Entanglement purification for arbitrary unknown ionic states via linear optics

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An entanglement purification scheme for arbitrary unknown (mixed and pure nonmaximally) entangled ionic states is proposed by using linear optical elements. The main advantage of the scheme is that not only two-ion maximally entangled pairs but also four-ion maximally entangled pairs can be extracted from the less entangled pairs. The scheme is within current technology.

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I. INTRODUCTION

Entanglement [1] presented by Schrödinger is a critical manifestation of quantum mechanics. Resulting from its nonlocality property, entanglement has become a more and more important resource in quantum-information processing (QIP). All of the applications [2–5] of entanglement work perfectly only with the pure maximally entangled states. Because quantum entanglement can only be produced locally [6], the entangled objects must be distributed among distant users for quantum communication purpose. Due to the impossibility that one quantum system can be isolated from the environment absolutely, the entanglement of the entangled objects will decrease exponentially with the propagating distance of the objects, and the practically available quantum entangled states are all nonmaximally entangled states or the more general case—mixed states. So, if nothing has been done on the distributed states before used in quantum communication, the long distance quantum communication [7] is impossible. To overcome the dissipation and decoherence, various schemes of entanglement distillation [8,9], entanglement concentration [10,11] and entanglement purification [12–24] have been proposed. Alternatively, a quantum repeater [25] also can be used to overcome this difficulty. The main processes of a quantum repeater are composed of entanglement purification [12] and entanglement swapping [11], and the main task of it is still to realize entanglement purification. So we will mainly discuss the entanglement purification process. Entanglement purification is a method that can extract a small number of entangled pairs with a relatively high degree of entanglement from a large number of less entangled pairs using only local operations and classical communication. In the original entanglement purification scheme [12], controlled-NOT (C-NOT) operations construct the main step of the purification process. But, in experiment, there is no implementation of C-NOT operations that can meet the error rate level, which is needed for the logic gates in long distance quantum communication [25]. So more and more attention is focused on finding the realizable schemes for entanglement purification. Pan *et al.* use the polarization beam splitter (PBS) [13] to replace the C-NOT gate needed in the original scheme [12], and can get the newly obtained polarization-entangled photon pairs with a larger fraction of fidelity. Most of the above entanglement purification schemes are theoretical ones. Recently, significant progress on entanglement purification has been achieved in experiment [8,14]. Kwiat *et al.* proposed an experimental entanglement distillation scheme for pure nonmaximally and mixed polarization-entangled photon states using partial polarizers [8]. Following the theoretical proposal [13], Pan *et al.* successfully realize the entanglement purification of general mixed states of polarization-entangled photon pairs using linear optics elements in experiment [14].

From the previous entanglement purification schemes, we conclude that most of them can only apply to the polarization-entangled photon pairs. There are few schemes for distillation [26,27] and purification of atomic and ionic entangled states in the literature. Although photons are the attractive carriers of information for the implementation of quantum communication, ions are also the preferred carrier for quantum information, because the realization of quantum computer and quantum computation relies on the optimal quantum carriers, which can be integrated. So the purification of ionic entangled states is of practical significance not only in quantum communication but also in quantum computation.

Inspired by Pan's proposal [13] for entanglement purification and Zhou's proposal for nondistortion quantum interrogation (NQI) [28], we will propose, in this paper, an entanglement purification scheme for arbitrary unknown [14,20] mixed entangled ionic states by using beam splitters (BS) and polarization-sensitive single photon detectors (D). For the arbitrary unknown nonmaximally entangled pure states, it also works. Through analysis, we can get near-perfect maximally entangled ionic states from the mixed entangled ionic states, provided we repeat the scheme several times. From the pure nonmaximally entangled states, we can get the perfect maximally entangled ionic states probabilistically. We can decide whether the purification procedure succeeds by operating single photon measurement on each side.

