

General implementation of all possible positive-operator-value measurements of single-photon polarization states

S. E. Ahnert and M. C. Payne

Theory of Condensed Matter Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom

(Received 21 September 2004; published 20 January 2005)

Positive-operator-value measures (POVM's) are the most general class of quantum measurements. We propose a setup in which all possible POVM's of a single-photon polarization state (corresponding to all possible sets of two-dimensional Kraus operators) can be implemented easily using linear optics elements. This method makes it possible to experimentally realize any projective orthogonal, projective nonorthogonal or nonprojective sets of any number of POVM operators. Furthermore, our implementation only requires vacuum ancillas and is deterministic rather than probabilistic. Thus it realizes every POVM with the correct set of output states. We give the settings required to implement two different well-known nonorthogonal projective POVM's.

DOI: 10.1103/PhysRevA.71.012330

PACS number(s): 03.67.-a, 03.65.Ta, 42.50.Dv

I. INTRODUCTION

The rapidly increasing interest in quantum information theory and its applications—a comprehensive overview of which can be found in [1]—has also generated significant interest in the theory and possible implementations of generalized measurement in the form of positive-operator-value measures (POVM's) [2–5]. Such measurements are particularly useful in the context of quantum cryptography [6–9].

Experimentally, a wide variety of quantum mechanical phenomena such as teleportation [10], interaction-free measurement [11], and nonlocality [12–14] have been demonstrated experimentally using photons [15–18]. Recently it was shown that the operations necessary for quantum computation can be implemented using linear optics [19,20], which makes photons a promising candidate for quantum information applications.

We propose here a single setup for the implementation of all possible POVM's of a single-photon polarization state. This includes POVM's with orthogonal projective, nonorthogonal projective, and nonprojective sets of Kraus operators. Similar to our previous work in [5], our setup is a deterministic as opposed to probabilistic implementation of a POVM, which means that the setup delivers one of the possible POVM output states of Eq. (2) in every measurement. In contrast to [5], however, the setup introduced here is much more general and at the same time far simpler. Furthermore, our method does not require any ancillas except for vacuum states. Finally, we illustrate how our setup can be used to implement two well-known POVM's, including one required for established quantum cryptography protocols [6–9].

II. POSITIVE OPERATOR VALUE MEASURES

The positive operator value measure is the most general formulation of quantum measurement [21]. Mathematically it corresponds to a positive-definite partition of unity in the space of operators on a given Hilbert space. A POVM is given by a set of positive-definite Hermitian operators $\{F_i\}$, which in turn can be expressed in terms of a set of so-called Kraus operators $\{M_i\}$, such that $F_i = M_i^\dagger M_i$ and for a POVM with n operators,

$$\sum_{i=1}^n M_i^\dagger M_i = \sum_{i=1}^n F_i = I, \quad (1)$$

where I is the unit matrix. After a POVM measurement is performed on a quantum state represented by a density matrix ρ , the state becomes ρ_i , where

$$\rho_i = \frac{M_i \rho M_i^\dagger}{\text{tr}(M_i \rho M_i^\dagger)}, \quad (2)$$

with probability p_i , where

$$p_i = \text{tr}(M_i \rho M_i^\dagger). \quad (3)$$

Note that a POVM can project the input state to a fixed set of (orthogonal or nonorthogonal) states, but can also be nonprojective, meaning that the set of possible final states is not fixed, but depends on the input state. In the case of a set of projective orthogonal operators all M_i can be written as outer products of pairs of orthogonal state vectors. For projective nonorthogonal operators all M_i can be written as outer products of general state vectors. Finally, nonprojective operators are all remaining sets—i.e., sets which contain at least one member which cannot be written as an outer product of state vectors.

A deterministic implementation (such as the setup we present here) of any of these types of POVM gives one of the possible POVM output states of Eq. (2) in every measurement, with probabilities of Eq. (3). A probabilistic implementation by contrast would only give the probability distribution of Eq. (3) and may not even be successful in performing the POVM every time.

