Depolarization channels with zero-bandwidth noises

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(Received 21 July 2004; published 18 January 2005)

A simple model describing depolarization channels with zero-bandwidth environment is presented and exactly solved. The environment is modeled by Lorentzian, telegraphic, and Gaussian zero-bandwidth noises. Such channels can go beyond the standard Markov dynamics and therefore can illustrate the influence of memory effects of the noisy communication channel on the transmitted information. To quantify the disturbance of quantum states the entanglement fidelity between arbitrary input and output states is investigated.

DOI: 10.1103/PhysRevA.71.012322 PACS number(s): 03.67.Hk, 89.70.+c

I. INTRODUCTION

One of the most important features concerning quantum communication is the capacity or the fidelity of quantum information transmitted in noisy quantum channels [1]. The key factor limiting the possibilities of communication using quantum states is an environment-induced noise. Uncontrolled interaction between the environment (E), and the transmitted quantum state can essentially affect the state and in consequence lower the communication capacity of the information channel [2,3].

For qubits, a well known class of quantum noisy channels consists of depolarizing channels [4]. Input information of such channels is stored in a density operator $\hat{\varrho}_{\rm in}$. Such channels can be characterized by a probability p that the quantum information is distorted, and with a probability $1-p$ that the information remains intact. In the simplest case of a single qubit transmitted through the noisy channel the influence of noise is usually decomposed into three interaction channels. Bit error channel $\hat{\sigma}_x$ flipping the values of bits: $|0\rangle\mapsto|1\rangle, |1\rangle\mapsto|0\rangle$; phase error channel $\hat{\sigma}_z$ flipping the phase: $|0\rangle\mapsto|0\rangle, |1\rangle\mapsto-|1\rangle$; and phase and bit error channel $\hat{\sigma}_y$ flipping both: $|0\rangle\mapsto i|1\rangle, |1\rangle\mapsto -i|0\rangle$.

The influence of these interaction channels can be written as an incoherent combination of three unbiased terms generating bit flip errors and phase flip errors in the form given by [5]

$$\hat{\varrho}_{\rm out} = (1-p)\hat{\varrho}_{\rm in} + \frac{p}{3}(\hat{\sigma}_x\hat{\varrho}_{\rm in}\hat{\sigma}_x + \hat{\sigma}_y\hat{\varrho}_{\rm in}\hat{\sigma}_y + \hat{\sigma}_z\hat{\varrho}_{\rm in}\hat{\sigma}_z). \quad (1)$$

This depolarizing channel is just an example of a general quantum channel characterized by a trace-preserving, linear map $\Phi:\hat{\varrho}_{\rm in}\mapsto\hat{\varrho}_{\rm out}$. The influence of the environment on the quantum state $\hat{\varrho}_{\rm in}$ can be represented in terms of a Kraus decomposition: $\Phi(\hat{\varrho}_{\rm in})=\Sigma_r\hat{K}_r\hat{\varrho}_{\rm in}\hat{K}_r^\dagger$, where $\Sigma_r\hat{K}_r^\dagger\hat{K}_r=\hat{\mathbb{1}}$ [6].

The unbiased depolarizing channel given by Eq. (1), corresponds to $\hat{K}_0=\sqrt{1-p}\hat{\mathbb{1}}$, $\hat{K}_1=\sqrt{p/3}\hat{\sigma}_x$, $\hat{K}_2=\sqrt{p/3}\hat{\sigma}_y$, and $\hat{K}_3=\sqrt{p/3}\hat{\sigma}_z$.

It is not difficult to derive the expression corresponding to Eq. (1), from a unitary evolution involving an extended Hilbert space $\mathcal{H}\otimes\mathcal{H}_E$, of the input qubit and an additional qubit (E), consisting the environment degree of freedom. Although mathematically correct, such a derivation has no simple or direct physical realization in terms of a realistic noise. In fact in most known cases, the environment is much more complex, and the additional fact that the three interaction channels corresponding to bit error, flip error, and phase error do not commute leaves the incoherent addition of these channels in the formula (1) questionable.

A more realistic approach to depolarization channels with bit errors requires a better understanding of the physics involved in the system-environment interactions. We shall assume that the system-environment interaction is characterized by a model Hamiltonian H. A unitary evolution of this combined system leads to a Kraus decomposition described by a time dependent map $\Phi_t:\hat{\varrho}_{\rm in}\mapsto\hat{\varrho}(t)$. This means that one should have a time-dependent $p(t)$ such that at the input time $t=0, p(0)=0$. Because of this the general property of the map, $\Phi_t|_{t=0}=\hat{\mathbb{1}}$, expresses the continuity at the origin and hence for all time.