II. ENTANGLEMENT PURIFICATION FOR MIXED STATES

For communication purpose, the two distant users Alice and Bob should share maximally entangled states:

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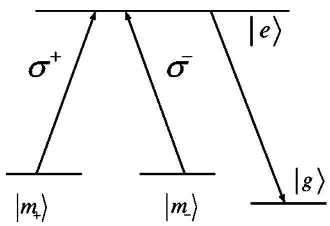

FIG. 1. Level configuration of the ions used in the scheme. The ions, which are in the degenerate states $|m_+\rangle$ and $|m_-\rangle$, can be excited into the unstable excited state $|e\rangle$ by absorbing one σ^+ or σ^- polarized photon, then it can decay to the stable ground state $|g\rangle$ with a scattered photon rapidly.

$$|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|m_+\rangle_1|m_+\rangle_2 + |m_-\rangle_1|m_-\rangle_2), \tag{1a}$$

$$|\Psi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|m_+\rangle_1|m_-\rangle_2 + |m_-\rangle_1|m_+\rangle_2). \tag{1b}$$

These are two Bell states for two ions. One of the two ions is at Alice's side, the other at Bob's. Here, $|m_+\rangle$ and $|m_-\rangle$ are two degenerate metastable states of ions. The ions can be excited from $|m_+\rangle$ or $|m_-\rangle$ to the excited states $|e\rangle$ by absorbing one σ^+ or σ^- circular polarization photon with unit efficiency. The excited state $|e\rangle$ is not a stable one, so the ions in that state will decay rapidly to the stable ground state $|g\rangle$ with a scattered photon $|S\rangle$. This process can be expressed as

$$a^+_\pm|0\rangle|m_\pm\rangle \rightarrow |S\rangle|g\rangle. \tag{2}$$

The level configuration of the ions is depicted in Fig. 1. But, for communication purpose, the two ions must be distributed to different locations. During the transmission process, entanglement will inevitably degrade. So the entangled states after distribution are usually mixed ones. Suppose that the mixed state to be purified is in the form

$$\rho_{AB} = F|\Phi^+\rangle_{AB}\langle\Phi^+| + (1-F)|\Psi^+\rangle_{AB}\langle\Psi^+|. \tag{3}$$

Because a general mixed state can be rotated into the form in Eq. (3), the discussion on the state in Eq. (3) applies to general mixed cases [13]. Further, to complete the purification scheme, we suppose that Alice and Bob have shared an ionic ensemble, each pair of which can be described by the state in Eq. (3). Here, $F=\langle\Phi^+|\rho_{AB}|\Phi^+\rangle$ is the fidelity of the pairs with respect to $|\Phi^+\rangle$.

Next, we will discuss the purification procedure in detail. To complete the purification process, we must carry out operations on two pairs of the ensemble. We denote the four ions of the two pairs as 1,2 and 3,4, and the total state of the two pairs before purification can be regarded as a probabilistic mixture of four pure states: $|\Phi^+\rangle_{12}|\Phi^+\rangle_{34}$ with probability F^2, $|\Phi^+\rangle_{12}|\Psi^+\rangle_{34}$ with probability $F(1-F)$, $|\Psi^+\rangle_{12}|\Phi^+\rangle_{34}$ with probability $(1-F)F$, and $|\Psi^+\rangle_{12}|\Psi^+\rangle_{34}$ with probability $(1-F)^2$.

The main setup, depicted in Fig. 2, are two Mach-Zehnder interferometers (M_A and M_B) located at Alice's and Bob's side, respectively.

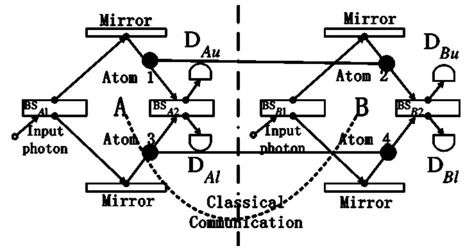

FIG. 2. The setup for purification scheme. Alice places two ions 1,3 which are at her side on the two arms of M_A, ion 1 on upper path and ion 3 on lower one, by using the trapping technology [29], and analogously for the two ions 2,4 at Bob's side. One σ^+ polarized photon at each side will be superimposed on the first two BS (BS_{A1} and BS_{B1}) of M_A and M_B, respectively. After the first BS the photon will take two possible paths (u denotes the upper path and l denotes the lower one). Reflected by two mirrors, the two possible paths will recombine at the second BS (BS_{A2} and BS_{B2}). Because the two ions at one side are initially placed on the two optical paths, the ions and the photon will interact. This interaction will generate a shift of the interference after the second BS (BS_{A2} and BS_{B2}). Then through single photon measurement after the two second BS ($\mathrm{BS}_{A2}, \mathrm{BS}_{B2}$), Alice and Bob can compare their measurement results via classical communication. If the two lower output ports (D_{Al}, D_{Bl}) all fire, the purification succeeds.

