Entanglement swapping via quantum state discrimination

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We study entanglement swapping through nonmaximally entangled states. The combination of the standard protocol with an optimal scheme for quantum state discrimination leads to a reliable protocol for the conclusive probabilistic generation of maximally entangled states. A possible setup based on cold trapped ions is proposed.

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I. INTRODUCTION

In quantum communications, entanglement plays the role of a fundamental resource [1]. It allows one to realize processes such as quantum teleportation [2] and quantum cryptography [3] which by classical means are only approximately implemented [4]. Most quantum communication protocols require a maximally entangled state for their implementation. The use of partially entangled states leads to a reduction in the quality and the reliability of these protocols.

Recently, the connection between unambiguous state discrimination and quantum state teleportation has been studied by several authors [5]. The main result is that the use of schemes for optimal unambiguous state discrimination [6] leads to an enhancement of quantum teleportation through a partially entangled pure state in such a way that perfect teleportation can be conclusively achieved with certain success probability. The optimal conclusive quantum teleportation protocol has recently been found by Roa *et al.* [7].

In this article we study entanglement swapping through partially entangled states of bipartite finite-dimensional quantum systems. Under these conditions, the standard protocol for entanglement swapping leads to partially entangled states, whose entanglement cannot be increased by local transformations. An analysis of the protocol indicates that use of nonmaximally entangled states for entanglement swapping leads to the problem of discriminating among a set of nonorthogonal quantum states. We show that, in this case, the known protocol for entanglement swapping can be enhanced by the use of optimal conclusive state discrimination. Thereby, it is possible to obtain probabilistically a maximally entangled state from a pair of partially entangled pure states. We characterize the success probability for conclusive entanglement swapping and propose a physical realization for this process based on cold ions in a linear trap inside an optical cavity.

This article is organized as follows: In Sec. II we briefly summarize the main tenets of entanglement swapping. Section III shows how the use of partially entangled states in entanglement swapping leads naturally to conclusive quantum state discrimination. Finally, Sec. IV outlines a physical proposal for this process. We summarize in Sec. V.

II. ENTANGLEMENT SWAPPING WITH QUDITS

By entanglement swapping we mean the process that generates entangled particles without a direct dynamical interaction among the involved particles [8]. This process has already been experimentally implemented [9].

Entanglement swapping can be succinctly described by the following identity:

$$XOR_{23}|\Psi_{0,0}\rangle_{12}\otimes|\Psi_{0,0}\rangle_{34}=\frac{1}{d}\sum_{u,l=0}^{d-1}|\Psi_{l,u}\rangle_{14}\otimes F^{-1}|l\rangle_2\otimes|u\rangle_3. \tag{1}$$

The maximally entangled states $|\Psi_{l,u}\rangle$ with $l,u=0,\ldots,d-1$ are a generalization of Bell states to two d-dimensional quantum systems (qudits) defined by

$$\begin{aligned}|\Psi_{l,u}\rangle_{ct}&=XOR_{ct}F|l\rangle_c\otimes|u\rangle_t\\&=\frac{1}{\sqrt{d}}\sum_{q=0}^{d-1}\exp\left(\frac{2\pi i}{d}ql\right)|q\rangle_c\otimes|q-u\rangle_t,\end{aligned} \tag{2}$$

where XOR_{ct} is the generalized control-NOT quantum gate, $XOR_{ct}|i\rangle|_c j\rangle_t=|_c i\rangle|i\ominus j\rangle_t$ [10], c and t denote the control and the target qudit, respectively, F is the Fourier transform—i.e., $F|l\rangle=(1/\sqrt{d})\Sigma_{k=0}^{d-1}\exp(i2\pi lk/d)|k\rangle$—and operations involving subindexes are carried out modulo d.