In general the problem of finding the evolution of the state interacting with the environment is very difficult, therefore the environment-induced noise is modeled using various simplified approaches including several assumptions such as the Markov property. In such case it is assumed that the evolution of the state at a given instant is fully determined by the state at that instant, so the process has no "memory" of its past. The Markov property means the quantum channel is such that for an infinitesimal time interval: $p(\Delta t)\simeq\Delta t$. In this case the channel map generates a completely positive dynamical semigroup: $\Phi_t\circ\Phi_{t'}=\Phi_{t+t'}$ for $t,t'\geqslant 0$, which defines a Markovian dynamics.

As it is well known, for such Markovian maps we can transform the time-dependent Kraus decomposition (1) into a local Lindblad equation [7]

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$$\frac{d\hat{\varrho}(t)}{dt} = \hat{\mathcal{L}}\hat{\varrho}(t), \tag{2}$$

with the initial condition $\hat{\varrho}(0) = \hat{\varrho}_{\text{in}}$, and where the Lindblad superoperator $\hat{\mathcal{L}}$ can be derived from the Kraus operators.

In this paper we present a simple model describing an evolution of a quantum state interacting with the environment and study various properties of the affected state. A physical picture behind the algebra can be for example a randomly fluctuating magnetic field acting on an electron's magnetic moment, a thermally fluctuating birefringence of a single mode fiber transmitting a polarization state of a single photon or any other fluctuating environment influencing the considered quantum system.

Using our model we will justify the assumption (1) and determine the conditions for its validity. We will show that the disturbed output state has the form (1) in the infinite interaction time limit only if the disturbance is unbiased and acts identically in all bases. In the general case the evolution leads to a different final state. It will also turn out that the simple model leads to a nontrivial evolution not obeying the Markov property. Therefore, our simple approach will lead us out of the no-memory approximation regime. Our model shows that in many cases one cannot fairly neglect the memory effects of the interaction and consequently the Markov property is not always valid [8,9].

The memory effects have already been a subject of interest in a completely different context [10], where the authors have studied properties of communication channels where the influence of the noise onto the consecutively transmitted states was correlated. In this paper we will be interested in studying memory effects affecting a single state of a transmitted quantum system.

The paper is organized as follows: in Sec. II we present the model of a qubit in a presence of an external noise. In Sec. III we show that if the environment of the channel consists of zero-bandwidth noises one can calculate exactly expressions determining the evolution of a single qubit in such channel. Section IV contains several examples of zero-bandwidth noises that can affect the input state. We show that in general the decoherence channel cannot be regarded as an incoherent superposition of independent interactions. Special cases involving Markovian and exactly soluble non-Markovian dynamics are derived. In Sec. V the efficacy of the quantum channel is quantified by the fidelity between the output state and the input state. An appropriate measure for assessing the fidelity of a mixed input state is the entanglement fidelity [11], which is the maximum fidelity of states being purifications of the input mixed state ρ and the output state $\Phi(\rho)$. Finally Sec. VI concludes the paper.

II. INTERACTION MODEL

The interaction of a qubit with an environment inducing random bit errors will be described by the following Hamiltonian

$$\hat{H} = \mathbf{r} \cdot \hat{\boldsymbol{\sigma}} = x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z, \tag{3}$$

where the three components $r_i = (x, y, z)$ will be uncontrolled "noisy" parameters characterizing the fluctuations of the environment.

We will assume that r_i are independent random variables. This situation is a simplification of a generally very difficult problem with $r_i(t)$ being time-dependent stochastic processes with arbitrary autocorrelations: $\langle r_i(t) r_j(t') \rangle = \delta_{ij}\Delta_i(t,t')$. In these applications, the autocorrelation functions $\Delta_i(t,t')$, have usually a Fourier limited spectrum with an effective bandwidth γ characterizing the environment noise. Even for the simplest form of the autocorrelations, the exact solution of the full time-dependent problem involving more that one $r_i(t)$ noise is not known. However our simple noise model can illustrate several properties of various completely positive maps.

