t_0 \Delta_{SB}[q] \leq 2t_0 \lambda_0$.

D. Overall perspective: Good and bad fault-paths

Since the bath may retain information about the time-evolution and error processes for arbitrary long times we cannot describe the decoherence process by sequences of superoperators on the system qubits. Instead, there is a single superoperator for the entire computation that is obtained by tracing over the bath at the end of the computation. Thus in our analysis we will consider the entire *unitary* evolution of system, bath, and ancillas. At time $t=0$ bath and ancilla and system are uncoupled and we may always purify the bath, i.e., find a pure state in a larger bath Hilbert space which, when the extra Hilbert space is traced out, yields the desired mixed state. We can then assume a pure initial product state for the combined system and bath, SB. The unitary evolution of the computation consists of a sequence and/or parallel application of the unitary gates $U[q_i, q_j](t+t_0, t)$ and $U[q_i](t+t_0, t)$. Each such gate, say for two qubits, can be written as a sum of an error-free evolution $U_0[q_i, q_j](t+t_0, t)$ and a fault term $E[q_i, q_j]$. Therefore the entire computation can be written as a sum over *fault-paths*, that is, a sum of sequences of unitary error-free operators interspersed with fault operators. This is very similar as in the fault-tolerance analysis for Markovian error models, where the superoperator during each gate-time t_0 can be expanded in an error-free evolution and an erroneous evolution so that the entire superoperator for the circuit is a sum over fault-paths.

The main idea behind the threshold result for fault-tolerance is then as follows, see [4]. There are *good* fault-paths with so-called *sparse* numbers of faults which keep being corrected during the computation and which lead to (approximately) correct answers of the computation; and there are *bad* fault-paths which contain too many faults to be corrected and imply a *crash* of the quantum computer.

Now the goal of our fault-tolerance derivation which is completely analogous in structure as the one in [4] is to show the following.

(A) Sparse fault-paths lead to sparse errors in the computation. This fact uses the formal distinction between faults that occur during the computation and the effects of these faults, the errors, that arise due to the subsequent evolution which can spread the faults. The fact that sparse fault-paths give rise to sparse errors is due to fundamental properties of fault-tolerant error-correcting circuitry, namely that there exists error-correcting codes and procedures that do not spread faults too much. It is independent of the choice of decoherence model, and can be applied to any model where one can make an expansion into fault-paths. See Lemma 3.

(B) Sparse errors give good final answers. This is a technical result whose derivation may differ slightly in one or the other decoherence model, but which is intuitively sound for all possible decoherence models. See Lemma 4.

(C) The norm of the operator corresponding to all bad nonsparse fault-paths is "small." This result depends crucially on the decoherence model that is chosen, in particular the spatial or temporal correlations that are allowed. Second, it depends on the strength of the errors, that is, only for small enough strength below some threshold value will the norm of the bad fault-path operator get small. See Sec. II B.

(A),(B)(,C)$\Rightarrow$ When the *bad* operator norm is small, the answer of the computation is close to what the good fault-path operator yields which is the correct answer according to item B. See Lemma 4 and Theorem 1.

Another small comment about our model is the following: In the usual model for error-correction (see Ref. [6] in [17]), measurements are performed to determine the error-syndrome or the correct preparation of the ancilla states. Since we prefer to view the entire computation as a unitary process, we may replace these measurements by coherent quantum operations. In the error-correction with measurement procedures it is assumed that faults can occur in the measurement itself or in the quantum gate that is performed that depends on this measurement record, but the measurement record by itself is stable since it is classical. If we replace measuring by coherent action for technical reasons in this derivation, it is then fair to assume that the qubit that carries the measurement record is error-free, in other words it does no longer interact with a bath. This modeling basically allows the standard fault-tolerance results in item A expressed in Lemma 3 to carry over in the simplest way to our model.

II. THRESHOLD RESULT

A. Nomenclature

Let the basic error-free quantum circuit denoted by M_0 consist of N *locations* [4]. Each location is given by a triple $(\{q\},G,t)$ where $\{q\}$ denotes the qubits (one or two at most) involved in some gate G (G could be **I**) at time t in the quantum circuit. In the following, $E[i]$ or $U[i]$ will denote operators that involve location i, i.e., if q_1 and q_2 interact at location i we will write $U_0[q_1,q_2]=U_0[i]$ instead of enumerating the qubits. For fault-tolerance one constructs a family of circuits M_r by concatenation. That is, we fix a computation code C (see definition 15 in Ref. [4]), for example a CSS code, encoding one qubit into (say) m qubits [18]. We obtain the circuit M_r by replacing each location in the circuit M_0 by a block of encoded qubits to which we apply an error-correcting procedure followed by a fault-tolerant implementation of G, see Fig. 2. Repeated substitution will give us a circuit M_r at concatenation level r.

Essential are the following definitions and a lemma taken from Ref. [4] which define sparseness of a set of locations and error-spread of a code:

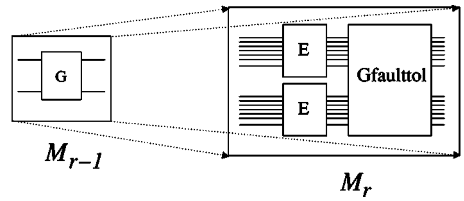

FIG. 2. Every single or two-qubit gate G in the circuit M_{r-1} gets replaced by an error-correcting procedure E followed by a fault-tolerant implementation of G, Gfaulttol (possibly involving ancillas).

Definitions from Ref. [4]:

(1) A set of qubits in M_r is called an s-block if they originate from 1 qubit in M_{r-s}. A s-working period in M_r is a time interval which originates from one time-step in M_{r-s}. An s-rectangle in M_r is a set of locations that originate from one location in M_{r-s}.

(2) Let B be a set of r-blocks in the circuit M_r. An $(r,1)$-sparse set of qubits A in B is a set of qubits in which for every r-block in B, there is at most one $(r-1)$-block such that the set A in this block is not $(r-1,1)$-sparse. A (0, 1)-sparse set of qubits in M_0 is an empty set of qubits.