The protocol for entanglement swapping can readily be read out of identity Eq. (1). Let us assume that particles 1 and 4 belong to Alice and Bob, respectively, and particles 2 and 3 belong to Charlie. If Charlie applies a generalized control-NOT gate onto his particles and measures them separately in bases $F^{-1}|l\rangle_2$ and $|u\rangle_3$, then the joint state of particles 1 and 4 will be projected onto the maximally entangled state $|\Psi_{l,u}\rangle_{14}$.

Thereby, the entanglement contained in states $|\Psi_{0,0}\rangle_{12}$ and $|\Psi_{0,0}\rangle_{34}$ has been transferred to state $|\Psi_{l,u}\rangle_{14}$. This state can be deterministically transformed into state $|\Psi_{0,0}\rangle_{14}$ by Alice and Bob by means of appropriate local unitary transformations conditional to the outcome of the measurement procedure.

III. ENTANGLEMENT SWAPPING VIA PARTIALLY ENTANGLED PURE STATES

A particularly interesting situation arises when the two pairs of particles are initially in a nonmaximally entangled state. To illustrate this let us consider the following initial state:

$$|\Psi\rangle_{12}\otimes|\Psi\rangle_{34}=\sum_m c_m|m\rangle_1\otimes|m\rangle_2\otimes\sum_n d_n|n\rangle_3\otimes|n\rangle_4, \tag{3}$$

where the coefficients c_m and d_n are positive and are in general different from $1/\sqrt{d}$. Clearly, the states associated with the pairs of particles are generally nonmaximally entangled. When the protocol for entanglement swapping described in the previous section is carried out, the final non-normalized state of particles 1 and 4 will be

$$\sum_{l,m=0}^{d-1} c_m d_{m-u}\exp\left(\frac{-2i\pi}{d}m(s+l)\right)|\Psi_{l,u}\rangle_{14}, \tag{4}$$

which is associated with a projection onto states $F|s\rangle_2$ and $|u\rangle_3$. In general, this state will only be partially entangled. Thus, the known protocol for entanglement swapping via partially entangled states cannot lead to maximally entangled states. Consequently, this protocol has to be reformulated.

The starting point for such a modification is the following identity:

$$XOR_{23}|\Psi\rangle_{12}\otimes|\Psi\rangle_{34}=\frac{1}{\sqrt{d}}\sum_{u,l=0}^{d-1}|\Psi_{l,u}\rangle_{14}\otimes|\nu_{l,u}\rangle_2\otimes|u\rangle_3. \tag{5}$$

This identity generalizes identity Eq. (1) to the case of two pairs of nonmaximally entangled states of Eq. (3). Comparing both identities, we see that states $F^{-1}|l\rangle_2$ have been replaced by states $|\nu_l\rangle_2$ defined by

$$|\nu_{l,u}\rangle_2=Z^{D-l}|\nu_{0,u}\rangle_2, \tag{6}$$

where state $|\nu_{0,u}\rangle_2$ is given by

$$|\nu_{0,u}\rangle_2=\sum_m c_m d_{m-u}|m\rangle_2 \tag{7}$$

and the spectral decomposition of the Z operator is

$$Z=\sum_m \exp\left(\frac{2i\pi}{d}m\right)|m\rangle\langle m|. \tag{8}$$

The identity (5) suggests a protocol for entanglement swapping. The maximally entangled states $|\Psi_{l,u}\rangle_{14}$ are in one-to-one correspondence with states $|\nu_{l,u}\rangle_2$ and $|u\rangle_3$. Therefore, by performing a suitable measurement on particles 2 and 3, it would be possible to project the states of particles 3 and 4 onto a maximally entangled state. Since states $\{|u\rangle_3\}$ are mutually orthogonal, a von Neumann measurement suffices to identify them. However, the states $\{|\nu_{l,u}\rangle_2\}$ are not mutually orthogonal and, consequently, it is not possible to distinguish with certainty among them.

