Family of concurrence monotones and its applications

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We extend the definition of concurrence into a family of entanglement monotones, which we call concurrence monotones. We discuss their properties and advantages as computational manageable measures of entanglement, and show that for pure bipartite states all measures of entanglement can be written as functions of the concurrence monotones. We then show that the concurrence monotones provide bounds on quantum information tasks. As an example, we discuss their applications to remote entanglement distributions (RED) such as entanglement swapping and remote preparation of bipartite entangled states (RPBES). We prove a powerful theorem which states what kind of (possibly mixed) bipartite states or distributions of bipartite states cannot be remotely prepared. The theorem establishes an upper bound on the amount of *G*-concurrence (one member in the concurrence family) that can be created between two single-qudit nodes of quantum networks by means of tripartite RED. For pure bipartite states the bound on the *G*-concurrence can always be saturated by RPBES.

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I. INTRODUCTION

Entanglement is one of the main ingredients of nonintuitive quantum phenomena. Besides being of interest from a fundamental point of view, entanglement has been identified as a nonlocal resource for quantum information processing [1]. In particular, shared bipartite entanglement is a crucial resource for many quantum information tasks such as teleportation [2], quantum cryptography [3], entanglement swapping [4], and remote state preparation (RSP) [5–8] that are employed in quantum information protocols.

One of the remarkable discoveries on bipartite entanglement is that for pure states, there is a unique and single measure of entanglement, called entropy of entanglement [9], that quantifies, *asymptotically*, the nonlocal resources of a large number of copies of a pure bipartite state. However, the generalizations of the entropy of entanglement to mixed states yields, even asymptotically, more than one measure of entanglement, such as entanglement of formation and distillation [10]. Despite the enormous efforts that have been made in the past years, mixed entanglement lacks a complete quantification [11].

For a finite number of shared pure states, the entropy of entanglement is not sufficient, and more measures of entanglement are required to quantify completely the nonlocal resources. These are called *entanglement monotones* [12] since they behave monotonically under local transformations of the system. The family of entanglement monotones E_k ($k=0,1,2,\ldots,d-1$) introduced in [13] were first defined over the set of pure states as

$$E_k(|\psi\rangle) = \sum_{i=k}^{d-1} \lambda_i, \qquad (1)$$

where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ are the Schmidt numbers of the $(d\times d)$-dimensional bipartite state $|\psi\rangle$, and then extended to mixed states by means of the convex roof extension. For a pure state $|\psi\rangle$ these measures of entanglement quantify *completely* the nonlocal resource since all the Schmidt coefficients of $|\psi\rangle$ are determined by them. The entanglement monotones defined in Eq. (1) play a central role in transformations of pure states by local operations and classical communications (LOCC) [13–15]. Moreover, each member of the family may quantify the possibility to perform a particular task in quantum information processing (for example, $E_2=1-\lambda_0$ quantifies the possibility to perform faithful teleportation with partially entangled states [16]).

Nevertheless, the family of entanglement monotones $E_k(\rho)$ is not enough to quantify completely the entanglement of a bipartite mixed state ρ. Furthermore, it will be argued here, that if ρ is a $(d\times d)$-dimensional mixed state with $d > 4$, in general, it is impossible to find analytical expression (i.e., an explicit formula like in [17,18]) for $E_k(\rho)$ (as well as for the entanglement of formation and other measures of entanglement). Thus, we are motivated to look for other sets of monotones which are more computationally manageable.

Such a computationally manageable measure of entanglement is the *concurrence*. The concurrence as a measure of entanglement was first introduced in [17,18] for an entangled pair of qubits and later on generalized to higher dimensions [19,20] (there are other generalizations of concurrence which we will not discuss here [21]). Already in [17,18] the importance of the concurrence monotone was recognized and the entanglement of formation of a mixed entangled pair of qubits was calculated explicitly in terms of the concurrence. In higher dimensions there is not yet an explicit formula for the generalized concurrence [19], but lower bounds have been found [20]. Recently, it has been shown [22] that the concurrence plays also a major role in remote entanglement distributions (RED) protocols such as entanglement swapping (ES) and remote preparation of bipartite entangled states (RPBES).

In this paper we introduce a family of entanglement

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monotones which we call *concurrence monotones*. We discuss its properties and show that for pure states *all* measures of entanglement can be written as functions of the concurrence monotones. We show that these concurrence monotones can serve as a powerful tool to rule out the possibility of certain tasks in quantum information processing. In particular, we find an upper bound on the entanglement that can be produced by tripartite RED protocols and show that the protocol given in [22] for RPBES saturates the bound. The measure of entanglement is taken to be one of the members in the concurrence family, which we give the name G-concurrence, since for pure states the G-concurrence is the *geometric mean* of the Schmidt numbers. In addition, we provide an operational interpretation of the G-concurrence as a type of entanglement capacity.

This paper is organized as follows. In Sec. II we define the family of concurrence monotones and then discuss its importance and advantages. In Sec. III we discuss its applications to RED protocols and in Sec. IV we summarize our results and conclusions.