We suppose the input photon at Alice's side is σ^+ polarized, and it is superimposed on BS_{A1} at the left lower input port of M_A, and analogously for the description of Bob's side. The effect of the BS on the input photon can be expressed as

$$a^+_{l,\pm}|0\rangle_i \overset{\mathrm{BS}}{\longrightarrow} \frac{1}{\sqrt{2}}(a^+_{u,\pm} + i a^+_{l,\pm})|0\rangle_i, \tag{4a}$$

$$a^+_{u,\pm}|0\rangle_i \overset{\mathrm{BS}}{\longrightarrow} \frac{1}{\sqrt{2}}(a^+_{l,\pm} + i a^+_{u,\pm})|0\rangle_i. \tag{4b}$$

where l and u denote optical paths (lower and upper), $i = A, B, a^+_{l,\pm}|0\rangle_A$ and $a^+_{l,\pm}|0\rangle_B$ denote two input photons of M_A and M_B, respectively, and $\pm$ denotes the direction of polarization. The BS splits the wave function of the input photon into two parts—the reflected part and the transparent one. There will be a $\pi/2$ phase shift between the input photon and the reflected wave function, and the transparent part is synchronized with the input photon. The BS takes no effect on the polarization of the input photon. These are critical to the purification process.

To analyze the evolution of the total system, we will consider the evolution of the following four product states of two ions. We will consider the ions 1,3 case, and the result for ions 2,4 are same as that of the ions 1,3 case

$$a_{l,+}^{+}|0\rangle_A|m_+\rangle_1|m_+\rangle_3 \xrightarrow{\mathrm{BS}_{A1},\mathrm{Ions}1,3,\mathrm{BS}_{A2}} \frac{1}{\sqrt{2}}(|S\rangle_1|g\rangle_1|m_+\rangle_3 + i|m_+\rangle_1|S\rangle_3|g\rangle_3), \tag{5a}$$

$$a_{l,+}^{+}|0\rangle_A|m_+\rangle_1|m_-\rangle_3 \xrightarrow{\mathrm{BS}_{A1},\mathrm{Ions}1,3,\mathrm{BS}_{A2}} \frac{1}{\sqrt{2}}|S\rangle_1|g\rangle_1|m_-\rangle_3 + \frac{i}{2}(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_A|m_+\rangle_1|m_-\rangle_3 \tag{5b}$$

$$a_{l,+}^{+}|0\rangle_A|m_-\rangle_1|m_+\rangle_3 \xrightarrow{\mathrm{BS}_{A1},\mathrm{Ions}1,3,\mathrm{BS}_{A2}} \frac{i}{\sqrt{2}}|m_-\rangle_1|S\rangle_3|g\rangle_3 + \frac{1}{2}(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_A|m_-\rangle_1|m_+\rangle_3, \tag{5c}$$

$$a_{l,+}^{+}|0\rangle_A|m_-\rangle_1|m_-\rangle_3 \xrightarrow{\mathrm{BS}_{A1},\mathrm{Ions}1,3,\mathrm{BS}_{A2}} ia_{u,+}^{+}|0\rangle_A|m_-\rangle_1|m_-\rangle_3. \tag{5d}$$