The problem of state distinguishability or state discrimination has been a recurrent subject of research both due to its implications for the understanding of qantum mechanics and to its practical applications. The main strategy consists of using generalized measurements. This allows one to define an optimal conclusive (or unambiguous) scheme for discriminating among a finite number of nonorthogonal states with given *a priori* probabilities. The price we pay in doing this is that the scheme occasionally leads to measurement results which do not enable the precise identification of the state. This idea was first proposed by Ivanovic [11] and has been studied by Dieks [12] and Peres [13] for two nonorthogonal states generated with equal *a priori* probabilities. The generalization to arbitrary *a priori* probabilities was obtained by Jäger and Shimony [14]. The case of a larger number of states was considered by Chefles [15] and Peres and Terno [16], where the former showed that results for the several states case simply generalize from those of the two-state case when the states involved are linearly independent. Moreover, Chefles [15] has shown that there exists a conclusive state discrimination scheme if and only if the states to be discriminated are linearly independent. However, for a linearly dependent set, if C copies of the state are available, then the resulting C-particle states may form a linearly independent set and be amenable to unambiguous discrimination [17,18].

In what follows we shall apply a conclusive state discrimination scheme to entanglement swapping. Let us start by discussing the properties of the states $|\nu_{l,u}\rangle_2$. In general, there are d^2 of these states defined in a d-dimensional Hilbert space. Clearly, these states cannot be linearly independent. However, for u fixed, these states can be split up into d sets Ω_u labeled by the value of u. According to Eq. (6), each state in set Ω_u is generated from a fiducial state by applying powers of the unitary transformation Z. Therefore, these states are symmetric. Furthermore, these states are linearly independent as long as the coefficients c_m and d_n are all nonzero. Therefore, the states belonging to a particular set Ω_u can be conclusively identified.

The inner product between two arbitrary states (l',l) in a particular set Ω_u is given by

$$\langle\nu_{l,u}|\nu_{l',u}\rangle=\sum_{m=0}^{d-1}|c_m|^2|d_{m-u}|^2=N_u. \tag{9}$$

Introducing the properly normalized states $|\tilde{\nu}_{l,u}\rangle=|\nu_{l,u}\rangle/\sqrt{N_u}$ into Eq. (5), we obtain

$$XOR_{23}|\Psi\rangle_{12}\otimes|\Psi\rangle_{34}=\frac{1}{\sqrt{d}}\sum_{u=0}^{d-1}\sqrt{N_u}\left(\sum_{l=0}^{d-1}|\Psi_{l,u}\rangle_{14}\otimes|\tilde{\nu}_{l,u}\rangle_2\right)\otimes|u\rangle_3. \tag{10}$$

The N_u coefficients entering into this equation correspond to the probabilities of projecting onto state $|u\rangle_3$. After such a projection, the normalized conditional state of the remaining particles is given by

$$|\Psi_u\rangle=\frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}|\Psi_{l,u}\rangle_{14}\otimes|\tilde{\nu}_{l,u}\rangle_2. \tag{11}$$

Thereby, the problem of entanglement swapping has been reduced to the discrimination of the states in a particular set Ω_u.

Previously we have mentioned that the states in Ω_u are symmetric. Thus, we have to study the probability $P_{l,u}$ of projecting onto these states from the conditional state $|\Psi_u\rangle$. This is given by

$$P_{l,u} = \sum_{m=0}^{d-1} |A_m^u|^4, \tag{12}$$

where the A_m^u coefficients are defined by

$$A_m^u = \frac{c_m d_{m-u}}{\sqrt{N_u}}. \tag{13}$$

Clearly this expression turns out to be independent of the label l. Therefore, the states in Ω_u have equal *a priori* probabilities. Moreover, since there is no entanglement between particles 2 and 3, the probability of projecting onto state $|\tilde{\nu}_{l,u}\rangle_2 \otimes |u\rangle_2$ corresponds simply to the product $N_u P_{l,u}$.