In the proposed scenario, the evolution of a single qubit given by the von Neumann equation has the following form:

$$\frac{d\hat{\varrho}}{dt} = i\sum_i r_i[\hat{\sigma}_i, \hat{\varrho}] = \hat{\mathcal{L}}_0\hat{\varrho}. \tag{4}$$

The appearing Liouville superoperator $\hat{\mathcal{L}}_0$ describes a unitary rotation of a qubit defined by the coefficients r_i. The solution involves a stochastic averaging with respect to the environment. As a result it can be written in the following compact and formal form:

$$\hat{\varrho}(t) = \langle T e^{\int_0^t ds \hat{\mathcal{L}}_0(s)} \rangle \hat{\varrho}_{\text{in}}. \tag{5}$$

There is no useful formula that can handle the chronological time-ordering of three Pauli matrices, and allows an exact stochastic averaging over the environment noises. There are however special cases when the exact solution of Eq. (5) can be obtained. We know of three cases. Case one involves an arbitrary stochastic noise and only one r_i. In this case the chronological ordering plays no role. In case two, fluctuations are Gaussian and the bandwidth characterizing the environment noise is infinite $\gamma = \infty$ (white noise). In this case an exact average of Eq. (5) exists, and a Lindbland equation (2) for the channel map can be derived. Case three, the one investigated in this paper, corresponds to arbitrary random fluctuations of r_i with the environment described by a zero-bandwidth environment noise: $\gamma = 0$. In this case all autocorrelations $\Delta_i(t,t')$, become time-independent, the chronological product plays no role, and an exact averaging over the environment with arbitrary statistics can be performed.

III. EXACT SOLUTION WITH ZERO BANDWIDTH

In order to find the evolution of a state under a time-independent Liouvillian $\hat{\mathcal{L}}_0$, we first find its eigenstates:

$$\hat{\boldsymbol{\mathcal{L}}}_0 \mathbf{A} \cdot \hat{\boldsymbol{\sigma}} = i r_i A_j [\hat{\sigma}_i, \hat{\sigma}_j] = -2 r_i A_j \epsilon_{ijk} \hat{\sigma}_k = \lambda \mathbf{A} \cdot \hat{\boldsymbol{\sigma}}. \tag{6}$$

Hence we obtain a set of equations:

$$-2r_iA_j\epsilon_{ijk} = \lambda A_k. \tag{7}$$

The solutions exist only for a set of eigenvalues $\lambda \in \{0, 2ir, -2ir\}$, where $r=\sqrt{x^2+y^2+z^2}$. For this set of eigenvalues we find the corresponding eigenvectors: for $\lambda_0=0$ we have $\mathbf{A}_0=\mathbf{r}$, and for $\lambda_\pm=\pm 2ir$ we have $\mathbf{A}_\pm=(\mp iyr - xz, \pm ixr - yz, x^2+y^2)$.

In order to determine the evolution of an arbitrary initial state

$$\hat{\varrho}_{\text{in}} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{a}\cdot\hat{\boldsymbol{\sigma}}), \tag{8}$$

it is helpful to decompose it into the calculated eigenvectors. Pauli operators $\hat{\sigma}_i$ written in the calculated eigenbasis have the following form:

$$\hat{\sigma}_x = \left(\frac{x}{r^2}\mathbf{A}_0 - \frac{xz(\mathbf{A}_+ + \mathbf{A}_-) - iyr(\mathbf{A}_+ - \mathbf{A}_-)}{2r^2(x^2+y^2)}\right)\cdot\hat{\boldsymbol{\sigma}},$$

$$\hat{\sigma}_y = \left(\frac{y}{r^2}\mathbf{A}_0 - \frac{yz(\mathbf{A}_+ + \mathbf{A}_-) - ixr(\mathbf{A}_+ - \mathbf{A}_-)}{2r^2(x^2+y^2)}\right)\cdot\hat{\boldsymbol{\sigma}},$$

$$\hat{\sigma}_z = \left(\frac{z}{r^2}\mathbf{A}_0 + \frac{\mathbf{A}_+ + \mathbf{A}_-}{2r^2}\right)\cdot\hat{\boldsymbol{\sigma}}. \tag{9}$$