Then we can give the evolution of the four probabilistic pure states:

$$F^2:\quad a_{l,+}^{+}|0\rangle_A a_{l,+}^{+}|0\rangle_B|\Phi^+\rangle_{12}|\Phi^+\rangle_{34} \xrightarrow{M_A,\mathrm{Ions}1,3,M_B,\mathrm{Ions}2,4} -\frac{1}{8}(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_A(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_B \times |m_+\rangle_1|m_+\rangle_2|m_-\rangle_3|m_-\rangle_4$$
$$+\frac{1}{8}(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_A(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_B \times |m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_+\rangle_4 - \frac{1}{2}a_{u,+}^{+}|0\rangle_A a_{u,+}^{+}|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_-\rangle_3|m_-\rangle_4 + \frac{\sqrt{10}}{4}|\mathrm{Scatter}\rangle. \tag{6a}$$

$$F(1-F):\quad a_{l,+}^{+}|0\rangle_A a_{l,+}^{+}|0\rangle_B|\Phi^+\rangle_{12}|\Psi^+\rangle_{34} \xrightarrow{M_A,\mathrm{Ions}1,3,M_B,\mathrm{Ions}2,4} \frac{i}{4}(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_A a_{u,+}^{+}|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_-\rangle_4$$
$$+\frac{i}{4}a_{u,+}^{+}|0\rangle_A(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_B|m_-\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4 + \frac{\sqrt{3}}{2}|\mathrm{Scatter}\rangle, \tag{6b}$$

$$(1-F)F:\quad a_{l,+}^{+}|0\rangle_A a_{l,+}^{+}|0\rangle_B|\Psi^+\rangle_{12}|\Phi^+\rangle_{34} \xrightarrow{M_A,\mathrm{Ions}1,3,M_B,\mathrm{Ions}2,4} -\frac{1}{4}(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_A a_{u,+}^{+}|0\rangle_B|m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_-\rangle_4 - \frac{1}{4}a_{u,+}^{+}|0\rangle_A$$
$$\times(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_B|m_-\rangle_1|m_+\rangle_2|m_-\rangle_3|m_-\rangle_4 + \frac{\sqrt{3}}{2}|\mathrm{Scatter}\rangle, \tag{6c}$$

$$(1-F)^2:\quad a_{l,+}^{+}|0\rangle_A a_{l,+}^{+}|0\rangle_B|\Psi^+\rangle_{12}|\Psi^+\rangle_{34} \xrightarrow{M_A,\mathrm{Ions}1,3,M_B,\mathrm{Ions}2,4} \frac{i}{8}(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_A(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_B \times |m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4$$
$$+\frac{i}{8}(a_{l,+}^{+} + ia_{u,+}^{+})|0\rangle_A(a_{u,+}^{+} + ia_{l,+}^{+})|0\rangle_B \times |m_-\rangle_1|m_+\rangle_2|m_+\rangle_3|m_-\rangle_4 + \frac{\sqrt{14}}{4}|\mathrm{Scatter}\rangle, \tag{6d}$$

where $|\mathrm{Scatter}\rangle$ denotes the normalized vectors describing the state of the scattered photons, which can be filtered out from the detector. After evolution, Alice and Bob will operate single photon measurements at the lower and upper output ports of M_A and M_B, respectively. In this purification scheme, the first ($|\Phi^+\rangle_{12}|\Phi^+\rangle_{34}$) and the fourth ($|\Psi^+\rangle_{12}|\Psi^+\rangle_{34}$) cases will lead to the measurement result that the two lower output ports (D_{Al} and D_{Bl}) fire simultaneously, but the second ($|\Phi^+\rangle_{12}|\Psi^+\rangle_{34}$) and the third ($|\Psi^+\rangle_{12}|\Phi^+\rangle_{34}$) cases never lead to. From the evolution result, we get that if the two lower output ports (D_{Al} and D_{Bl}) fire simultaneously, Alice and Bob will get the four-ion maximally entangled state $(1/\sqrt{2})(|m_+\rangle_1|m_+\rangle_2|m_-\rangle_3|m_-\rangle_4+|m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_+\rangle_4)$ with probability $F^2/32$, and get another four-ion maximally entangled state $(1/\sqrt{2})(|m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4+|m_-\rangle_1|m_+\rangle_2|m_+\rangle_3|m_-\rangle_4)$ with probability $(1-F)^2/32$. If Alice and Bob measure the ions 3 and 4 in the $|\pm\rangle$ basis, where $|+\rangle=(1/\sqrt{2})(|m_+\rangle+|m_-\rangle)$, $|-\rangle=(1/\sqrt{2})(|m_+\rangle-|m_-\rangle)$, the maximally entangled state $(1/\sqrt{2})(|m_+\rangle_1|m_+\rangle_2|m_-\rangle_3|m_-\rangle_4+|m_-\rangle_1|m_-\rangle_2|m_+\rangle_3|m_+\rangle_4)$ will collapse into different states corresponding to various measurement results. For the results $|+\rangle_3|+\rangle_4$ and $|-\rangle_3|-\rangle_4$, the four-ion maximally entangled state will collapse into the state $|\Phi^+\rangle_{12}$. But for the results $|+\rangle_3|-\rangle_4$ and $|-\rangle_3|+\rangle_4$, it will collapse into $|\Phi^-\rangle_{12}$, then Alice can operate a phase rotation operation on ion 1 to convert $|\Phi^-\rangle_{12}$ into $|\Phi^+\rangle_{12}$. For the four-ion maximally entangled state $(1/\sqrt{2})(|m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4+|m_-\rangle_1|m_+\rangle_2|m_+\rangle_3|m_-\rangle_4)$ case, the measurement results and the needed operations have been synchronized with the first case naturally. So after the evolution, the single photon measurement and the single ion measurement on each side, the two remaining ions will be left in the new states expressed by the new density operator:

$$\rho_{12} = F'|\Phi^+\rangle_{12}\langle\Phi^+| + (1-F')|\Psi^+\rangle_{12}\langle\Psi^+|. \tag{7}$$

where $F' = F^2/[F^2+(1-F)^2]$, is the new fidelity. If the fidelity of the initial shared entangled ensemble satisfies $F > \frac{1}{2}$, $F' > F$, the initial entangled state is purified [13,14]. Because F can be an arbitrary number between 0.5 and 1.0, the iteration of our scheme can extract a near-perfect maximally entangled state from the ensemble shared by Alice and Bob.

III. ENTANGLEMENT CONCENTRATION FOR PURE NONMAXIMALLY ENTANGLED STATES

Here concludes the discussion of the entanglement purification for mixed ionic states. We find that the above scheme can also be used to concentrate the nonmaximally entangled pure states. The setup and the ionic level structure are all the same to the mixed states case. We can suppose the nonmaximally entangled pure state is in the form

$$|\Psi\rangle_{AB} = a|m_+\rangle_A|m_-\rangle_B + b|m_-\rangle_A|m_+\rangle_B, \tag{8}$$

where $|a|^2+|b|^2=1$. Just like the mixed state case, two pairs of ions (1,2 and 3,4) will be placed on M_A, M_B. The evolution of the total state of the system can be expressed as

$$a^+_{l,+}|0\rangle_A a^+_{l,+}|0\rangle_B|\Psi\rangle_{12}|\Psi\rangle_{34} \xrightarrow{M_A,\text{Ions } 1,3,M_B,\text{Ions } 2,4} \frac{iab}{4}(a^+_{u,+}+ia^+_{l,+})|0\rangle_A(a^+_{l,+}+ia^+_{u,+})|0\rangle_B$$
$$\times|m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4 + \frac{iab}{4}(a^+_{l,+}+ia^+_{u,+})|0\rangle_A(a^+_{u,+}+ia^+_{l,+})|0\rangle_B|m_-\rangle_1|m_+\rangle_2|m_+\rangle_3|m_-\rangle_4 + \sqrt{\frac{2-|a|^2|b|^2}{2}}|\text{Scatter}\rangle. \tag{9}$$

After evolution, if the detectors D_{Al} and D_{Bl} fire, the four ions are left in a maximally entangled state: $(1/\sqrt{2})(|m_+\rangle_1|m_-\rangle_2|m_-\rangle_3|m_+\rangle_4+|m_-\rangle_1|m_+\rangle_2|m_+\rangle_3|m_-\rangle_4)$ with probability $|a|^2|b|^2/8$. Although we probably can get the four-ion maximally entangled states corresponding to the measurement results, D_{Au} and D_{Bu}, D_{Al} and D_{Bu}, D_{Au} and D_{Bl}, we should omit these results for the reason that the fire at the upper outport (D_{Au}, D_{Bu}) probably means the ions are not precisely placed on the optical paths.