III. POVM OPERATOR MODULE

The implementation proposed here uses $(n-1)$ linear optics modules for an n operator POVM. One such module is depicted in Fig. 1. It consists of five polarizing beam splitters, arranged as shown in this figure. The photon enters at the bottom left of Fig. 1 and is split into its horizontal and vertical polarization states $|H\rangle$ and $|V\rangle$, respectively. These

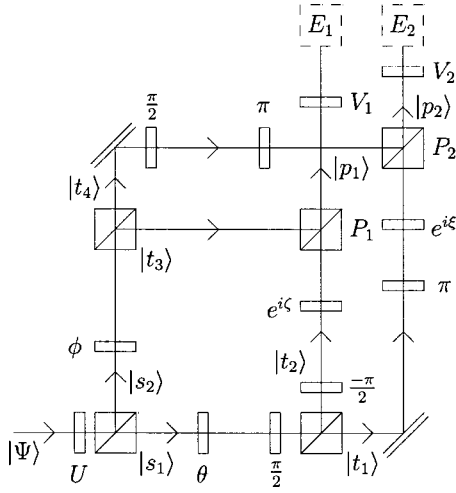


FIG. 1. The module implementing measurement operators F_1 and F_2 . The photon enters in state $|\Psi\rangle$ at the bottom left corner and exits either at E_1 or E_2 , where it can be detected. All beam splitters are polarizing beam splitters with the same polarization basis and transmit photons in the $|H\rangle$ state, while reflecting photons in the $|V\rangle$ state. The angles θ , ϕ , $\pi/2$, and π of the polarization rotators are measured relative to this basis. U , V_1 , and V_2 are unitary operators, and $e^{i\zeta}$ and $e^{i\xi}$ signify phase shifters.

components, now in the path (or “which-path”) states $|s_1\rangle$ (H component) and $|s_2\rangle$ (V component) are then rotated by angles θ (H component) and ϕ (V component), using variable polarization rotators. Then both of these amplitudes are in turn split by two further polarizing beam splitters and form a superposition of the four path states $|t_1\rangle$, $|t_2\rangle$, $|t_3\rangle$, and $|t_4\rangle$. The beam splitters P_1 then reunify the path states $|t_2\rangle$ and $|t_3\rangle$ in $|p_1\rangle$, and P_2 recombines $|t_1\rangle$ and $|t_4\rangle$ in $|p_2\rangle$. Furthermore, the setup contains another five polarization rotators (rotating by angles $+\pi/2$, $-\pi/2$, and π), three variable unitary operators U , V_1 , and V_2 , and two variable phase shifts $e^{i\zeta}$ and $e^{i\xi}$. All these elements are placed as shown in Fig. 1. Ancillas, often required in linear optics implementations of quantum information processes, are here only present in the form of vacuum states in the ports of the polarizing beam splitters.

Consider the case where $U=V_1=V_2=I$ (the unit matrix) and $\zeta=\xi=0$. Then a photon incident on the apparatus in the state

$$|\Psi\rangle = a|H\rangle + b|V\rangle, \quad (4)$$

where $|a|^2 + |b|^2 = 1$ evolves to

$$\begin{aligned} |\Psi\rangle &\rightarrow a|H\rangle|s_1\rangle + b|V\rangle|s_2\rangle \rightarrow a(\cos\theta|H\rangle + \sin\theta|V\rangle)|s_1\rangle \\ &\quad + b(\cos\phi|V\rangle - \sin\phi|H\rangle)|s_2\rangle \rightarrow a(\cos\theta|V\rangle - \sin\theta|H\rangle) \\ &\quad \times |s_1\rangle + b(\cos\phi|V\rangle - \sin\phi|H\rangle)|s_2\rangle \rightarrow a(\cos\theta|V\rangle|t_2\rangle \\ &\quad - \sin\theta|H\rangle|t_1\rangle) + b(\cos\phi|V\rangle|t_3\rangle - \sin\phi|H\rangle|t_4\rangle) \\ &\rightarrow a(\cos\theta|H\rangle|t_2\rangle + \sin\theta|H\rangle|t_1\rangle) + b(\cos\phi|V\rangle|t_3\rangle \\ &\quad + \sin\phi|V\rangle|t_4\rangle). \end{aligned} \quad (5)$$

The beam splitter P_1 then recombines path states $|t_2\rangle$ and $|t_3\rangle$ and, similarly, P_2 recombines $|t_1\rangle$ and $|t_4\rangle$, so that

$$\begin{aligned} |\Psi\rangle &\rightarrow (a \cos\theta|H\rangle + b \cos\phi|V\rangle)|p_1\rangle + (a \sin\theta|H\rangle \\ &\quad + b \sin\phi|V\rangle)|p_2\rangle, \end{aligned} \quad (6)$$

where $|p_1\rangle$ and $|p_2\rangle$ denote the path states of amplitudes emerging from beam splitters P_1 and P_2 , respectively.