The optimal scheme for conclusive state discrimination was discussed in detail by Roa *et al.* [7] for the case of symmetric states generated with equal *a priori* probabilities. Let us now apply the results in [7] to the discrimination of the states in set Ω_u. The main idea is that conclusive or unambiguous discrimination among the elements of a set of linearly independent states is possible if we allow for successful distinguishability with some probability. In the particular case we are dealing with, the Hilbert space of particle two is enlarged by considering unused dimensions or including ancillary systems to allow the existence of a unitary transformation U_u such that its action is defined by

$$U_u|\tilde{\nu}_{l,u}\rangle_2 = \sqrt{p_{l,u}}|e_{l,u}\rangle_2 + |\Phi_{l,u}\rangle_2 \tag{14}$$

for u fixed. States $\{|e_{l,u}\rangle\}$ are a d-dimensional basis which generates the $\mathcal{U}^u$ subspace; and the $d-1$ unnormalized, linearly dependent states $\{|\Phi_{l,u}\rangle\}$ belong to the $\mathcal{A}^u$ subspace spanned by basis $\{|a_{l,u}\rangle\}$. Both subspaces are orthogonal and are such that $\mathcal{H}=\mathcal{U}^u \oplus \mathcal{A}^u$. The application of U_u follows a measurement which projects particle two onto one of these two subspaces. A projection onto state $|e_{l,u}\rangle$ in $\mathcal{U}^u$ allows conclusive identification of state $|\tilde{\nu}_{l,u}\rangle$ with probability $p_{l,u}$. A projection onto state $|a_{k,u}\rangle$ leads to no certain knowledge about state $|\tilde{\nu}_{l,u}\rangle$. It can be shown [7,15] that the optimal success probability $p_{l,u}$ is

$$p_{l,u} = |A_{\min}^u|^2, \tag{15}$$

where $|A_{\min}^u| = \min\{|A_k^u|\}_{k=0,\ldots,d-1}$ and that states $|\Phi_{l,u}\rangle$ are

$$|\Phi_{l,u}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \left[\sum_{m=0}^{d-1} \exp\left(\frac{2\pi i}{d} m(k-l)\right) \times \sqrt{|A_m^u|^2 - |A_{\min}^u|^2} \right] |a_{k,u}\rangle. \tag{16}$$

Now we can proceed to clearly state the entanglement swapping protocol combined with conclusive state distinguishability. Let us recall the identity (10):

$$|\Psi\rangle = XOR_{23}|\Psi\rangle_{12} \otimes |\Psi\rangle_{34} = \frac{1}{\sqrt{d}} \sum_{u=0}^{d-1} \sqrt{N_u} \left(\sum_{l=0}^{d-1} |\Psi_{l,u}\rangle_{14} \otimes |\tilde{\nu}_{l,u}\rangle_2 \right) \otimes |u\rangle_3. \tag{17}$$

A measurement on particle 3 projects the remaining particles onto the state

$$|\Psi_u\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} |\Psi_{l,u}\rangle_{14} \otimes |\tilde{\nu}_{l,u}\rangle_2. \tag{18}$$

Conditional on the measurement's outcome, the operator U_u is applied on particle 2 leading to the state

$$U_u|\Psi^u\rangle = \frac{1}{d} \sum_{l=0}^{d-1} |\Psi_{l,u}\rangle_{14} \otimes (\sqrt{p_{l,u}}|e_{l,u}\rangle_2 + |\Phi_{l,u}\rangle_2). \tag{19}$$

Following the application of U_u, a measurement is carried out on particle 3 in the basis $\{|e_{l,u}\rangle, |a_{k,u}\rangle\}$ (with $l,k = 0,\ldots,d-1$). This projects particles 1 and 4 onto the maximally entangled state $|\Psi_{l,u}\rangle_{1,4}$ when state $|e_{l,u}\rangle_2$ is detected or onto the normalized superposition of maximally entangled states,

$$|\Phi\rangle_{14} = \sum_{l=0}^{d-1} \left(\sum_{m=0}^{d-1} \frac{\exp\left(\frac{2\pi i}{d} m(k-l)\right)}{\sqrt{d(1-d|A_{\min}^u|^2)}} \sqrt{|A_m^u|^2 - |A_{\min}^u|^2} \right) |\Psi_{l,u}\rangle_{14}, \tag{20}$$

when state $|a_{k,u}\rangle_2$ is detected.