II. DEFINITION OF CONCURRENCE MONOTONES

In the following, we will use the definition of concurrence as given in [17,18] for the (2×2)-dimensional case, and its generalization to higher dimensions as given in [19] (see also [20]). The concurrence of a pure bipartite normalized state $|\psi\rangle$ is defined as

$$C(|\psi\rangle) \equiv \sqrt{\frac{d}{d-1}(1-\mathrm{Tr}\hat{\rho}_r^2)}, \tag{2}$$

where the reduced density matrix $\hat{\rho}_r$ is obtained by tracing over one subsystem. In the definition above we added the factor $\sqrt{d/(d-1)}$ so that $0\leqslant C(|\psi\rangle)\leqslant 1$. For $d=2$ Eq. (2) also coincides with the definition given in [17,18] by means of the "spin flip" transformation. The concurrence of a mixed state, $\hat{\rho}$, is then defined as the average concurrence of the pure states of the decomposition, minimized over all decompositions of $\hat{\rho}$ (the convex roof):

$$C(\hat{\rho}) = \min \sum_i p_i C(|\psi_i\rangle) \quad \left(\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|\right). \tag{3}$$

In the following definition of the family of concurrence monotones, the concurrence defined in Eqs. (2) and (3) is denoted by C_2 since it is the second member of the family.

Definition 1. (a) Consider a $(d\times d)$-dimensional bipartite pure state $|\psi\rangle$ with Schmidt numbers $\lambda\equiv(\lambda_0,\lambda_1,\ldots,\lambda_{d-1})$. The d concurrence monotones, $C_k(|\psi\rangle)$ $(k=1,2,\ldots,d)$, of the state $|\psi\rangle$ are defined as follows (see also [23,24] for similar definitions):

$$C_k(|\psi\rangle) \equiv \left(\frac{S_k(\lambda_0,\lambda_1\ldots,\lambda_{d-1})}{S_k(1/d,1/d,\ldots,1/d)}\right)^{1/k}, \tag{4}$$

where $S_k(\lambda)$ is the kth elementary symmetric function of $\lambda_0,\lambda_1,\ldots,\lambda_{d-1}$. That is,

$$S_1(\lambda) = \sum_i \lambda_i, \quad S_2(\lambda) = \sum_{i<j} \lambda_i\lambda_j,$$

$$S_3(\lambda) = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k, \ldots, S_d(\lambda) = \prod_{i=0}^{d-1} \lambda_i. \tag{5}$$

(b) Consider a $(d\times d)$-dimensional bipartite mixed state ρ. The d concurrence monotones, $C_k(\rho)$, of the state ρ are then defined as the average C_k of the pure states of the decomposition, minimized over all decompositions of ρ (the convex roof):

$$C_k(\rho) = \min \sum_i p_i C_k(|\psi_i\rangle) \quad \left(\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|\right). \tag{6}$$

The functions $S_k(\lambda)$ and $[S_k(\lambda)]^{1/k}$ are Schur-concave (see pp. 78,79 in [25]). Moreover,

$$S_k(\lambda) \leqslant S_k(1/d,1/d,\ldots,1/d) = \frac{1}{d^k}\binom{d}{k}, \tag{7}$$

since the vector $(1/d,1/d,\ldots,1/d)$ is majorized by *all* vectors $\lambda=(\lambda_0,\ldots,\lambda_{d-1})$ with non-negative components that sum to 1. Thus, $0\leqslant C_k(|\psi\rangle)\leqslant 1$ and $C_k(|\psi\rangle)=1$ only when all the Schmidt numbers of $|\psi\rangle$ equal to $1/d$ (i.e., $|\psi\rangle$ is a maximally entangled state).

Equation (4) together with the convex roof extension of C_k to mixed states [see Eq. (6)] defines an entanglement monotone for each k. To see that, first note that

$$C_k(|\psi\rangle) = f_k(\mathrm{Tr}_B|\psi\rangle\langle\psi|), \tag{8}$$

where the trace is taken over one subsystem (say Bob's system) and $f_k(\sigma)\equiv[S_k(\lambda(\sigma))/S_k(1/d,\ldots,1/d)]^{1/k}$ [$\lambda(\sigma)$ is the vector of eigenvalues of the density matrix σ]. According to theorem 2 in [12] C_k is an entanglement monotone if $f_k(\sigma)$ is a unitarily invariant, concave function of σ. The concavity of $f_k(\sigma)$ follows from two facts. First (see p. 79 in [25]), for any two vectors x and y with $x_i,y_i\geqslant 0$ $(i=0,1,\ldots,d-1)$

$$[S_k(x+y)]^{1/k} \geqslant [S_k(x)]^{1/k} + [S_k(y)]^{1/k}. \tag{9}$$

Second, for two Hermitian matrices A and B, $\lambda(A+B)\prec\lambda(A)+\lambda(B)$ (see p. 245 in [25]). Thus, given two density matrices σ_1 and σ_2 we have $(0\leqslant t\leqslant 1)$

$$\begin{aligned} f_k[t\sigma_1+(1-t)\sigma_2] &= \left[\frac{S_k[\lambda(t\sigma_1+(1-t)\sigma_2)]}{S_k(1/d,\ldots,1/d)}\right]^{1/k} \\ &\geqslant \left[\frac{S_k[\lambda(t\sigma_1)+\lambda((1-t)\sigma_2)]}{S_k(1/d,\ldots,1/d)}\right]^{1/k} \\ &\geqslant \left[\frac{S_k[\lambda(t\sigma_1)]}{S_k(1/d,\ldots,1/d)}\right]^{1/k} \\ &\quad + \left[\frac{S_k[\lambda((1-t)\sigma_2)]}{S_k(1/d,\ldots,1/d)}\right]^{1/k} \\ &= tf_k(\sigma_1)+(1-t)f_k(\sigma_2). \end{aligned} \tag{10}$$

Thus Eqs. (4) and (6) define entanglement monotones.