(3) A set of locations in a r-rectangle is $(r,1)$-sparse when there is at most 1 $(r-1)$-rectangle such that the set is not $(r-1,1)$-sparse in that $(r-1)$-rectangle. A fault-path in M_r is $(r,1)$-sparse if in each r-rectangle, the set of faulty locations is $(r,1)$-sparse.

(4) A computation code C has spread s if one fault which occurs in a particular 1-rectangle affects at most s qubits in each 1-block at the end of that 1-rectangle, i.e., causes at most s errors in each 1-block.

(5) Let A_C be the number of locations in a 1-rectangle for a given code C.

We state the basic lemma about properties of computation codes which was proved in Ref. [4] (with a correction).

Lemma 3 (A: Lemma 8 in [4] with a correction). Let C be a computation code that can correct 2 errors and has spread s=1. Consider a computation M_r subjected to a $(r,1)$-sparse fault-path. At the end of each r-working period the set of errors is $(r,1)$-sparse.

Thus for simplicity we will be using a quantum computation code that encodes one qubit and can correct two errors and has spread $s=1$. We denote the entire unitary evolution of M_r including the bath as Q^r. We may write $Q^r=Q^r_G+Q^r_B$ where Q^r_G is a sum over *good* $(r,1)$-sparse fault-path operators and Q^r_B contains the *bad* nonsparse terms. A fault-path operator E_{SB} that is $(r,1)$-sparse is a sequence of free evolutions $U_0[i]$ for all locations except that in every r-rectangle there is a $(r,1)$-sparse set of locations where a fault operator $E[i]$ occurs.

Definition 1 (Operators in the interaction picture). Let $U_0(t_2,t_1)=U_S(t_2,t_1)\otimes U_B(t_2,t_1)$ be the free uncoupled evolution of system and bath in the time-interval $[t_1,t_2]$. We define a fault-operator $E(t_2,t_1)$ in the interaction picture as

$$E(t_2,t_1) = U_0(t_2,t_1)EU_0^{\dagger}(t_2,t_1). \tag{14}$$

The interpretation is that $E(t_2,t_1)$ is the spread of a fault E that occurs at t_1 due to the subsequent free evolution.

Then it is simple to see the following:

Proposition 1 (Error spread in the interaction picture). Consider a quantum circuit M. Let $U_0(t_F,t_I)$ be the uncoupled evolution for M. Faults occur at a set of "time-resolved" locations

$$\mathcal{T}=\{(i_1,t_1),(i_2,t_2),\ldots,(i_k,t_k)\},$$

where $i_1,\ldots,i_k$ is the set of distinct locations of the faults and $t_1,\ldots,t_k$ label the specific times that the faults occur at the locations. Let $E_{\text{SB}}(\mathcal{T})$ be a particular fault-path operator in which at every faulty location $(i,t)\in\mathcal{T}$ we replace $U_0[i]$ by a fault-operator $E[i]$. We have

$$E_{\text{SB}}(\mathcal{T})U_0^{\dagger}(t_F,t_I)=E[i_k](t_F,t_k)\cdots E[i_1](t_F,t_1). \tag{15}$$

We note that the system part of $E_{\text{SB}}U_0^{\dagger}$ is $\mathbf{I}$ everywhere except for the qubits that are in the causal cone of the faulty locations, i.e., the qubits to which the errors potentially have spread.

Proof. This can be shown by inserting $\mathbf{I}=U_0^{\dagger}(t_F,t_i)U_0(t_F,t_i)$ in the appropriate places and then using the definition of fault operators in the interaction picture. ■

Now we include error-correction and differentiate between the ancilla systems A used for error-correction which may contain noise and the registers R in which the errors remain sparse. Note that all these ancillas are in principle discarded after being used, but we may as well leave them around. Let $K|_C$ be the restriction of the operator K to vectors in the code-space of C, i.e., $K|_C=K\mathbf{P}_C$ where $\mathbf{P}_C$ is the projector on the codespace.

Let us consider a fault-path operator E_{SB} representing a single fault E at time t on some block that is subsequently corrected by an error-free error-correcting procedure. Let $|\text{IN}\rangle$ be the initial state of the computer, bath and ancillas and $U_0(t_F,t_I)$ be the perfect evolution. We have

$$E_{\text{SB}}|\text{IN}\rangle=E_{\text{SB}}U_0^{\dagger}U_0|\text{IN}\rangle=E_{\text{SB}}U_0^{\dagger}|\psi_C(t_F)\rangle, \tag{16}$$

where $|\psi_C(t_F)\rangle$ is the final perfect state of the computer, prior to decoding and therefore in the code-space. E_{SB} is the sequence $U_0(t_F,t)EU_0(t,t_0)$ where $U_0(t_F,t)$ includes the error correction operation. In other words, in the interaction picture, we can write

$$E_{\text{SB}}|\text{IN}\rangle=E(t_F,t)|\psi_C(t_F)\rangle. \tag{17}$$

The error-correcting conditions (see [15], par. 10.3) imply that when acting on the code space *and* an ancilla state set to $|00\cdots0\rangle$ the operator $E(t_F,t)$ will be $E(t_F,t)=\mathbf{I}|_C\otimes(\mathcal{J})_{\text{AB}}$ where $\mathcal{J}$ is some arbitrary operator on the ancilla (that receives the error syndrome in the error-correcting procedure) and bath. In Eq. (17) the final error-free state has all ancillas set to $|00\cdots0\rangle$ and the system state is in the code-space and thus the error acts as $\mathbf{I}$ on the system.