At this point it is easy to find the action of the evolution operator $\exp(\hat{\mathcal{L}}_0 t)$ on the given input state. We simply multiply the eigenvectors appearing in our decomposition by the proper factors $\exp(\lambda t)$ with corresponding eigenvalues λ. This yields

$$\begin{aligned} e^{\hat{\mathcal{L}}_0 t}\hat{\varrho}_{\text{in}} = \frac{1}{2}\Bigg[&\hat{\mathbb{1}} + a_x\left(\frac{x}{r^2}\mathbf{A}_0 - \frac{xz(e^{2irt}\mathbf{A}_+ + e^{-2irt}\mathbf{A}_-) - iyr(e^{2irt}\mathbf{A}_+ - e^{-2irt}\mathbf{A}_-)}{2r^2(x^2+y^2)}\right)\cdot\hat{\boldsymbol{\sigma}} \\ &+ a_y\left(\frac{y}{r^2}\mathbf{A}_0 - \frac{yz(e^{2irt}\mathbf{A}_+ + e^{-2irt}\mathbf{A}_-) + ixr(e^{2irt}\mathbf{A}_+ - e^{-2irt}\mathbf{A}_-)}{2r^2(x^2+y^2)}\right)\cdot\hat{\boldsymbol{\sigma}} + a_z\left(\frac{z}{r^2}\mathbf{A}_0 + \frac{e^{2irt}\mathbf{A}_+ + e^{-2irt}\mathbf{A}_-}{2r^2}\right)\cdot\hat{\boldsymbol{\sigma}}\Bigg]. \end{aligned} \tag{10}$$

The above formula expresses the state of the qubit evolving under the action of the Liouvillian defined by the arbitrary vector $\mathbf{r}$.

In our model of the noisy channel, the interaction between the environment and the qubit can be described by a randomly chosen vector $\mathbf{r}$. Therefore, to model the evolution of the qubit under the influence of the environment-induced noise we will average the obtained output state over all possible realizations of the dynamics characterized by arbitrary vectors $\mathbf{r}$. For simplicity we will be interested in an averaged evolution of the qubit with an even in $\mathbf{r}$ probability distribution $p(\mathbf{r})=p(-\mathbf{r})$. In this case a nonvanishing contribution to the averaged output state will come only from the symmetric part of the expression (10):

$$\begin{aligned} \left\{e^{\hat{\mathcal{L}}_0 t}\hat{\varrho}_{\text{in}}\right\}_{\text{sym}} = \frac{1}{2}\Bigg[&\hat{\mathbb{1}} + a_x\hat{\sigma}_x\left(\frac{x^2}{r^2} + \frac{y^2+z^2}{r^2}\cos 2rt\right) \\ &+ a_y\hat{\sigma}_y\left(\frac{y^2}{r^2} + \frac{x^2+z^2}{r^2}\cos 2rt\right) \\ &+ a_z\hat{\sigma}_z\left(\frac{z^2}{r^2} + \frac{x^2+y^2}{r^2}\cos 2rt\right)\Bigg]. \end{aligned} \tag{11}$$

From the above formula it follows that the initial state (8) evolves into an averaged output state

$$\hat{\varrho}_{\text{out}}(t) = \frac{1}{2}\left[\hat{\mathbb{1}} + a_i\Lambda_i(t)\hat{\sigma}_i\right]. \tag{12}$$

The dynamics of the output state is completely described at any time by a set of time-dependent functions $\Lambda_i(t)$, which have the following form:

$$\Lambda_i(t) = 1 - 2\int d^3r\, p(\mathbf{r})\left(1 - \frac{r_i^2}{r^2}\right)\sin^2 rt. \tag{13}$$

The output state can be also equivalently represented in terms of Kraus operators $\hat{K}_r$:

$$\hat{\varrho}_{\text{out}}(t) = \sum_r \hat{K}_r(t)\hat{\varrho}_{\text{in}}\hat{K}_r^\dagger(t), \tag{14}$$

where [9]

$$\hat{K}_0 = \frac{1}{2}\hat{\mathbb{1}}\sqrt{1+\Lambda_x+\Lambda_y+\Lambda_z} = \hat{\mathbb{1}}\sqrt{\int d^3r\, p(\mathbf{r})\cos^2 rt},$$

$$\hat{K}_1 = \frac{1}{2}\hat{\sigma}_x\sqrt{1+\Lambda_x-\Lambda_y-\Lambda_z} = \hat{\sigma}_x\sqrt{\int d^3r\, p(\mathbf{r})\frac{x^2}{r^2}\sin^2 rt},$$

$$\hat{K}_2 = \frac{1}{2}\hat{\sigma}_y\sqrt{1-\Lambda_x+\Lambda_y-\Lambda_z} = \hat{\sigma}_y\sqrt{\int d^3r\, p(\mathbf{r})\frac{y^2}{r^2}\sin^2 rt},$$

$$\hat{K}_3 = \frac{1}{2}\hat{\sigma}_z\sqrt{1-\Lambda_x-\Lambda_y+\Lambda_z} = \hat{\sigma}_z\sqrt{\int d^3r\, p(\mathbf{r})\frac{z^2}{r^2}\sin^2 rt}. \tag{15}$$