Advantages of concurrence monotones

There are several advantages and applications for these particular measures of entanglement. First, the family of concurrence monotones as defined in Eqs. (4) and (6) is *complete* in the sense that all the Schmidt coefficients of a given pure state can be determined by the d concurrence monotones. To see that, let us define the characteristic polynomial $f_\lambda(x)=(x-\lambda_0)(x-\lambda_2)\cdots(x-\lambda_{d-1})$ whose singular values are the Schmidt numbers. It is easy to see that $f_\lambda(x)$ can be written as

$$f_\lambda(x)=\sum_{k=0}^{d}\frac{(-1)^k}{d^k}\binom{d}{k}x^{d-k}[C_k(\lambda)]^k, \tag{11}$$

where $C_{k=0}(\lambda)\equiv 1$ and $C_{k=1}(\lambda)\equiv\Sigma_i\lambda_i=1$. Hence, the singular values of $f_\lambda(x)$ (i.e., the Schmidt numbers) are determined completely by the concurrence monotones C_k.

Furthermore, consider a pure $(d\times d)$-dimensional state

$$|\psi\rangle=\sum_{ij}a_{ij}|i\rangle_A|j\rangle_B, \tag{12}$$

where $|i\rangle_A$ and $|j\rangle_B$ are some d-dimensional bases in Alice and Bob systems, respectively. The Schmidt numbers are the nonzero eigenvalues of the matrix $A^\dagger A$ (or $AA^\dagger$), where the matrix elements of A are a_{ij}. Thus, in general, for $d>4$, according to Abel's impossibility theorem (also Galois) there is no analytical expression for the Schmidt numbers in terms of a_{ij}. The advantage of our family of concurrence monotones is that one can always express analytically $C_k(|\psi\rangle)$ in terms of a_{ij}:

$$C_k(|\psi\rangle)=d\left[\frac{\mathrm{Tr}B^{(k)}}{\binom{d}{k}}\right]^{1/k}, \tag{13}$$

where $B^{(k)}$ is the kth compound of the matrix $A^\dagger A$ (see p. 502 in [25] for the definition of compound matrices). Such an explicit formula (in terms of a_{ij}) is not available for most of the measures of entanglement discussed in literature (including the entropy of entanglement, α-entropy or Renyi entropy, and the family of entanglement monotones given in [13]).

As an example, consider the entropy of entanglement $E(|\psi\rangle)=-\mathrm{Tr}\rho_\mathrm{r}\log\rho_\mathrm{r}$, where $\rho_\mathrm{r}\equiv\mathrm{Tr}_B|\psi\rangle\langle\psi|$ is the reduced density matrix. If $|\psi\rangle$ is given in terms of a_{ij} as above, then in order to calculate the entropy of entanglement, one must be able to write ρ_r in its diagonal form. However, for $d>4$, in general, it is impossible to solve the equation $f_\lambda(x)=0$ analytically [$f_\lambda(x)$ is defined in Eq. (11)].

For $d\leqslant 4$ the entropy of entanglement can be expressed in terms of the concurrence monotones. For $d=2$, the solution to the quadratic equation $f_\lambda(x)=0$ is simple and the entropy of entanglement is given by

$$E(|\psi\rangle)=h\left(\frac{1+\sqrt{1-[C_2(|\psi\rangle)]^2}}{2}\right), \tag{14}$$

where $h(x)=-x\log x-(1-x)\log(1-x)$. This formula holds for mixed states where the concurrence for mixed states is defined in Eq. (6) and the LHS is replaced by the entanglement of formation [18].

For $d=3$, the solutions to the cubic equation $f_\lambda(x)=0$ are more complicated (although possible) and the entropy of entanglement is given by

$$E(|\psi\rangle)=H\left(\frac{1}{3}+\frac{2}{3}\sqrt{1-[C_2(|\psi\rangle)]^2}\cos(\theta/3),\right.$$
$$\left.\frac{1}{3}+\frac{2}{3}\sqrt{1-[C_2(|\psi\rangle)]^2}\cos((\theta+2\pi)/3)\right),$$

$$\cos\theta\equiv\frac{1-\frac{3}{2}[C_2(|\psi\rangle)]^2+\frac{1}{2}[C_3(|\psi\rangle)]^3}{(1-[C_2(|\psi\rangle)]^2)^{3/2}}, \tag{15}$$

where $H(x,y)=-x\log x-y\log y-(1-x-y)\log(1-x-y)$. Similarly, for $k=4$, it is possible to find the solutions to the quartic equation $f_\lambda(x)=0$ and express the entropy of entanglement in terms of the concurrence monotones.