Similarly, let E_{SB} contain two faults at times $t_1<t_2$ that have not spread (say) and are then corrected by a perfect error-correcting procedure. We have

$$E_{\text{SB}}|\text{IN}\rangle=E_2(t_F,t_2)E_1(t_F,t_1)|\psi_C(t_F)\rangle. \tag{18}$$

Let us assume, for example, that E_1 occurs prior to error-correction and E_2 occurs during error-correction. Then due to the error correction $E_1(t_F,t_1)$ acts as $\mathbf{I}$ on the code space when the ancilla used for error-correction is set to $|00\cdots0\rangle$ and acts as $\mathcal{J}$ on this ancilla and the bath. The error E_2 will not be corrected and may still be present (but will not have spread to more qubits in the block due to the spread properties of the code that is used) after error-correction. Thus in total we can write for this process that $E_1(t_F,t_1)$ acts as $\mathbf{I}$ on the code space, whereas $E_2(t_F,t_2)$ is an operator that acts on the code space as at most one error per block.

Alternatively, both faults could occur prior to error-correcting so they can both be corrected by our code. This implies that both $E_1(t_F,t_1)$ and $E_2(t_F,t_2)$ act as $\mathbf{I}$ on the code-space. Note that after the first fault the ancilla will be partially filled (i.e., not be $|00\cdots0\rangle$) but since the code can correct two errors there is still space to put the second error syndrome in. However, a third operator $E_3(t_F,t_3)$ would no longer act as $\mathbf{I}$ on the code-space since the code cannot correct three errors.

In other words, with these examples we can see how Lemma 3 can be translated in terms of the sparseness of the errors in the interaction picture, i.e., the sparseness of places where they act as nonidentity on the final encoded state of the register qubits. In the next lemma we need to consider the effect of such sparse fault-path operators E_{SB} on the final state of the computer. This is the state of the computer obtained after fault-tolerant decoding which is as follows. The fault-tolerant decoding procedure for a single level of encoding takes a code word $|c\rangle$ and "copies" (by doing CNOT gates) the codeword m times. Then on each "copy" we determine what state it encodes and then we take the majority of the m answers. This procedure is done recursively when more levels of encoding are used.

In the fault-tolerant decoding procedure faults can occur on the code words, i.e., as incoming faults, during the copying procedures and during the determination of what is encoded by the code word. The last procedure will usually be a conversion from a quantum state to a classical bit string since this will be the most efficient. This implies that the step of taking the recursive majority of these bits is basically error-free since it only involves classical data. In the next lemma we model this by coherent quantum operations that output superpositions of decoded bit strings followed by an error-free measurement that takes the recursive majority of these bits.

Lemma 4 (B: Sparse faults give almost correct answers). Let $Q^r=Q_G^r+Q_B^r$ be the unitary evolution of M_r and let $\|Q_B^r\|\leq\epsilon<1/2$. Let $\mathbb{P}_0(i)$ be the output probability distribution under measurement of some set of qubits of the error-free original computation M_0. Let $\mathbb{P}(i)$ be the simulated output distribution of the encoded computation M_r with evolution Q^r. We have

$$\|\mathbb{P}_0-\mathbb{P}\|_1\leq\sqrt{2\epsilon}+16\epsilon. \tag{19}$$

Proof. The initial state of the computer is $|\text{IN}\rangle_{\text{RAB}}=|00\cdots0\rangle_{\text{RA}}\otimes|\text{IN}_{\text{B}}\rangle$ for some arbitrary state $|\text{IN}_{\text{B}}\rangle$. Let U_0^r be the error-free evolution of M_r including the final decoding operation. Thus let $U_0^r|\text{IN}\rangle_{\text{RAB}}=|\text{OUT}_0\rangle_{\text{R}}\otimes|\text{REST}\rangle_{\text{AB}}$. Let $Q^r|\text{IN}\rangle_{\text{RAB}}=|\text{OUT}\rangle_{\text{RAB}}$ and $Q_{B/G}^r|\text{IN}\rangle_{\text{RAB}}=|\text{OUT}_{B/G}\rangle_{\text{RAB}}$. We will drop the label RAB from now on. The norm of $|\mathrm{OUT}_G\rangle$ will be denoted as $\|\mathrm{OUT}_G\|$. We have

$$1=\|Q^r|\mathrm{IN}\rangle\| \leq \|Q^r_G|\mathrm{IN}\rangle\| + \|Q^r_B|\mathrm{IN}\rangle\|, \quad (20)$$

so that $\|\mathrm{OUT}_G\| \geq 1-\|Q^r_B|\mathrm{IN}\rangle\| \geq 1-\epsilon$. On the other hand $\|Q^r_G\|=\|Q^r-Q^r_B\| \leq 1+\epsilon$.

Let G be the set of $(r,1)$-sparse fault-paths. We have $Q^r_G=\Sigma_{\mathcal{T}\in G}E_{\mathrm{SB}}(\mathcal{T})$ where $E_{\mathrm{SB}}(\mathcal{T})$ is the fault-path operator of a $(r,1)$-sparse fault-path labeled by location and time index set $\mathcal{T}$. We can write

$$|\mathrm{OUT}_G\rangle=\sum_{\mathcal{T}\in G}E_{\mathrm{SB}}(\mathcal{T})U_0^{r\dagger}|\mathrm{OUT}_0\rangle_{\mathrm{R}}\otimes|\mathrm{REST}\rangle_{\mathrm{AB}}. \quad (21)$$