If the initial nonmaximally entangled state is in the following form, $a|m_+\rangle_A|m_+\rangle_B+b|m_-\rangle_A|m_-\rangle_B$, the concentration will similarly succeed, provided that the D_{Al} and D_{Bl} fire, and the successful probability is also $|a|^2|b|^2/8$. After obtaining the four-ion maximally entangled states, Alice and Bob can make single ion measurement on ions 3,4 in the basis $|\pm\rangle$ just like in the mixed states case. Then the remaining ions 1,2 will be left in two-ion maximally entangled state. From analysis, the successful probability for obtaining a two-ion maximally entangled state is still $|a|^2|b|^2/8$.

If we want to get four-ion maximally entangled states, there is no need for us to operate the ionic measurement in the basis $|\pm\rangle$. So our purification and concentration scheme can not only generate two-ion maximally entangled states but also can generate four-ion maximally entangled states. In this sense, the present scheme is more efficient than the previous scheme [13]. In Pan's scheme, the four-photon entangled states can not be extracted, because the measurement on one pair of photons is needed to complete the purification procedure, otherwise we can not get to know whether the purification succeeds or not, while, in our scheme, after the single photon measurement, the purification process can conclude if we need four-ion maximally entangled states. Then the four-ion maximally entangled states can be used as a robust entanglement resource in quantum communication. That is to say, our scheme is a purification scheme without postselection measurement [30].

IV. DISCUSSION

After the discussion on the purification itself, we now discuss the practical implementation of it. Singly positively charged alkaline ions have only one electron outside a closed shell, so they are commonly used in the quantum information experiments using trapped ions [31,32]. Here we discuss a possible implementation of our purification scheme using $^{40}Ca^+$ as an example. The relevant levels of $^{40}Ca^+$ have been depicted in Fig. 3 [33].

$D_{5/2}$ and $D_{3/2}$ are two metastable levels of $^{40}Ca^+$ with lifetimes of the order of $1s$. s_1 and s_2 are two sublevels of $D_{5/2}$ with $m=-\frac{5}{2}$ and $m=-\frac{1}{2}$, and these two sublevels are coupled to $|e\rangle$ by σ_- and σ_+ light at 854 nm. Here e, S_1, S_2, $S_{1/2}$ correspond to e, m_-, m_+, g in Fig. 1, respectively. That is to say, we use the $S_{1/2}$ as stable ground state, S_1, S_2 as two degenerate metastable state and $P_{3/2}$ as excited state. The arbitrary superposition state of these two degenerate metastable states can be realized by applying a laser pulse of

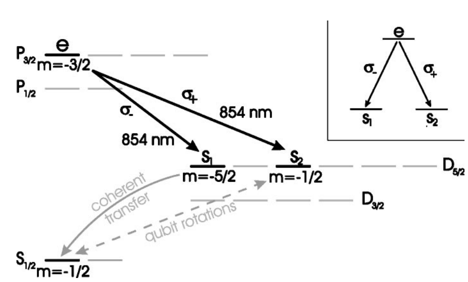

FIG. 3. Relevant levels of $^{40}Ca^+$ ions [33].

appropriate length, which can be realized in a few microseconds [34]. The $^{40}Ca^+$ in state S_1 or S_2 can be excited into the excited state $P_{3/2}$ by applying one σ_- or σ_+ light at 854 nm. Then decay from $|e\rangle$ to S_1, S_2, to $D_{3/2}$, and to $S_{1/2}$ are all possible. But Refs. [32,33] give the transition probability for $P_{1/2} \rightarrow S_{1/2}$(397 nm) as 1.3×10^8 s^{-1} and the branching ratio of $P_{1/2} \rightarrow D_{3/2}$(866 nm) versus $P_{1/2} \rightarrow S_{1/2}$(397 nm) as 1:15, while the branching ratio for $P_{3/2} \rightarrow D_{5/2}$(854 nm) versus $P_{3/2} \rightarrow S_{1/2}$(393 nm) can be estimated as 1:30, giving 0.5×10^7 s^{-1} for the transition probability. So, in most cases, the $^{40}Ca^+$ in the excited state will decay into the stable ground state $S_{1/2}$. The detection of the internal states of $^{40}Ca^+$ can be realized by using a cycling transition between $S_{1/2}$ and $P_{1/2}$(397 nm) [31,32].