This corresponds to the matrix transformations

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow D_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad (7)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow D_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\phi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad (8)$$

in the respective Hilbert spaces of $|p_1\rangle$ and $|p_2\rangle$. Note that, as required, $D_1^2 + D_2^2 = I$ where I is the unit matrix. It is due to the vacuum state ancilla entering the first beam splitter that two of the four outputs of the two final beam splitters remain dark, giving a partition of unity into two operators rather than four.

An arbitrary $n \times n$ matrix A can be written as $A = VDU$ where V and U are unitary matrices and D is a diagonal matrix. Thus we can write any general Kraus operator M for quantum measurement as

$$M_i = V_i D_i U_i. \quad (9)$$

The moduli of the elements of the diagonal matrix D_i are confined to lie between 0 and 1. (Note that in general $M_i \neq M_i^\dagger$.)

Let us consider the two operators in our module, with $U_1 = U_2$ implemented by a variable polarization rotator, placed before the entrance of the module. Also we introduce phase shifts ζ and ξ for the sake of generality, as the $\{D_i\}$ in Eq. (9) are in general complex. Hence,

$$D_1 = \begin{pmatrix} e^{i\zeta} \cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix}, \quad D_2 = \begin{pmatrix} e^{i\xi} \sin\theta & 0 \\ 0 & \sin\phi \end{pmatrix}. \quad (10)$$

Thus,

$$F_1 = M_1^\dagger M_1 = U_1^\dagger D_1^2 U_1, \quad (11)$$

where $D_1^2 \equiv D_1^\dagger V_1^\dagger V_1 D_1 = D_1^\dagger D_1$. Hence,

$$\begin{aligned} F_2 &= I - F_1 = I - U_1^\dagger D_1^2 U_1 = U_1^\dagger U_1 - U_1^\dagger D_1^2 U_1 \\ &= U_1^\dagger (I - D_1^2) U_1 = U_1^\dagger D_2^2 U_1, \end{aligned} \quad (12)$$

as required by $\sum_{i=1}^n M_i^\dagger M_i = \sum_{i=1}^n F_i = I$. This also makes it clear that $U_1 = U_2$ follows naturally and does not place an additional constraint on the space of possible operators that can be implemented. Hence the arrangement illustrated in Fig. 1 provides a physical implementation of a completely general positive-definite bipartition of unity, which is what a two-operator POVM represents in mathematical terms. This apparatus is deterministic, which means that all output states are given by Eq. (2) with probabilities given in Eq. (3). It therefore does not only implement the POVM operators, but also specific Kraus operators, chosen by the operators $\{V_i\}$ in

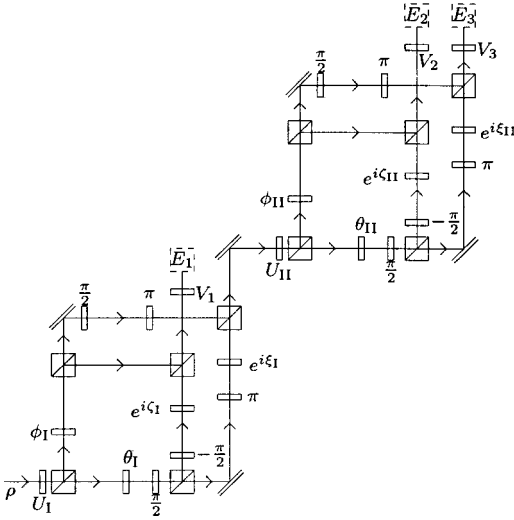


FIG. 2. The full setup for three measurement operators, using two modules. Note the phase shifters with phase factors ζ_1 , ζ_{II} , ξ_1 , and ξ_{II} , as well as additional unitary operators U_I , U_{II} , V_1 , V_2 , and V_3 introduced in order to make the setup completely general. A general state ρ entering the setup becomes the output state $\rho_i = M_{i\rho} M_i^\dagger / \text{tr}(M_{i\rho} M_i^\dagger)$ at exit E_i with probability $p_i = \text{tr}(M_{i\rho} M_i^\dagger)$.