These exact expressions for the Kraus operators are the main result of our investigations. Before we discuss various statistical models, we note that an unbiased incoherent addition of bit-error channels in most cases is not justified. The formula above shows that the time-dependent $\Lambda_i(t)$ functions couple in a highly nontrivial way the three channels. The simplified expression (1) does not reflect this complicated relation between various bit-error channels.

IV. EXAMPLES OF A NOISE

A. Markov noise

Consider a simple case of a completely positive map determined by a Lorentzian probability distribution

$$p(\mathbf{r}) = \frac{1}{3\pi}\left(\frac{\Gamma/2}{x^2+\Gamma^2/4}\delta(y)\delta(z) + \delta(x)\frac{\Gamma/2}{y^2+\Gamma^2/4}\delta(z) + \delta(x)\delta(y)\frac{\Gamma/2}{z^2+\Gamma^2/4}\right) \tag{16}$$

characterized by a width Γ. The corresponding Kraus operators (15) in this case read

$$\hat{K}_0 = \hat{\mathbb{1}}\sqrt{\frac{1+\exp(-\Gamma t)}{2}},$$

$$\hat{K}_1 = \hat{\sigma}_x\sqrt{\frac{1-\exp(-\Gamma t)}{2}},$$

$$\hat{K}_2 = \hat{\sigma}_y\sqrt{\frac{1-\exp(-\Gamma t)}{2}},$$

$$\hat{K}_3 = \hat{\sigma}_z\sqrt{\frac{1-\exp(-\Gamma t)}{2}}. \tag{17}$$

The resulting dynamics of such a channel is

$$\hat{\varrho}(t) = \frac{1+\exp(-\Gamma t)}{2}\hat{\varrho}(0) + \frac{1-\exp(-\Gamma t)}{6} \times [\hat{\sigma}_x\hat{\varrho}(0)\hat{\sigma}_x + \hat{\sigma}_y\hat{\varrho}(0)\hat{\sigma}_y + \hat{\sigma}_z\hat{\varrho}(0)\hat{\sigma}_z]. \tag{18}$$

Let us note that this expression is equivalent to Eq. (1) if the probability is time-dependent i.e.,

$$p(t) = \frac{1-\exp(-\Gamma t)}{2}. \tag{19}$$

In the steady state $p(\infty)=\frac{1}{2}$, the quantum channel reduces to a very simple expression [5]:

$$\hat{\varrho}_{\text{out}} = \frac{1}{2}\left(\hat{\varrho}_{\text{in}} + \frac{1}{3}\hat{\sigma}_x\hat{\varrho}_{\text{in}}\hat{\sigma}_x + \frac{1}{3}\hat{\sigma}_y\hat{\varrho}_{\text{in}}\hat{\sigma}_y + \frac{1}{3}\hat{\sigma}_z\hat{\varrho}_{\text{in}}\hat{\sigma}_z\right). \tag{20}$$

One can easily check that the infinitesimal time evolution of the last three Kraus operators is $\hat{K}_i(\Delta t)\simeq\sqrt{\Delta t}\hat{\sigma}_i$. This behavior is typical for a diffusion process and consequently the state evolution clearly obeys the Markov no-memory property, and as a consequence has the form of the Lindblad equation (2):

$$\frac{d\hat{\varrho}}{dt} = \mathcal{L}\hat{\varrho}(t) = -\frac{\Gamma}{2}\left[\hat{\varrho} - \frac{1}{3}(\hat{\sigma}_x\hat{\varrho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\varrho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\varrho}\hat{\sigma}_z)\right]. \tag{21}$$