By the arguments above and the fundamental Lemma 3 we know that $E_{\mathrm{SB}}(\mathcal{T})U_0^{\dagger}$ is $\mathbf{I}$ everywhere except on a $(r,1)$-sparse set of qubits. Let w be the number of output qubits of M_0. The ideal state $|\mathrm{OUT}_0\rangle_{\mathrm{R}}$ has the property that all qubits in an r-block have the same value in the computational basis, i.e.,

$$|\mathrm{OUT}_0\rangle_{\mathrm{R}}=\sum_{i_1,\ldots,i_w}\alpha_{i_1,\ldots,i_w}|i_1\rangle^{\otimes m^r}\cdots|i_w\rangle^{\otimes m^r}, \quad (22)$$

where m is the number of qubits in a 1-block. The final step of the computation is a measurement of all output qubits that takes the recursive majority on the block to get the final output string i of length w with probability $\mathbb{P}^{tot}(i)$. We model this measurement using POVM elements E_k, $-\Sigma_k E_k=\mathbf{I}$. Since not all of these w output bits may be relevant output bits of M_0, we may use the fact that trace-distance is nonincreasing over tracing [15] so that

$$\|\mathbb{P}_0-\mathbb{P}\|_1 \leq \|\mathbb{P}_0^{tot}-\mathbb{P}^{tot}\|_1, \quad (23)$$

where $\mathbb{P}^{tot}(k)=\mathrm{Tr}\,E_k|\mathrm{OUT}\rangle\langle\mathrm{OUT}|_{\mathrm{RAB}}$ and $\mathbb{P}_0^{tot}(k)=\mathrm{Tr}\,E_k|\mathrm{OUT}_0\rangle\langle\mathrm{OUT}_0|_{\mathrm{R}}$. Let us also define $\mathbb{P}_G^{tot}$, the distribution of outcomes if the state of the computer would be the normalized state $|\mathrm{OUT}_G^N\rangle \equiv |\mathrm{OUT}_G\rangle/\|\mathrm{OUT}_G\|$. The triangle inequality and the properties of the trace-norm imply that

$$\begin{aligned}\|\mathbb{P}^{tot}-\mathbb{P}_0^{tot}\|_1 &\leq \|\mathbb{P}^{tot}-\mathbb{P}_G^{tot}\|_1+\|\mathbb{P}_G^{tot}-\mathbb{P}_0^{tot}\|_1\\ &\leq \||\mathrm{OUT}\rangle\langle\mathrm{OUT}|-|\mathrm{OUT}_G^N\rangle\langle\mathrm{OUT}_G^N|\|_1+\|\mathbb{P}_G^{tot}-\mathbb{P}_0^{tot}\|_1.\end{aligned} \quad (24)$$

Here the first term can be bounded, using the relation of the trace norm to the fidelity $F(\psi,\phi)=|\langle\psi|\phi\rangle|$ [15], as

$$\begin{aligned}&\||\mathrm{OUT}\rangle\langle\mathrm{OUT}|-|\mathrm{OUT}_G^N\rangle\langle\mathrm{OUT}_G^N|\|_1\\ &\leq \sqrt{1-F(\mathrm{OUT},\mathrm{OUT}_G^N)^2}\\ &\leq \sqrt{1-\|\mathrm{OUT}_G\|^2} \leq \sqrt{2\epsilon-\epsilon^2}.\end{aligned} \quad (25)$$

Now consider the second trace norm on the right-hand side of Eq. (24). We note that all states that are linear combinations of $(r,1)$-sparse error sets applied to the state $|k_1\rangle^{\otimes m^r}\cdots|k_w\rangle^{\otimes m^r}$ will give rise to the measurement outcome k since we are taking majorities. We can model $E_k=P_k$ where P_k is the projector onto the space of computational basis states that give rise to the majority output string k. Thus we have

$$P_k|\mathrm{OUT}_G\rangle=\alpha_{k_1,\ldots,k_w}\sum_{\mathcal{T}\in G}E_{\mathrm{SB}}(\mathcal{T})U_0^{r\dagger}|k_1\rangle^{\otimes m^r}\cdots|k_w\rangle^{\otimes m^r}\otimes|\mathrm{REST}\rangle_{\mathrm{AB}}, \quad (26)$$

which can be written as $\alpha_{k_1,\cdots,k_w}Q^r_G|\psi_k\rangle$ for some normalized state $|\psi_k\rangle_{\mathrm{RAB}}$. This implies that the second term in Eq. (24) can be bounded as

$$\begin{aligned}\|\mathbb{P}_G^{\mathrm{tot}}-\mathbb{P}_0^{\mathrm{tot}}\|_1 &= \sum_k|\alpha_{k_1,\ldots,k_w}|^2\left|\frac{\|Q^r_G|\psi_k\rangle\|^2}{\|\mathrm{OUT}_G\|^2}-1\right|\\ &\leq \sum_k|\alpha_{k_1,\ldots,k_w}|^2\max_k\left|\frac{\|Q^r_G|\psi_k\rangle\|^2}{\|\mathrm{OUT}_G\|^2}-1\right| \leq \frac{4\epsilon}{(1-\epsilon)^2},\end{aligned} \quad (27)$$

using the bounding inequalities of $\|Q^r_G\|$ and $\|\mathrm{OUT}_G\|$. All bounds put together, using $\epsilon<1/2$, give the result, Eq. (4).■

B. Step C: Nonsparse fault-paths have small norm

Consider the evolution Q^r which can be viewed as a sequence of unitary evolutions, one for each r-rectangle, since qubits in different rectangles do not interact. The number of locations in M_0 is N. The computation Q^r is bad when at least one r-rectangle is bad, or using Lemma 1

$$\|Q^r_B\| \leq N\|R^r_B\|\|R^r_G\|^{N-1}, \quad (28)$$

where R^r_B and R^r_G are the good and bad parts of the unitary evolution R^r for some r-rectangle. The unitarity of R^r implies that we can bound $\|R^r_G\| \leq 1+\|R^r_B\|$. In each rectangle we can view the entire evolution as a sequence of unitary evolutions, one for each $(r-1)$-rectangle. Note that we are again using the fact that noninteracting qubits have noninteracting baths. An r-rectangle is bad when there are at least two $(r-1)$-rectangles which contain sets of faulty locations which are not $(r-1,1)$ sparse. This implies, using Lemma 1 again, that

$$\|R^r_B\| \leq \binom{A_C}{2}\|R^{r-1}_B\|^2\|R^{r-1}_G\|^{A_C-2}, \quad (29)$$

where we can use that $\|R^{r-1}_G\| \leq 1+\|R^{r-1}_B\|$. When $r=1$, R^1_G is a unitary operation and thus $\|R^1_G\|=1$. This recurrence in r is identical to the one in Lemma 11 in Ref. [4] and thus the solution and results are the same if we replace η in Ref. [4] by $\lambda_0 t_0$. Thus the critical error threshold value is