Eq. (9). These operators are implemented at the exits of the operator modules, as shown in Figs. 1 and 2.

IV. GENERALIZATION TO N MODULES

If we wish to perform any POVM consisting of three measurement operators, we implement the first POVM operator as F_1 in the first module and redirect the amplitude emerging from the other exit into a second module, which acts upon it with different initial rotation and different θ and ϕ parameters, implementing the remaining operators F_2 and F_3 , so that

$$F_1 = U_1^\dagger D_1^2 U_1,$$

$$F_2 = U_1^\dagger \tilde{D}_1^\dagger U_{II}^\dagger D_{II}^2 U_{II} \tilde{D}_1 U_1,$$

$$F_3 = U_1^\dagger \tilde{D}_1^\dagger U_{II}^\dagger (I - D_{II}^2) U_{II} \tilde{D}_1 U_1 = I - F_1 - F_2, \quad (13)$$

where U_I and U_{II} are the unitary operators implemented on the photon before modules I and II, respectively, D_I and D_{II} correspond to D_1 in the modules I and II, respectively, and \tilde{D}_I plays the role of D_2 so that $\tilde{D}_I^\dagger \tilde{D}_I = I - D_I^2$. Figure 2 shows the complete setup for performing any POVM with three measurement operators.

The generalization to n operators, using $n-1$ modules, is straightforward. In general, for n operators and $n-1$ modules we have, for $j < n$,

$$F_j = \left[\prod_{i=1}^{j-1} U_{(i)}^\dagger \tilde{D}_{(i)}^\dagger \right] U_{(j)}^\dagger D_j^2 U_{(j)} \left[\prod_{i=1}^{j-1} \tilde{D}_{(j-i)} U_{(j-i)} \right] \quad (14)$$

and the last operator ($j=n$) is given by

$$F_n = \left[\prod_{i=1}^{n-1} U_{(i)}^\dagger \tilde{D}_{(i)}^\dagger \right] \left[\prod_{i=1}^{n-1} \tilde{D}_{(n-i)} U_{(n-i)} \right], \quad (15)$$

where we have introduced the notation $U_I = U_{(1)}$, $U_{II} = U_{(2)}$, ..., $U_I = U_{(1)}$, $U_{II} = U_{(2)}$, ..., etc.

Note that any probabilistic POVM on a single-photon polarization state can be realized with a statistical mixture of at most four POVM operators. If however our setup is to be used for realizing a fully deterministic n -operator POVM, then $n-1$ modules are needed.

V. EXAMPLES

As an example of the simplicity of applying this procedure, we implement the nonorthogonal projective POVM with the three axes of projection separated by 120° , as discussed in our previous work [5]. However, in contrast to our previous example, we now do not require any delay lines in order to recycle photons around the apparatus in order to implement these operators with unit probability of success.

If we consider

$$U_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_{II} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (16)$$

$$D_I = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_{II} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (17)$$

which corresponds to $\theta_I = \arccos(\sqrt{2/3})$, $\theta_{II} = 0$, $\phi_I = \pi/2$, and $\phi_{II} = \pi/2$. The angles of initial polarization rotation at modules I and II are 0 and $\pi/4$, respectively.