B. Telegraphic non-Markov noise

Now, let us assume that the noise introduced to the system is a random telegraphic noise [12], so that the disturbance of the qubit induced by the environment is discrete, and jumps between two values $\pm a$. For concreteness we consider the following probability distribution:

$$p(\mathbf{r}) = \frac{1}{2}[\delta(x-a)+\delta(x+a)]\delta(y)\delta(z). \tag{22}$$

In this case the Kraus operators (15) equal

$$\hat{K}_0 = \hat{\mathbb{1}}\sqrt{\cos^2 at},$$

$$\hat{K}_1 = \hat{\sigma}_x\sqrt{\sin^2 at},$$

$$\hat{K}_2 = \hat{K}_3 = 0 \tag{23}$$

and the disturbed qubit at instant t is in the state

$$\hat{\varrho}(t) = \cos^2 at\hat{\varrho}(0) + \sin^2 at\hat{\sigma}_x\hat{\varrho}(0)\hat{\sigma}_x. \tag{24}$$

The periodic result is very straightforward, however it reveals something interesting. Although our model is quite simple, it leads to nontrivial dynamics, which becomes apparent when we analyze the evolution of the density operator $d\hat{\varrho}/dt$. One can easily find that the time evolution is given by a nonlocal in time Lindblad equation:

$$\frac{d\hat{\varrho}(t)}{dt} = -\frac{a^2}{2}\int_0^t ds[\hat{\varrho}(s) - \hat{\sigma}_x\hat{\varrho}(s)\hat{\sigma}_x]. \tag{25}$$

This shows that the time dynamics of $\hat{\varrho}(t)$ at the given instant t depends not only on the state at this instant, but also on the state at all earlier times. This behavior can be seen already from the form of the Kraus operator, which for the infinitesimal time evolves as $\hat{K}_1(\Delta t)\simeq\Delta t\hat{\sigma}_x$ and such an evo-

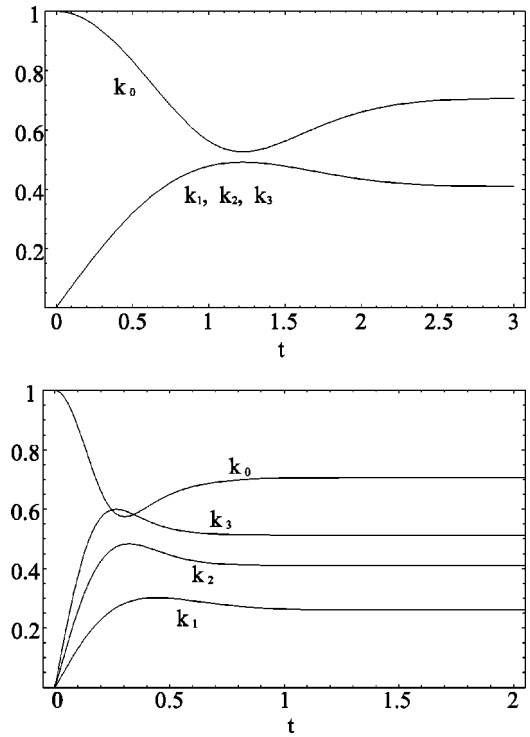

FIG. 1. Kraus operator's coefficients k_i (defined via the relation $\hat{K}_i = k_i\hat{\sigma}_i$) evolving in dimensionless time for a Gaussian probability distribution $p(\mathbf{r}) = (1/\sqrt{\pi^3 d_x^2 d_y^2 d_z^2})\exp(-x^2/d_x^2 - y^2/d_y^2 - z^2/d_z^2)$, where on the upper figure $d_x = d_y = d_z = 1$ and on the lower $d_x = 1$, $d_y = 2$, and $d_z = 3$.

lution characterizes non-Markov processes with zero bandwidth [9].

From this example we conclude that our model in general does not obey the "no-memory" approximation and the evolution of the state is non-Markovian. One could also think of studying multidimensional telegraphic noise, however in this case the analysis becomes much more complicated and there is no simple, linear integral kernel as the one in Eq. (25).

C. Gaussian noise

Although in general the expressions (15) are not analytically integrable, one can find explicitly the Kraus operators in the asymptotic steady-state limit $t \to \infty$. In this limit the square of rapidly oscillating trigonometric functions appearing in the integrals (15) can be approximated by their average value $\frac{1}{2}$. In the simplest case of an arbitrary, spherically symmetric probability distribution $p(r)$ the coefficients k_i of the Kraus operators (defined via the relation $\hat{K}_i = k_i\hat{\sigma}_i$) equal $k_0 = 1/\sqrt{2}$, $k_1 = k_2 = k_3 = 1/\sqrt{6}$ and consequently the output quantum state reads as in Eq. (20).