$$(\lambda_0 t_0)_c=\frac{1}{\mathbf{e}A_C(A_C-1)}. \quad (30)$$

Here we can observe a difference with the simplest error model with error probability p for which the critical value is $p_c=1/\binom{A_C}{2}$ [4]. The dimensionless quantity $\lambda_0 t_0$ plays the role of an *amplitude*, see Sec. V, which implies that this threshold value may be more stringent than in the simple probabilistic error model (see also the critique by Alicki [19] on our results). However, we believe that this analysis is too course to really give information about the value of the threshold. The fact is that in practice, baths do not have infinite memory times since they are coupled to many other degrees of freedom. Representing the coupling between bath and system as a pure coherent evolution was needed in this analysis to deal with the non-Markovian dynamics; however, we do not expect this formal procedure to give rise to an *optimal* error threshold.

The idea of the remaining derivation given in Ref. [4] is to show that when $\lambda_0 t_0 < (\lambda_0 t_0)_c$ for large enough concatenation level r, $\|Q_B^r\| \leq \epsilon$ for arbitrary small ϵ. Lemma 4 then tells how much our quantum computation errs from the error-free computation. Summarizing we get the following, as in Ref. [4].

Theorem 1 (Threshold theorem for local non-Markovian noise). Let N be the number of locations of an error-free quantum computation M that outputs samples from a probability distribution $\mathbb{P}$. There exists a quantum circuit M' subjected to noise according to the Hamiltonian $H_{\rm SB}$ and bath Hamiltonian $H_{\rm B}$ that obeys the "noninteracting qubits have noninteracting baths" assumption which outputs the probability distribution $\mathbb{P}'$ such that

$$\|\mathbb{P}' - \mathbb{P}\|_1 \leq \epsilon, \tag{31}$$

when $\Delta_{\rm SB} t_0 < 1/[\mathbf{e} A_C (A_C - 1)]$ and A_C is the number of locations in a 1-rectangle of a (computation) code C that can correct two errors and has spread $s=1$. The number of locations in M' is N polylog $N/\sqrt{\epsilon}$.

Proof. This follows from the basic results in Ref. [4] and Lemma 4 and step C above. It was proved in Ref. [4] for $\lambda_0 t_0$ below the threshold $(\lambda_0 t_0)_c$ in Eq. (30) when the concatenation level $r = c_1 \log[\log(N/\epsilon') + c_2] + c_3$, for constants c_1, c_2, and c_3 we have $\|Q_B^r\| \leq \epsilon'$. So we choose $M' = M_r$, the computation at this concatenation level r which implies that $\|\mathbb{P}' - \mathbb{P}\|_1 \leq \sqrt{2\epsilon'} + 16\epsilon' \equiv \epsilon$. The number of resources (time and space, related to A_C) in M' scales exponentially, i.e., the number of locations in M_r is $N A_C^r$. With the dependence of r on N and ϵ' this implies the polylogarithmic overhead in terms of N and $\sqrt{\epsilon}$. ∎

III. EXTENSION TO DECOHERENCE MODELS WITH CLUSTERED QUBITS

In the most general noise model we start with a Hamiltonian description of system and bath. We will assume that such Hamiltonians are 1–system local, that is, the interaction Hamiltonian between system and bath is a sum of terms each of which couples a *single* qubit to some part of the bath. This covers many interaction Hamiltonians in systems that are being considered for quantum computation (see [20]).

We have seen that basically the only place where the noise model enters the derivation of fault-tolerance is in Sec. II B, i.e., the derivation that the total amplitude/probability/norm for nonsparse fault-paths at concatenation level r goes (doubly exponentially fast in r) to zero when the initial error strength is below the threshold. Locality of the interaction Hamiltonian is an important (and necessary) ingredient in the derivation of fault-tolerance since it implies—without any further assumptions on the structure of the bath or the (non)-Markovian character of the system—that fault-path operators with k faults have a norm bounded by $(2\lambda_0 t_0)^k$ (see the Appendix). This bound is not strong enough by itself to derive that $\|Q_B^r\|$ becomes arbitrarily small for sufficiently small $\lambda_0 t_0$. We find that there are technical and potentially fundamental problems in the derivation of step C, for the most general local Hamiltonian model both *in the Markovian case* as well as in the non-Markovian case. The problems are due to the fact that all qubits of the computer potentially couple at a given time to the same bath which was prevented in the derivation of Theorem 1 by assuming that "noninteracting qubits have noninteracting baths." The problem is basically due to the fact that the unitary evolution of a working period cannot be written as a product of unitary evolutions for each rectangle in the working period since different rectangles may share their bath.

We thus need to consider restricted models that are still physically very relevant:

Clustered qubits at encoding level $r=1$

We can generalize the model in Sec. I A, i.e., noninteracting qubits have noninteracting baths, to one in which a cluster of qubits can share a bath. The model is depicted in Fig. 3. We will assume that qubits that are contained in a 1-rectangle of M_r may share a common bath whereas qubits in different 1-rectangles do not share a bath. We imagine that baths are attached to physical locations, so that the interaction regions of different r-rectangles are physically separate. This means that from one 1-working period to the next one, qubits have to be moved around, i.e., qubits that participate in one 1-rectangle have to be brought together.

Let us for the moment neglect the machinery that is necessary to move qubits around. Then we can observe that the entire computation M_r can be viewed as a sequence of unitary gates each involving a single 1-rectangle. In the 1-rectangle we cannot decompose the evolution as a sequence of unitary transformations for each location; but at this lowest level $r=1$ it is simple to derive a bound on the *bad* part R_B^1 of the unitary operation R^1. Given this bound we can insert it in the previous recurrence of Eq. (29) and determine a threshold which is the same as before. Here is the bound on R_B^1:

Lemma 5. Let $R^1 = R_B^1 + R_G^1$ be the unitary transformation of a 1-rectangle where R_B^1 is a sum of nonsparse fault-path operators, i.e., each such operator contains at least two locations with faults. Then

$$\|R_B^1\| \leq 2(A_C t_0 \lambda_0)^2 \tag{32}$$

and $\|R_G^1\| \leq 1 + \|R_B^1\|$.