The analytical expression for $C_k(|\psi\rangle)$ in terms of the reduced density matrix $\rho_r\equiv\mathrm{Tr}_B|\psi\rangle\langle\psi|$ is given by

$$C_k(|\psi\rangle)=d\left[\frac{1}{\binom{d}{k}}\sum_{\{N_m\}}(-1)^{k-\Sigma_{m=1}^{k}N_m}\prod_{m=1}^{k}\frac{1}{N_m!}\left(\frac{\mathrm{Tr}\rho_r^m}{m}\right)^{N_m}\right]^{1/k}, \tag{16}$$

where the sum is taken over all the non-negative integers $N_1,N_2,\ldots,N_k$ that satisfy the constraint $N_1+2N_2+\cdots+kN_k=k$. This expression (see also [26,27]) follows directly from multinomial formulas given in [28]. As an example, for $k=2,3,4$, Eq. (16) gives

$$C_2(|\psi\rangle)=\sqrt{\frac{d}{d-1}(1-\mathrm{Tr}\rho_r^2)},$$

$$C_3(|\psi\rangle)=\left[\frac{d^2}{(d-1)(d-2)}(1-3\mathrm{Tr}\rho_r^2+2\mathrm{Tr}\rho_r^3)\right]^{1/3},$$

$$C_4(|\psi\rangle)=\left[\frac{d^3}{(d-1)(d-2)(d-3)}[1-6\mathrm{Tr}\rho_r^2+8\mathrm{Tr}\rho_r^3-6\mathrm{Tr}\rho_r^4+3(\mathrm{Tr}\rho_r^2)^2]\right]^{1/4}. \tag{17}$$

We can see that for $k=2$ Eq. (16) is reduced to the expression for the concurrence given in [19]. Note also that $C_k(|\psi\rangle)=0$ if k is greater than the Schmidt number of $|\psi\rangle$.

The G-concurrence monotone

The last member of the family $C_{k=d}$ is of a particular importance and we denote it by G_d since it is the *geometric mean* of the Schmidt numbers

$$G_d(|\psi\rangle)\equiv C_{k=d}(|\psi\rangle)=d(\lambda_0\lambda_1\cdots\lambda_{d-1})^{1/d}. \tag{18}$$

Note that for $d=2$ the G-concurrence coincides with the original definition of concurrence given by Hill and Wootters [17].

The G-concurrence has several interesting features.

A computational manageable measure of entanglement: for the $d\times d$ bipartite pure state $|\psi\rangle$ in Eq. (12), the G-concurrence is given simply by [29] [cf. Eq. (13)]

$$G_d(|\psi\rangle) = d[\mathrm{Det}(A^\dagger A)]^{1/d}, \tag{19}$$

where the matrix elements of A are a_{ij}.

Multiplicativity: first, given a $d_1\times d_1$ $(d_2\times d_2)$ bipartite entangled state, $|\psi_1\rangle$ $(|\psi_2\rangle)$, we have

$$G_{d_1d_2}(|\psi_1\rangle\otimes|\psi_2\rangle) = G_{d_1}(|\psi_1\rangle)G_{d_2}(|\psi_2\rangle). \tag{20}$$

Note that although in both sides of the equation above we take the geometric means of the Schmidt numbers of the relevant states, $G_{d_1d_2}=C_{k=d_1d_2}$ is not the *same* measure of entanglement as $G_{d_1}=C_{k=d_1}$ [30]. Second, given a bipartite state $|\psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$, a complex number c and operators (complex matrices) $\hat{A}\in\mathcal{H}_A$ and $\hat{B}\in\mathcal{H}_B$ we have [31]

$$G_d(c|\psi\rangle) = |c|^2 G_d(|\psi\rangle), \tag{21}$$

$$G_d(\hat{A}\otimes\hat{B}|\psi\rangle) = |\mathrm{Det}(\hat{A})|^{2/d}|\mathrm{Det}(\hat{B})|^{2/d}G_d(|\psi\rangle), \tag{22}$$

where we have used Eq. (19).

A lower bound: the G-concurrence monotone provides a lower bound for all the other concurrence monotones. First, for *pure* bipartite states we have the inequalities (cf. p.224 in [25])

$$[C_2(|\psi\rangle)]^2 \geqslant [C_3(|\psi\rangle)]^3 \geqslant \cdots \geqslant [C_d(|\psi\rangle)]^d \equiv [G_d(|\psi\rangle)]^d. \tag{23}$$

Second, given a *mixed* bipartite state ρ we have [32]

$$G_d(\rho) \leqslant C_k(\rho)\ \forall\ \ k=1,2,\ldots,d. \tag{24}$$

Note that the relations in Eqs. (23) and (24) may be useful in finding lower bounds on measures of entanglement such as entanglement of formation. In addition, as we will see in the following section, the G-concurrence monotone plays a central role in tripartite RED protocols.

III. REMOTE ENTANGLEMENT DISTRIBUTION

As mentioned in the Introduction, shared bipartite entanglement is a crucial shared resource for many quantum information tasks such as teleportation [2], entanglement swapping [4], and remote state preparation (RSP) [5–8] that are employed in quantum information protocols.

Remote preparation of bipartite entangled states [22] (RPBES) is another important quantum information task in which a quantum network (QNet) have a single supplier (named "Sapna") who shares entangled states with nodes via quantum channels, then performs LOCC to produce pairwise entangled states between any two nodes, say, Alice and Bob. A crucial feature of RPBES is that Alice and Bob end up sharing a *unique* bipartite entangled state. A more general scheme, in which Alice and Bob end up sharing a *distribution* of entangled states is called remote entanglement distribution [22] (RED).