This result reproduces the steady-state Markov limit justifying its validity. However it is valid only when the probability distribution $p(\mathbf{r})$ is spherically symmetric, i.e., the three channels are unbiased. In general the input state evolves to a different limit.

In Fig. 1 we have shown the numerically calculated evolution of the Kraus operator's coefficients for a Gaussian probability distribution

$$p(\mathbf{r}) = \frac{1}{\sqrt{\pi^3 d_x^2 d_y^2 d_z^2}}\exp\left(-\frac{x^2}{d_x^2} - \frac{y^2}{d_y^2} - \frac{z^2}{d_z^2}\right) \tag{26}$$

for the following two cases: when the probability $p(r)$ is spherically symmetric with $d_x = d_y = d_z = 1$ (upper plot) and for the asymmetric distribution with $d_x = 1$, $d_y = 2$, $d_z = 3$ (lower plot). It is seen that after some characteristic time, the state becomes stationary, however the limit depends on the characteristics of the probability distribution $p(\mathbf{r})$. The example discussed above provides an illustration of unbiased and biased Gaussian depolarization channels.

V. FIDELITY FOR MIXED INPUT STATES

In order to judge the quality of a communication channel and the role of the introduced noise one needs a tool to investigate the state disturbance during the transmission. To quantify the influence of the external noise onto the transmitted quantum state we use an entanglement fidelity measure defined as the following overlap between the input and output density matrix [4]:

$$\mathcal{F}(\hat{\varrho}_{\text{in}}, \hat{\varrho}_{\text{out}}) = \left(\text{Tr}\left\{\sqrt{\hat{\varrho}_{\text{in}}^{1/2}\hat{\varrho}_{\text{out}}\hat{\varrho}_{\text{in}}^{1/2}}\right\}\right)^2. \tag{27}$$

This fidelity is in general very difficult or impossible to calculate. For an arbitrary input state of a single qubit given by Eq. (8), and with an arbitrary spherically symmetric probability distribution $p(r)$ characterizing the external noise this fidelity can be calculated exactly and is equal to

$$\mathcal{F}(\hat{\varrho}_{\text{in}}, \hat{\varrho}_{\text{out}}) = \frac{1}{2}\left(\xi + \sqrt{\chi(1-a^2)}\right), \tag{28}$$

where $\xi = 1 + a_x^2\Lambda_x + a_y^2\Lambda_y + a_z^2\Lambda_z$ and $\chi = 1 - a_x^2\Lambda_x^2 - a_y^2\Lambda_y^2 - a_z^2\Lambda_z^2$. For pure states ($a = 1$) the formula simplifies to $\mathcal{F} = \xi/2$. On the other hand it is not very surprising that the maximally mixed state ($a = 0$) remains unchanged under the influence of the noise, while the communication fidelity decreases with increasing purity of the input state.

Using the same approach one may study also the evolution of multidimensional systems. Of course the general expressions become very complicated, even when we consider a two-qubit Hilbert space, however it is possible to find some compact solutions for special cases of pure states.

Consider an arbitrary two-qubit initial pure state:

$$|\Psi_{\text{in}}\rangle = a|\updownarrow\updownarrow\rangle + b|\updownarrow\leftrightarrow\rangle + c|\leftrightarrow\updownarrow\rangle + d|\leftrightarrow\leftrightarrow\rangle. \tag{29}$$

Using the same approach as above, one can calculate that for the independent disturbance of each mode with the same type of noise characterized by the spherically symmetric probability distribution $p(r)$ the fidelity of the transformation is

$$\begin{aligned}\mathcal{F}(|\Psi\rangle_{\text{in}}, \hat{\varrho}_{\text{out}}) &= \left(\text{Tr}\left\{\sqrt{|\Psi_{\text{in}}\rangle\langle\Psi_{\text{in}}|\hat{\varrho}_{\text{out}}|\Psi_{\text{in}}\rangle\langle\Psi_{\text{in}}|}\right\}\right)^2 \\ &= \langle\Psi_{\text{in}}|\hat{\varrho}_{\text{out}}|\Psi_{\text{in}}\rangle \\ &= \left(\frac{1+\Lambda}{2}\right)^2 - 4\Lambda\frac{1-\Lambda}{2}|bc - ad|^2.\end{aligned} \tag{30}$$