Proof. We do a Trotter expansion for R^1 as in Lemma 6 and obtain a tree with infinite depth. We combine branches of the tree in the following way: (1) After a location has become faulty we append the full unitary for the remaining time of the location and (2) if two faults have occurred at two different locations we do no longer branch the tree and just append the entire remaining unitary transformation to that branch. In this way the norm of every time-resolved branch with at least two faults $E_{2^+}(\mathcal{T})$ is bounded by $\|E_{2^+}(\mathcal{T})\| \leq (2\lambda_0 t_0/n)^2$. There are $\binom{A_C t_0 n/t}{2}$ such branches and thus

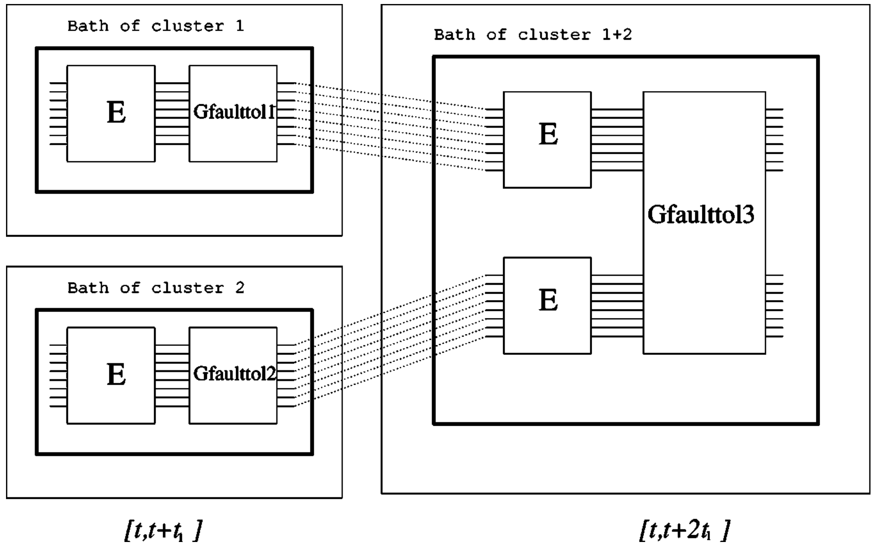

FIG. 3. Schematic representation of a decoherence model where clusters of qubits can share a common bath. Logical qubits 1 and 2 are encoded once in a block of qubits. In the original circuit these qubits first undergo single qubit gates G_1 and G_2 and then interact in G_3. In encoded form this implies three 1-rectangles that each take some time t_1; these are denoted by the boxes with fat lines in the figure. Each 1-rectangle or cluster has its own bath. These baths may change over time, that is, the bath of cluster 1 may evolve or change and not be the same as the environment that this block of qubits sees later.

$$\|R_B^1\| \leqslant (2\lambda_0 t/n)^2 \binom{A_C t_0 n/t}{2} \leqslant 2(A_C t_0 \lambda_0)^2. \qquad (33)$$

■

A physical example of this decoherence model is the proposal for scalable ion-trap computation [21]. A few qubits are stored in an ion-trap where they may share a common bath. The states of qubits can be moved around to let them interact. A small cluster of ion-traps may be used to carry out the fault-tolerant circuits and error-correcting at level $r=1$ of encoding.

The issue of moving qubit states around is not entirely trivial and will be addressed in detail in a future paper [22].

IV. OVERVIEW

We would like to summarize the known results, including the ones in this paper, on threshold results for different decoherence models. The simplest model is one in which we assume that each location undergoes an error with probability p and undergoes no error with probability $1-p$. This is a specific example of a Markovian model in which every location has its own separate environment, i.e., we have "Single location baths," see the upper left entry in Table I. Generalizations of this model exist [3,4]; in these models a superoperator $\mathcal{S}(\rho)=\mathcal{S}_0(\rho)+\mathcal{E}(\rho)$ where $\mathcal{S}_0$ corresponds to the error-free evolution and $\mathcal{E}$ to the erroneous part, is associated with each location. Again this corresponds to the upper left entry in the table. This model has been generalized to allow for more general correlations in space and time in the following manner. In Ref. [4] fault-tolerance was derived in a model where it is assumed that the probability for a fault-path with k faults is bounded by $Cp^k(1-p)^{N-k}$ where N is the

TABLE I. A check (√) indicates that a fault-tolerance result exists whereas a question mark (?) indicates that it is not known to exist so far (neither has it been disproved). The results for non-Markovian baths assume a 1–system local interaction Hamiltonian that can be bounded in norm. They also assume that we can do two-qubit gates between any two qubits in the circuit (that is, we do not take physical locality constraints into account). The assumptions on the structure of the system-bath interaction and the bath Hamiltonian are given by the three columns. Single location baths implies that the interaction and the baths are constrained so that for each elementary time-interval (clock-cycle) $[t, t+t_0]$ the following condition is obeyed: qubits that do not interact can only interact with baths which do not interact, see Fig. 1(b). Note that the particular baths with which the qubits interact may change over time. Cluster location baths is the extension of this model covered in Sec. III where a cluster of qubits can share the same bath, see Fig. 3. In the last column there is no constraint on the bath.

		Spatial correlations		
		Single location baths	Cluster location baths at $r=1$	Arbitrary baths
Temporal correlations	Markovian within gate-time t_0	√	√	?
	Non-Markovian with finite memory time $\tau > t_0$	√	√	?
	Non-Markovian with unlimited memory time	√	√	?

total number of locations in the circuit [note the difference with Eq. (A1) in the Appendix]. Similarly, in Ref. [3] fault-tolerance was derived under the assumption that a fault-path with *at least* k faults has probability bounded by Cp^k for some constant C. Let us call these conditions the exponential decay conditions. Note in Table I that it is not known whether one can derive fault-tolerance for a entirely Markovian model but *with* extended spatial correlations between the baths, i.e., for every clock-cycle we have a superoperator that acts on all qubits of the system; the point is that it is not clear whether such a superoperator would obey some sort of exponential decay conditions.