The scheme for tripartite RED, introduced in [22], commences with a four-way shared state, $\hat{\rho}_{1234}=\hat{\rho}_{12}\otimes\hat{\rho}_{34}$ with $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ bipartite entangled states, and with Sapna (the supplier) holding shares 2 and 3, and Alice and Bob holding shares 1 and 4, respectively. Each share has a corresponding d-dimensional Hilbert space. The three parties Alice, Bob, and Sapna perform LOCC to create a set of outcomes

$$O \equiv \{\hat{\sigma}_{14}^j = \mathrm{Tr}_{23}\hat{\sigma}_{1234}^j, Q_j; j=1,\ldots,s\}, \tag{25}$$

with Q_j the probability that Alice and Bob share the mixed state $\hat{\sigma}_{14}^j$ which is obtained by reducing the four-way shared state $\hat{\sigma}_{1234}^j$ over Sapna's shares. In general RED, the states $\{\hat{\sigma}_{14}^j\}$ may be inequivalent under LOCC whereas in RBESP the states $\hat{\sigma}_{14}^j$ shared by Alice and Bob must be equivalent under LOCC, so Alice and Bob can always transform $\hat{\sigma}_{14}^j$ into a unique entangled state (i.e., independent on j) via LOCC.

In this section, we address the issue of which distributions of states, O, can or cannot be created via LOCC by Alice, Bob, and Sapna. The G-concurrence monotone plays a major role in the following theorem that establishes which distributions of states cannot be produced by RED.

Theorem 1. If Alice, Bob and Sapna perform LOCC on the initial 4-qudit state $\hat{\rho}_{12}\otimes\hat{\rho}_{34}$ with O [in Eq. (25)] the resultant distribution of states shared between Alice and Bob, then

$$G_{14} \equiv \sum_{j=1}^{s} Q_j G_d(\hat{\sigma}_{14}^j) \leqslant G_{12}G_{34}, \tag{26}$$

with $G_{12}\equiv G_d(\hat{\rho}_{12})$ and $G_{34}\equiv G_d(\hat{\rho}_{34})$.

(In the next subsection we will show that the equality in the above equation can always be achieved by RBESP if $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ are pure.)

Proof. Let us write $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ in their *optimal* decompositions

$$\hat{\rho}_{12} = \sum_{l=0}^{d^2-1} p_l|\psi^{(l)}\rangle_{12}\langle\psi^{(l)}|, \quad \hat{\rho}_{34} = \sum_{l=0}^{d^2-1} q_l|\chi^{(l)}\rangle_{34}\langle\chi^{(l)}|. \tag{27}$$

We can always choose optimal decompositions with no more then d^2 elements [33]. The states $|\psi^{(l)}\rangle_{12}$ and $|\chi^{(l)}\rangle_{34}$ are given in their Schmidt decomposition:

$$|\psi^{(l)}\rangle_{12} = \sum_{k=0}^{d-1}\sqrt{\lambda_k^{(l)}}|k^{(l)}k^{(l)}\rangle_{12},$$

$$|\chi^{(l)}\rangle_{34} = \sum_{k=0}^{d-1}\sqrt{\eta_k^{(l)}}|k^{(l)}k^{(l)}\rangle_{34}, \tag{28}$$

with $\lambda_k^{(l)}$ and $\eta_k^{(l)}$ the Schmidt coefficients of $|\psi^{(l)}\rangle_{12}$ and $|\chi^{(l)}\rangle_{34}$, respectively. The index l in the states $\{|k^{(l)}\rangle_i\}$ represents d^2 different bases for each system $i=1,2,3,4$. Note that in this notation

$$G_{12} = d\sum_{l=0}^{d^2-1} p_l(\lambda_0^{(l)}\lambda_1^{(l)}\cdots\lambda_{d-1}^{(l)})^{1/d},$$

$$G_{34} = d\sum_{l=0}^{d^2-1} q_l(\eta_0^{(l)}\eta_1^{(l)}\cdots\eta_{d-1}^{(l)})^{1/d}. \qquad (29)$$

Since the entanglement between Alice and Bob remains zero unless Sapna perform a measurement, we assume that the first measurement is performed by Sapna and is described by the Kraus operators $\hat{M}^{(j)}$ and their components

$$M_{mm',kk'}^{(j,ll')} \equiv {}_{23}\langle m^{(l)}m'^{(l')}|\hat{M}^{(j)}|k^{(l)}k'^{(l')}\rangle_{23}, \qquad (30)$$

with k, k', m, $m'=0,1$ and l, $l'=1,2,3,4$.

The probability to obtain an outcome j is thus

$$Q_j \equiv \mathrm{Tr}(\hat{M}^{(j)}\hat{\rho}_{12}\otimes\hat{\rho}_{23}\hat{M}^{(j)\dagger}) = \sum_{l=0}^{d^2-1}\sum_{l'=0}^{d^2-1} p_l q_{l'} N^{(j,ll')}, \quad (31)$$

with $N^{(j,ll')} \equiv \Sigma_{m,m'} r_{mm'}^{(j,ll')}$ and

$$r_{mm'}^{(j,ll')} \equiv \sum_{k,k'} \lambda_k^{(l)}\eta_{k'}^{(l')}|M_{kk',mm'}^{(j,ll')}|^2. \qquad (32)$$