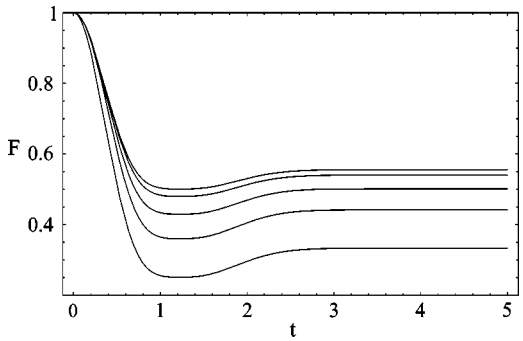

FIG. 2. Fidelity measure as a function of dimensionless time t for the input state $\hat{\varrho}_{\rm in}=\frac{1}{2}(|\updownarrow\updownarrow\rangle\langle\updownarrow\updownarrow|+|\leftrightarrow\leftrightarrow\rangle\langle\leftrightarrow\leftrightarrow|)+(m/2)(|\updownarrow\updownarrow\rangle\times\langle\leftrightarrow\leftrightarrow|+|\leftrightarrow\leftrightarrow\rangle\langle\updownarrow\updownarrow|)$ for several values of the parameter m. From bottom to top, respectively: $m=1$, $m=0.9$, $m=0.7$, $m=0.4$, and $m=0$.

What is interesting in the above result is that the fidelity is the highest for separable states (for example for $a=b=0$ and any c,d) and it drops down when the input state becomes more entangled.

Another compact result can be found for the following two-mode mixed input state:

$$\hat{\varrho}_{\rm in}=\frac{1}{2}\Big(|\updownarrow\updownarrow\rangle\langle\updownarrow\updownarrow|+|\leftrightarrow\leftrightarrow\rangle\langle\leftrightarrow\leftrightarrow|\Big)+\frac{m}{2}(|\updownarrow\updownarrow\rangle\langle\leftrightarrow\leftrightarrow|+|\leftrightarrow\leftrightarrow\rangle\langle\updownarrow\updownarrow|). \tag{31}$$

With a similar analysis one obtains the fidelity measure given by

$$F(\hat{\varrho}_{\rm in},\hat{\varrho}_{\rm out})=\frac{1}{4}\Big[1+\Lambda_z^2+m^2(\Lambda_x^2+\Lambda_y^2)+\sqrt{1-m^2}\times\sqrt{(1+\Lambda_z^2)^2-m^2(\Lambda_x^2+\Lambda_y^2)^2}\Big]. \tag{32}$$

In Fig. 2 we have plotted the dynamics of fidelity for several parameters m and the unbiased Gaussian probability distribution $p(r)=\pi^{-3/2}e^{-r^2}$. We find a not very surprising result, that the transformation fidelity is a decreasing function of the purity of the input state. For $m=0$ (which of course does not yet correspond to the maximally mixed state) the fidelity is the highest, while for the pure state ($m=1$) the fidelity is the lowest.

VI. CONCLUSIONS

Uncontrolled interaction between the environment and the transmitted quantum state can essentially affect the state and in consequence lower the communication capacity of an information channel. Several ideas has been put forth to overcome this problem. One of the most promising is the use of so-called decoherence-free subspaces [13,14]. This idea can be applied when the noise present in the system is correlated between consecutive uses of the communication channel [10]; however this is not always possible and therefore one needs a careful study of the properties of various types of noise and their influence on the quantum state.

In this paper we have introduced a dynamical model of interaction between the quantum state and its environment and shown that although based on simple assumptions, it leads to nontrivial solutions. Using the model we have analyzed properties of zero-bandwidth noise with Lorentzian, telegraphic, and Gaussian distributions and have shown that only the first of them obeys the Markov property, while the others exhibit memory effects and are non-Markovian. Our approach allowed us to solve a simplified version of a general problem when the noise is an arbitrary time-dependent stochastic process, whose solution is not known. We have calculated transformation fidelities for a collection of input states and analyzed their dynamics according to their entanglement or purity.

ACKNOWLEDGMENTS

This work was partially supported by KBN Grant No. 2PO3B 02123. A.D. thanks the Foundation for Polish Science for support.

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