V. MEASURES OF COUPLING STRENGTH AND DECOHERENCE

In our analysis the role of error amplitude is played by the dimensionless number $\lambda_0 t_0$ which captures the relative strength of the interaction Hamiltonian as compared to the system Hamiltonian. It is this quantity, $\lambda_0 t_0$, that should be $O(10^{-4})$ as was determined for some codes. In a purely Markovian analysis we typically replace λ_0 by an inverse T_2 or T_1 time and this may give a more optimistic idea of the regime of fault-tolerance. Let us consider a few examples of decoherence mechanisms and see how sensible it is to use Δ_{SB} as a bound for decoherence. A good example of a non-Markovian decoherence mechanism is a small finite dimensional environment localized in space, for example, a set of spins nearby the system of interest. An example is the decoherence in NMR due to interactions with nuclear spins in the same molecule. In NMR the nuclear exchange coupling between spins a and b is given by

$$H_{\mathrm{SB}} = J_{ab}\vec{I}_a \cdot \vec{I}_b. \tag{34}$$

If the J-coupling is treated as a source of decoherence as compared to the Zeeman-splitting ω_0 for an individual spin, then J/ω_0 can be $\sim 10^{-6}$ (see [23]).

For some physical systems a source of decoherence is a bath of spins, each of which couples to a single qubit. An example is the electron spin qubit in a single quantum dot which couples via the hyperfine coupling to a large set of nuclear spins in the semiconductor [24]. The interaction Hamiltonian is as follows:

$$H_{\mathrm{SB}} = \sum_{i=1}^{N} a(i)\vec{\sigma} \cdot \vec{I}[i], \tag{35}$$

where $a(i) = Av_0|\psi_s(i)|^2$ and A is the hyperfine coupling constant, v_0 is the volume of the crystal cell, and $|\psi_s(i)|^2$ is the probability of the electron to be at the position of nuclear spin i. If we bound $\Sigma_i |\psi_s(i)|^2 \leqslant 1$ we have that $\|H_{\mathrm{SB}}\| \leqslant CAv_0$ where C is a small constant (of order 1). This may give a somewhat weak upper bound on the decoherence, since we are basically adding the effects of each nuclear spin separately.

A third type of decoherence mechanism exists which is essentially troublesome in our analysis. This is the example of a single qubit, or spin, coupled to a bosonic bath. The interaction Hamiltonian is that of the spin-boson model [25]

$$H_{\mathrm{SB}} = \sigma_z \otimes \sum_{i=1}^{N} (c_i a_i + c_i^* a_i^\dagger), \tag{36}$$

where i labels the ith bosonic mode characterized by frequency ω_i. The ith bosonic mode has Hamiltonian $H_{\mathrm{B}_i} = \omega_i[a_i^\dagger a_i + (1/2)]$. In order to represent a continuous bath spectrum, one lets N go to infinity. In that limit the coupling constants c_i are determined by the spectral density $J(\omega) = \Sigma_i |c_i|^2 \delta(\omega - \omega_i)$. The spectral density can have various forms, matching the phenomenology of the particular physical system, an example is the ohmic form in which $J(\omega) = \alpha\omega e^{-\omega/\omega_c}$ where α is a weak coupling constant that has physical relevance and ω_c is a cutoff frequency that is also determined by the physics. It is clear that $\|H_{\mathrm{SB}}\|$ has no physical meaning since it is infinite, the reason being that there are infinitely excited bath states with infinitely high energy. We can determine an energy-dependent upper bound on this norm; using properties of the norm, we can estimate

$$\begin{aligned} \|H_{\mathrm{SB}}|\psi\rangle_{\mathrm{SB}}\| &= \left\| \sum_i (c_i a_i + c_i^* a_i^\dagger)|\psi\rangle_{\mathrm{SB}} \right\| \\ &\leqslant \sum_i |c_i| (\|a_i|\psi\rangle_{\mathrm{SB}}\| + \|a_i^\dagger|\psi\rangle_{\mathrm{SB}}\|) \\ &\leqslant \sum_i \frac{|c_i|}{\sqrt{\omega_i}} \sqrt{4\langle\psi|H_{\mathrm{B}_i}|\psi\rangle_{\mathrm{SB}}}. \end{aligned} \tag{37}$$

Using the Schwartz inequality we get

$$\begin{aligned} \|H_{\mathrm{SB}}|\psi\rangle_{\mathrm{SB}}\| &\leqslant 2\sqrt{\sum_i \frac{|c_i|^2}{\omega_i} \langle\psi|H_{\mathrm{B}}|\psi\rangle_{\mathrm{SB}}} \\ &= 2\sqrt{\langle H_{\mathrm{B}}\rangle_{\psi_{\mathrm{SB}}} \int_0^\infty d\omega \frac{J(\omega)}{\omega}} \end{aligned} \tag{38}$$

for some state of system and bath $|\psi\rangle_{\mathrm{SB}}$ where the bath Hamiltonian $H_{\mathrm{B}} = \Sigma_i H_{\mathrm{B}_i}$. The idea is that for the physically relevant states of the bath $\langle H_{\mathrm{B}}\rangle_\psi$ is bounded. The problem remains that this bound will in general be too poor to be physically relevant, since this energy bound may be quite large. Also, for ohmic coupling (for example) we have the integral $\int_0^\infty d\omega[J(\omega)/\omega] = \alpha\omega_c$, i.e., linear in ω_c. The cutoff ω_c may be quite large and it is more typical to see decoherence rates depend on $\log \omega_c$ as in the non-Markovian analysis of Ref. [26] for example.