The density matrix shared between Alice, Bob, and Sapna after outcome j occurs is

$$\hat{\sigma}_{1234}^j = \frac{1}{Q_j}\sum_{l,l'} p_l q_{l'} N^{(j,ll')}|\Phi^{(j,ll')}\rangle_{1234}\langle\Phi^{(j,ll')}|, \qquad (33)$$

where

$$|\Phi^{(j,ll')}\rangle_{1234} = \frac{1}{\sqrt{N^{(j,ll')}}}\sum_{k,k'}\sum_{m,m'}\sqrt{\lambda_k^{(l)}\eta_{k'}^{(l')}} \times M_{kk',mm'}^{(j,ll')}|k^{(l)}k'^{(l')}\rangle_{14}|m^{(l)}m'^{(l')}\rangle_{23}. \qquad (34)$$

Tracing over Sapna's subsystems yields

$$\hat{\sigma}_{14}^j = \frac{1}{Q_j}\sum_{l,l'}\sum_{m,m'} p_l q_{l'} r_{mm'}^{(j,ll')}|\phi_{mm'}^{(j,ll')}\rangle_{14}\langle\phi_{mm'}^{(j,ll')}|, \qquad (35)$$

where

$$|\phi_{mm'}^{(j,ll')}\rangle_{14} \equiv \frac{1}{\sqrt{r_{mm'}^{(j,ll')}}}\sum_{k,k'}\sqrt{\lambda_k^{(l)}\eta_{k'}^{(l')}}M_{kk',mm'}^{(j,ll')}|k^{(l)}k'^{(l')}\rangle_{14}. \qquad (36)$$

From the definition of the G-concurrence for mixed states (i.e., the convex roof extension), it follows that $G_d(\hat{\sigma}_{14}^j)$ cannot exceed the average of the G-concurrence over the decomposition in Eq. (35). Thus

$$G_d(\hat{\sigma}_{14}^j) \leqslant \frac{1}{Q_j}\sum_{l,l'}\sum_{m,m'} p_l q_{l'} r_{mm'}^{(j,ll')} G(|\phi_{mm'}^{(j,ll')}\rangle_{14}). \qquad (37)$$

Using Eq. (19) we find

$$G(|\phi_{mm'}^{(j,ll')}\rangle_{14}) = \frac{d(\prod_{k=0}^{d-1}\lambda_k^{(l)}\eta_k^{(l')})^{1/d}|\mathrm{Det}(\mathcal{M}_{mm'}^{(j,ll')})|^{2/d}}{r_{mm'}^{(j,ll')}}, \qquad (38)$$

where the d^2 elements of each matrix $\mathcal{M}_{mm'}^{(j,ll')}$ are $M_{kk',mm'}^{(j,ll')}$. Thus, substituting this result in Eq. (37) yields

$$G_{14} \equiv \sum_{j=1}^{s} Q_j G(\hat{\sigma}_{14}^j) \leqslant d\sum_{l,l'} p_l q_{l'}\left(\prod_{k=0}^{d-1}\lambda_k^{(l)}\eta_k^{(l')}\right)^{1/d} \times \sum_j\sum_{m,m'}|\mathrm{Det}(\mathcal{M}_{mm'}^{(j,ll')})|^{2/d}. \qquad (39)$$

Now, from the geometric-arithmetic inequality we have

$$\sum_{m,m'}|\mathrm{Det}(\mathcal{M}_{mm'}^{(j,ll')})|^{2/d} \leqslant \frac{1}{d}\sum_{m,m'}\mathrm{Tr}(\mathcal{M}_{mm'}^{(j,ll')\dagger}\mathcal{M}_{mm'}^{(j,ll')}) = \frac{1}{d}\mathrm{Tr}(\hat{M}^{(j)\dagger}\hat{M}^{(j)}). \qquad (40)$$

Hence, from Eq. (39) and Eq. (29) we get

$$G_{14} \leqslant \frac{1}{d^2}G_{12}G_{34}\sum_j \mathrm{Tr}(\hat{M}^{(j)\dagger}\hat{M}^{(j)}). \qquad (41)$$

Thus, from the completeness relation, $\Sigma_j\hat{M}^{(j)\dagger}\hat{M}^{(j)}=I$, we obtain Eq. (26).

Consider now the following LOCC: after Sapna's first measurement, she sends the result j to Alice and Bob. Based on this result, Alice then performs a measurement represented by the Kraus operators $\hat{A}_j^{(k)}$ and sends the result k to Bob and Sapna. Based on the results j, k from Sapna and Alice, Bob performs a measurement represented by the Kraus operators $\hat{B}_{jk}^{(n)}$ and send the result n to Sapna. In the last step of this scheme, Sapna performs a second measurement with Kraus operators denoted by $\hat{F}_{jkn}^{(j)}$ and send the result i to Alice and Bob. The final distribution of entangled states shared between Alice and Bob is denoted by $\{N_{jkni},\sigma_{14}^{jkni}\}$, where N_{jkni} is the probability for outcome j, k, n, i and $\hat{\sigma}_{14}^{jkni}=\mathrm{Tr}_{23}\hat{\sigma}_{1234}^{jkni}$ with

$$\hat{\sigma}_{1234}^{jkni} = \frac{1}{N_{jkni}}(\hat{A}_j^{(k)}\otimes\hat{F}_{jkn}^{(i)}\hat{M}^{(j)}\otimes\hat{B}_{jk}^{(n)})[\hat{\rho}_{12}\otimes\hat{\rho}_{34}] \times(\hat{A}_j^{(k)}\otimes\hat{F}_{jkn}^{(i)}\hat{M}^{(j)}\otimes\hat{B}_{jk}^{(n)})^\dagger. \qquad (42)$$