To enhance the emission efficiency of the photons from the ions, we can introduce cavities. Then the following three items will affect the emission efficiency of the photon from the ions:

(i) The coupling between cavity mode and the $P_{3/2} \rightarrow S_{1/2}$(393 nm) transition.

(ii) Decay from $P_{3/2}$ to $D_{5/2}$.

(iii) Cavity decay.

From Ref. [35], the probability p_{cav} for a photon to be emitted into the cavity mode after excitation to e can be expressed as $p_{cav} = 4\gamma\Omega^2/(\gamma+\Gamma)(\gamma\Gamma+4\Omega^2)$, where $\gamma = 4\pi c/F_{cav}L$ is the decay rate of the cavity, F_{cav} is its finesse, L is its length, $\Omega = (D/\hbar)\sqrt{hc/2\epsilon_0\lambda V}$ is the coupling constant between the transition and the cavity mode, D is the dipole element, λ is the wavelength of the transition, V is the mode volume (which can be made as small as $L^2\lambda/4$ for a confocal cavity with waist $\sqrt{L\lambda/\pi}$), and Γ is the non-cavity-related loss rate [33]. From the discussion of Ref. [33], the photon package is about 100 ns, which is a relatively long time for the purification scheme to be completed.

When calculating the total efficiency of the purification scheme, we must consider the following items:

(i) The emission efficiency of photon: p_{cav}, which has included the cavity decay. To maximize the p_{cav}, we have chosen $F_{cav} = 19\,000$, $L = 3$ mm. Then $\gamma = 9.9 \times 10^6$ s^{-1}, $p_{cav} = 0.01$ [33].

(ii) The effect of the photon detectors is expressed as $\eta^2/2$. Here we let a detection efficiency $\eta = 0.7$, which is a level that can be reached within the current technology.

(iii) The asynchronism of the two users will also introduce some errors, which can be denoted by ζ. The difficulty caused by the use of the two independent photon sources can be solved by the following method. We can connect the two users with an optical fiber. The photon source is placed in one side, and the two twin photons produced by this source can be led to two MZIs by this optical fiber. This method will also introduce some error. Because the photon will transmit at the velocity of light in the optical fiber, this error will be rather small. Then the efficiency of the purification scheme will only be affected slightly. So in numerical calculation, we can let $\zeta = 0.9$. In addition, we also can make use of a pair of classical pulses to synchronize the two photon sources, where the pulses are used to excite the optical switches.

(iv) Coupling the photon out of the cavity will introduce another error expressed as ξ, which can be modulated to be close to unit.

After considering the above factors, the total success probability can be expressed as follows:

$P = \{[F^2 + (1-F)^2]/32\} p_{cav}{}^2 (\eta^2/2)\zeta\xi$ for mixed state, that is to say, if we input photon with the rate of 3 000 000/s, we can get 70 pairs of purified entangled $^{40}Ca^+$ ions per minute for $F = 0.7$.

$P = [a^2(1-a^2)/8] p_{cav}{}^2 (\eta^2/2)\zeta\xi$ for pure state, that is to say, if we input photon with the rate of 3 000 000/s, we can get 100 pairs of purified entangled $^{40}Ca^+$ ions per minute for $a^2 = 0.7$.

In conclusion, we present an entanglement purification scheme, which can purify the general ionic mixed entangled stats by using a Mach-Zehnder interferometer. The most important advantage of the scheme is that it can extract not only two-ion maximally entangled states but also four-ion maximally entangled states from less entangled pure or mixed states. In addition, the operations carried out here are all simple and can be realized within the current technology. The main drawback of the scheme is how to place the ions on the optical paths precisely.

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