Then, using Eq. (13), we find that these give

$$F_1 = \frac{2}{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (18)$$

$$F_2 = \frac{1}{6} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}, \quad (19)$$

$$F_3 = \frac{1}{6} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \quad (20)$$

as required for this POVM. Note that if we want our apparatus to output photon states whose polarizations are separated by 120° , we can simply implement the appropriate Kraus operators while leaving $\{F_j\}$ unchanged, by choosing

$$V_1 = I, \quad V_2 = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad V_3 = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}. \quad (21)$$

A second example which illustrates the flexibility of this approach is the implementation of the POVM employed in the quantum cryptography protocol proposed by Ekert *et al.* [6–9]. The operators can be considered in terms of two polarization states with polarizations at angles α and β :

$$F_1 = \frac{1}{1 + \cos(\beta - \alpha)} \begin{pmatrix} \sin^2 \alpha & -\sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha & \cos^2 \alpha \end{pmatrix}, \quad (22)$$

$$F_2 = \frac{1}{1 + \cos(\beta - \alpha)} \begin{pmatrix} \sin^2 \beta & -\sin \beta \cos \beta \\ -\sin \beta \cos \beta & \cos^2 \beta \end{pmatrix}, \quad (23)$$

and

$$F_3 = I - F_1 - F_2. \quad (24)$$

These can be implemented using

$$U_I = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \quad U_{II} = \begin{pmatrix} \cos \alpha' & \sin \alpha' \\ -\sin \alpha' & \cos \alpha' \end{pmatrix}, \quad (25)$$

where $\alpha' = \text{arccot}\left\{\left[\sqrt{1 + 1/\cos(\beta - \alpha)}\right]\cot(\beta - \alpha)\right\}$ and

$$D_I = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{1}{1 + \cos(\beta - \alpha)}} \end{pmatrix}, \quad D_{II} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (26)$$

Note that for projective POVM's (i.e., where the Kraus operators are outer products), one of the elements of D will always be zero, as the set of output states is independent of the input state. Our setup, however, is more general and en-

compasses all POVM's, including nonprojective sets, whose output states depend on the input states.

VI. CONCLUSION

We have presented a linear optics setup for the deterministic implementation of an arbitrary positive-operator-value measurement of single-photon polarization states, with any number of operators. The only ancillas required for this implementation are vacuum states. Our method is completely general and includes all possible sets of POVM operators: namely, projective orthogonal, projective nonorthogonal, and nonprojective.

Thus any possible generalized quantum measurement of single-photon polarization states can be easily performed using this setup.

Due to its deterministic nature, this POVM setup could be used to perform POVM's on members of entangled states with the correct multipartite postmeasurement state as given by Eq. (2). Another interesting question arising from this is to ask what the most general setup for multipartite POVM's could be.

ACKNOWLEDGMENTS

The authors wish to thank T. Rudolph, G. Pryde, N. Peters, P. G. Kwiat, and S. Virmani for helpful discussions. S.E.A. was supported by Sidney Sussex College, Cambridge.

-
- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
- [2] A. Peres, Phys. Lett. A **128**, 19 (1988).
- [3] B. Reznik, e-print quant-ph/0203055.
- [4] J. Calsamiglia, Phys. Rev. A **65**, 030301(R) (2002).
- [5] S. E. Ahnert and M. C. Payne, Phys. Rev. A **69**, 012312 (2004).
- [6] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
- [7] A. K. Ekert, B. Huttner, G. M. Palma, and A. Peres, Phys. Rev. A **50**, 1047 (1994).
- [8] H. E. Brandt, J. M. Myers, and S. J. Lomonaco, Jr., Phys. Rev. A **56**, 4456 (1997); **58**, 2617(E) (1998).
- [9] H. E. Brandt, Am. J. Phys. **67**, 434 (1999).
- [10] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
- [11] A. C. Elitzur and L. Vaidman, Found. Phys. **23**, 987 (1993).
- [12] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964); reprinted in *Speakable and Unsayable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987).
- [13] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
- [14] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
- [15] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **49**, 91 (1982).
- [16] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).
- [17] P. G. Kwiat, A. G. White, J. R. Mitchell, O. Nairz, G. Weihs, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **83**, 4725 (1999).
- [18] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) **403**, 515 (2000).
- [19] E. Knill, R. Laflamme, and G. J. Milburn, Nature (London) **409**, 46 (2001).
- [20] T. C. Ralph, A. G. White, W. J. Munro, and G. J. Milburn, Phys. Rev. A **65**, 012314 (2002).
- [21] K. Kraus, *Lecture Notes: States, Effects and Operations* (Springer, New York, 1983).