Since the G-concurrence of any bipartite state satisfies Eq. (22), the analog of Eq. (41) for this LOCC protocol is therefore

$$G_{14} \equiv \sum_{j,k,n,i} N_{jkni}G(\hat{\sigma}_{14}^{jkni}) \leqslant \frac{1}{d^2}G_{12}G_{34}\sum_{j,k}|\mathrm{Det}(\hat{A}_j^{(k)})|^{2/d} \times\sum_n|\mathrm{Det}(\hat{B}_{jk}^{(n)})|^{2/d}\sum_i\mathrm{Tr}(\hat{M}^{(j)\dagger}\hat{F}_{jkn}^{(i)\dagger}\hat{F}_{jkn}^{(i)}\hat{M}^{(j)}). \qquad (43)$$

Moreover, from the geometric-arithmetic inequality we have

$$\sum_n |\mathrm{Det}(\hat{B}_{jk}^{(n)})|^{2/d} \leq \frac{1}{d}\sum_n \mathrm{Tr}\hat{B}_{jk}^{(n)\dagger}\hat{B}_{jk}^{(n)} = 1 \quad (44)$$

and a similar relation for $\hat{A}_j^{(k)}$. These results, together with the completeness relation $\Sigma_i \hat{F}_{jkn}^{(i)\dagger}\hat{F}_{jkn}^{(i)}=1$, lead us back to Eq. (41). As we can see, all operations that are performed by Alice, Bob, and Sapna after the first measurement by Sapna cannot increase the bound on C_{14}. ■

Theorem 1 concerns one supplier and two nodes, but in fact applies to one supplier and *any* pair of nodes; thus the result of theorem 1 is applicable to an arbitrarily large QNet with one supplier and many nodes. In fact theorem 1 can be extended to more than one supplier, as stated in the following corollary.

Corollary. Consider an align chain of N mixed bipartite states, $\rho_{0,1}, \rho_{1,2}, \ldots, \rho_{N-1,N}$, where the state $\rho_{k-1,k}$ ($k=1,2,\ldots,N$) is shared between party $k-1$ and party k. If the $N+1$ parties perform LOCC on the initial state $\rho_{0,1}\otimes\rho_{1,2}\otimes\cdots\otimes\rho_{N-1,N}$ with the resultant distribution of states between party 0 and N denoted by $\{P_j, \hat{\sigma}_{0N}^j\}$ (P_j is the probability to have the state $\hat{\sigma}_{0N}^j$), then

$$G_{0N} \equiv \sum_j P_j G_d(\hat{\sigma}_{0N}^j) \leq G_{01}G_{12}\cdots G_{N-1N}, \quad (45)$$

with $G_{k-1\ k} \equiv G_d(\rho_{k-1,k})$ ($k=1,2,\ldots,N$).

Theorem 1 and its corollary suggest an operational interpretation of the G-concurrence as a form of *entanglement capacity*. In the following subsection we show that if both $\hat{\rho}_{12}$ and $\hat{\rho}_{23}$ are $(d\times d)$-dimensional *pure* states, then the equality in Eqs. (26) and (45) can always be achieved.

An optimal protocol for RPBES

In this section we show that by LOCC Sapna can prepare a bipartite pure state between Alice and Bob with *any* value of the concurrence monotone G which is less or equal to $G_{12}G_{34}$. For this purpose, we introduce the protocol for RBESP that has been first introduced in [22]. In this protocol the supplier Sapna shares the initial $(d\times d)$-dimensional pure states $|\psi\rangle_{12}=\Sigma_{k=0}^{d-1}\sqrt{\lambda_k}|kk\rangle_{12}$ and $|\chi\rangle_{34}=\Sigma_{k=0}^{d-1}\sqrt{\eta_k}|kk\rangle_{34}$ (which are expressed in the Schmidt decomposition) with Alice and Bob, respectively.

The steps of the protocol are as follows.

(i) Sapna performs a projective measurement

$$\hat{P}^{(j,j')} = |P^{(j,j')}\rangle_{23}\langle P^{(j,j')}|, \quad j,j' = 0,1,\ldots,d-1, \quad (46)$$

with

$$|P^{(j,j')}\rangle_{23} \equiv \frac{1}{d}\sum_{m,m'=0}^{d-1} e^{i[(2\pi/d^2)(dj+j')(dm+m')+\theta_{mm'}]}|mm'\rangle_{23}, \quad (47)$$

with $\theta_{mm'}\in\mathbb{R}$ chosen freely. Note that the d^2 states $|P^{(j,j')}\rangle_{23}$ are orthonormal, regardless of the choice of $\theta_{mm'}$.

(ii) After the outcomes j, j' have been obtained, the state of the system can be written as $|P^{(j,j')}\rangle_{23}|\phi^{(j,j')}\rangle_{14}$, where

$$|\phi^{(j,j')}\rangle_{14} = \sum_{m=0}^{d-1}\sum_{m'=0}^{d-1}\sqrt{\lambda_m\eta_{m'}} \times e^{-i[(2\pi/d^2)(dj+j')(dm+m')+\theta_{mm'}]}|mm'\rangle_{14}. \quad (48)$$

(iii) Sapna sends the results j and j' to Bob ($2\log_2 d$ bits of information) and the result j' ($\log_2 d$ bits of information) to Alice. Bob then performs the unitary operation

$$\hat{U}_b^{(j,j')}|m'\rangle_4 = \exp\left(i\frac{2\pi}{d^2}(dj+j')m'\right)|m'\rangle_4, \quad (49)$$

and Alice performs the unitary operation

$$\hat{U}_a^{(j')}|m\rangle_1 = \exp\left(i\frac{2\pi}{d}j'm\right)|m\rangle_1. \quad (50)$$

(iv) The final state shared between Alice and Bob is

$$|F\rangle_{14} = \sum_{m=0}^{d-1}\sum_{m'=0}^{d-1}\exp(-i\theta_{mm'})\sqrt{\lambda_m\eta_{m'}}|mm'\rangle_{14} \quad (51)$$

(which is separable for $\theta_{mm'}=0$).