Cooling assumption

Some progress can be made in finding good bounds for $\|H_{\mathrm{SB}}\|$ in the case of a bosonic environment if additional assumptions about its state can be made. What is troublesome about the potential nonequilibrium state of the bath is that expectation values such as $\mathrm{Tr}\, a_i a_j |\psi\rangle\langle\psi|_{\mathrm{SB}}$ and $\mathrm{Tr}\, a_i^2 |\psi\rangle\langle\psi|_{\mathrm{SB}}$ may not be zero since the bath state may not be diagonal in the energy or boson number basis. On the other hand, interaction with other environments, for example, by means of cooling, will dephase the state of the bath (due to energy exchange) and drive it to a state that is diagonal in the energy eigenbasis. Under that assumption only the terms $\mathrm{Tr}\, a_i^\dagger a_i |\psi\rangle\langle\psi|_{SB}$ and $\mathrm{Tr}\, a_i a_i^\dagger |\psi\rangle\langle\psi|_{SB}$ are nonzero. In that scenario, Eq. (37) simplifies to

$$\|H_{SB}|\psi\rangle_{SB}\| \approx \sqrt{\sum_i |c_i|^2 \langle\psi|a_i^\dagger a_i + \mathbf{I}/2|\psi\rangle_{SB}}. \qquad (39)$$

Still, $\|H_{SB}|\psi\rangle_{SB}\|$ can be very large if some modes of the environment are highly excited, $n_i = a_i^\dagger a_i \gg 1$. However, in a realistic setting, this will be prevented by cooling the bath, i.e., by constantly removing energy from it. Without making a Markov approximation, we can thus assume that the occupation numbers n_i of the bath are upperbounded by those of a thermal distribution with an effective maximal temperature T_{eff}. This gives

$$\|H_{SB}|\psi\rangle_{SB}\| \lessapprox \sqrt{\int_0^\infty J(\omega)\coth(\beta_{\text{eff}}\omega/2)/2}, \qquad (40)$$

where $\beta_{\text{eff}} = 1/k_B T_{\text{eff}}$. We can evaluate Eq. (40) for the ohmic case using MATHEMATICA

$$\|H_{SB}|\psi\rangle_{SB}\| \lessapprox \sqrt{\frac{\alpha}{2}} \sqrt{-\omega_c^2 + \frac{2\Psi'\left(\frac{1}{\beta_{\text{eff}}\omega_c}\right)}{\beta_{\text{eff}}^2}}, \qquad (41)$$

where $\Psi'(x)$ is the first derivative of the digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$ where $\Gamma(x)$ is the gamma function. For $1/\beta_{\text{eff}}\omega_c \ll 1$ we do a series expansion and obtain

$$\|H_{SB}|\psi\rangle_{SB}\| \lessapprox \sqrt{\frac{\alpha}{2}} \sqrt{\omega_c^2 + \frac{1}{\beta_{\text{eff}}^2}\left[\frac{\pi^2}{3} + O\left(\frac{1}{\beta_{\text{eff}}\omega_c}\right)\right]}. \qquad (42)$$

Unlike Eq. (38), this bound does not involve extensive quantities, such as the total energy $\langle H_B\rangle_{\psi_{SB}}$ of the bath. However, Eq. (42) still involves the high-frequency cutoff ω_c because of the zero-point fluctuations of the bath.

VI. CONCLUSION

Some important open questions remain in the area of fault-tolerant quantum computation. Most importantly, is there a threshold result for non-Markovian error models with system-local Hamiltonians and no further assumptions on the bath? Is this a technical problem, i.e., how can one efficiently estimate Q_B^r, or are there specific malicious system-bath Hamiltonians that have such effect that the norm of the bad faults does not become smaller when increasing r? The next question is whether a better analysis is possible for the spin-boson model, which is a highly relevant decoherence model. One would like to evaluate the effect of H_{SB} in the sector of *physical* states, but the characterization of these physical states is unclear due to the non-Markovian dynamics. For real systems one is probably interested in a finite memory time $\tau > t_0$ which may be simpler to solve. For example, one can derive the superoperator for a single spin qubit coupled to a bosonic bath in the Born approximation [26], however, a derivation involving more than one system qubit may be too hard to do analytically.

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APPENDIX: BOUNDS ON FAULT-PATH NORMS

Lemma 6. (fault-path norms). Consider the entire unitary evolution Q^r of a quantum computation on SB. Let $\forall\, q \in S, \Delta_{SB}[q] \leq \lambda_0$. We expand Q^r as a sum over fault-paths which are characterized by a set of faulty locations $\mathcal{I}$. A fault-path operator with k faults has norm bounded by

$$\|E(\mathcal{I}_k)\| \leq (2\lambda_0 t_0)^k. \qquad (A1)$$

Proof. We do a Trotter-expansion of Q^r and obtain a tree with an infinite number of branches each of which corresponds to a certain time-resolved fault-path. Every time a fault occurs at some time t_m and location i_m we append unitary evolutions for the remaining time of the location, since we do not care that more faults occur in that time-interval, the location has failed anyway. These time-resolved fault-paths are characterized by an index set $\mathcal{T} = \{(i_1,t_1),(i_2,t_2),\ldots,(i_k,t_k)\}$ where $i_1,\ldots,i_k$ is the set of locations of the faults and $t_1,\ldots,t_k$ label the specific times that the faults occur at the locations. Every such time-resolved fault-operator with k faults has norm bounded

$$\|E(\mathcal{T}_k)\| \leq \left(\frac{2t\lambda_0}{n}\right)^k. \qquad (A2)$$

Now we need to group these time-resolved fault-paths corresponding to faults at sets of locations. For fixed n faults can occur in time-intervals of length t/n and thus during a time t_0 $(t_0 n/t)$ time-resolved faults can occur. This implies that

$$\|E(\mathcal{I}_k)\| \leq \sum_{\mathcal{T}_k \to \mathcal{I}_k} \|E(\mathcal{T}_k)\| \leq (2\lambda_0 t_0)^k. \qquad (A3)$$

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