We will show now that by choosing the phases $\theta_{mm'}$ appropriately, Sapna can prepare the state $|F\rangle_{14}$ with *any* value of $G(|F\rangle_{14})$ in the range $[0,G_{12}G_{34}]$. For this purpose, we define the square $(d\times d)$ complex matrix A with elements $a_{mm'}=\sqrt{\lambda_m\eta_{m'}}\exp(-i\theta_{mm'})$. Thus

$$G(|F\rangle_{14}) = d[\mathrm{Det}(A^\dagger A)]^{1/d} = G_{12}G_{34}[\mathrm{Det}(V^\dagger V)]^{1/d}, \quad (52)$$

where $G_{12}=d(\lambda_0\lambda_1\cdots\lambda_{d-1})^{1/d}$ $G_{34}=d(\eta_0\eta_1\cdots\eta_{d-1})^{1/d}$ and the matrix elements of V are $v_{mm'}=\exp(-i\theta_{mm'})/\sqrt{d}$. Note that for the choice $\theta_{mm'}=2\pi mm'/d$ the matrix V is unitary and therefore $G(|F\rangle_{14})=G_{12}G_{34}$. For other choices of $\theta_{mm'}$, Sapna can prepare the final state $|F\rangle_{14}$ with any value of the G-concurrence monotone in the range $[0,G_{12}G_{34}]$.

It is important to emphasize here that the choice $\theta_{mm'}=2\pi mm'/d$ maximizes *only* the G-concurrence. In fact, for other measures of entanglement the values of $\theta_{mm'}$ that maximize the entanglement depend explicitly on the Schmidt numbers λ_m and η_m. For example, the concurrence monotone $C_{k=2}$ of the final state $|F\rangle_{14}$ is

$$C_2(|F\rangle_{14}) = 2\left\{\sum_{k>k'}\sum_{m>m'}\lambda_k\lambda_{k'}\eta_m\eta_{m'} \times|e^{i(\theta_{km}+\theta_{k'm'})}-e^{i(\theta_{km'}+\theta_{k'm})}|^2\right\}^{1/2}. \quad (53)$$

Thus, in this case we see that the values of θ_{km} that maximize $C_2(|F\rangle_{14})$ depend explicitly on the Schmidt coefficients λ_k and η_m.

IV. SUMMARY AND CONCLUSIONS

In summary, we have introduced a family of entanglement monotones that extend the definition of concurrence. We have shown that for a finite number of copies of pure states (i.e., the deterministic case) the family characterizes completely the nonlocal resource. We have also discussed the advantage of the concurrence monotones over other measures of entanglement (such as the entropy of entanglement, the Renyi entropies, etc.) and showed that for a given bipartite state, $|\psi\rangle=\Sigma_{ij}a_{ij}|i\rangle|j\rangle$, the concurrence monotones can always be expressed analytically in terms of the coefficients a_{ij}. We also gave an analytical expression of the concurrence monotones (for pure states) in terms of the reduced density matrix [see Eq. (16)].

We then discussed a particular member of the family which we called the *G*-concurrence. The *G*-concurrence for pure states is defined as the geometric mean of the Schmidt numbers. It has several unique properties that make it extremely useful. In particular, we have proved a powerful theorem that establishes an upper bound on the amount of *G*-concurrence that can be created between two single-qudit nodes of quantum networks by means of RED. The theorem also suggests an operational interpretation of the G-concurrence as a type of entanglement capacity. We have proved that it is always possible to saturate the *G*-concurrence bound in the theorem if both of the entangled states are pure, and also suggested an operational interpretation of the *G*-concurrence as a type of entanglement capacity. An open question is left if it is possible to saturate the bound when the states are mixed.

The concurrence monotones are defined in terms of the symmetric functions of the Schmidt numbers [see Eq. (5)]. These symmetric functions have many interesting mathematical properties which were not introduced here (some of the properties can be found in [25]) and which are related to the field of majorization. Thus we believe that further investigations of these monotones will contribute to our understanding of entanglement.

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[29] Despite the simple expression in Eq. (19) for pure states, the

convex roof for the G-concurrence on mixed states is yet unknown. On the other hand, the multipartite, two level generalizations of concurrence [21] do admit an explicit formula for the convex roof.
[30] For example, if $d_1=d_2=2$ then it is clear from Eq. (17) that $C_{k=d_1=2}$ is a completely different measure than $C_{k=d_1d_2=4}$.
[31] The determinant of an operator, like its trace, is basis independent.
[32] For pure states, Eq. (24) follows from the geometric-arithmetic inequality, and for mixed states from the convex roof extension.
[33] Although the optimal decompositions in Eq. (27) are taken with d^2 elements, it is not necessary for the proof; we could instead write the optimal decompositions with any number of elements.