The probability $Q_{l,u}$ of obtaining the maximally entangled state $|\Psi_{l,u}\rangle_{1,4}$ is given by the product of N_u, the probability of projecting particle 3 onto a particular $|u\rangle$ state, and probability $p_{l,u}$ of successfully discriminating state $|\tilde{\nu}_{l,u}\rangle$—that is,

$$Q_{l,u} = N_u p_{l,u}. \tag{21}$$

Summing over both l and u we obtain the overall success probability Q of the protocol. The optimal value for this probability is given by the expression

$$Q = d \sum_{u=0}^{d-1} \min\{|c_k d_{k-u}|^2\}_{k=0,\ldots,d-1}, \tag{22}$$

which is a function of coefficients c_n and d_m which define the partially entangled states of the two pairs of qudits. If we assume that the c_m and d_m coefficients have been ordered from the lower to the maximum value—i.e., $c_m \leqslant c_{m+1}$ and $d_m \leqslant d_{m+1}$ for $0 \leqslant m \leqslant d$—this expression can be cast in the form

$$Q = d \sum_{u=0}^{d-1} \min\{|c_0 d_{d-u}|^2, |d_0 c_{d-u}|^2\}, \tag{23}$$

which allows us to find the following upper and lower bounds for the overall success probability:

$$\frac{d}{2}(\min\{|c_m|^2\} + \min\{|d_m|^2\}) \geqslant Q \geqslant d \min\{|c_m|^2\}\min\{|d_m|^2\}. \tag{24}$$

The states of particles 1 and 4 are in a partially entangled state even when the discrimination process leads to an ambiguous result. Since these states, Eq. (20) , are pure, the entanglement can easily be calculated. This is given by

$$E=\frac{d}{(d-1)}\left\{1-\frac{\sum_m |A_m^u|^4+d|A_{\min}^u|^2(|A_{\min}^u|^2-2)}{(1-d|A_{\min}^u|^2)^2}\right\}. \tag{25}$$

Here we have used the expression $E=d[1-\mathrm{Tr}(\rho_1)^2]/(d-1)$ to calculate the entanglement. Let us note that we have introduced an extra factor so that $0\leqslant E\leqslant 1$.

Clearly, the entanglement depends on the value of the outcome of the measurement carried out on particle 3, but not on the measurement's result k. Thus, the states arising from a failure in the discrimination process have the same amount of entanglement. This can be explained by considering that the $|\Phi\rangle_{14}$ states can be cast in the form

$$|\Phi\rangle_{14}=Z_1^k\otimes 1_4\sum_{m=0}^{d-1}\frac{\sqrt{|A_m^u|^2-|A_{\min}^u|^2}}{\sqrt{d(1-d|A_{\min}^u|^2)}}|m\rangle_1\otimes|m-u\rangle_4. \tag{26}$$

Thereby, these states turn out to be symmetric under the local, unitary operator $Z_1^k\otimes 1_4$ and, consequently, they have the same amount of entanglement.

Let us now consider the case of entanglement swapping without resorting to state discrimination. It can be shown that a measurement on particle 3. and a measurement on particle 2 in the Fourier basis lead to the state

$$|\tilde{\Phi}\rangle_{14}=\sum_{l=0}^{d-1}\left[\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{2\pi i}{d}m(s-l)\right)A_m^u\right]|\Psi_{l,u}\rangle_{14} \tag{27}$$

when the measurement outcomes are s and u, respectively. The entanglement of particles 1 and 4 in this state is

$$\tilde{E}=\frac{d}{(d-1)}\left(1-\sum_m |A_m^u|^4\right). \tag{28}$$

This entanglement is also dependent only on the outcome u of a measurement on particle 3 As was mentioned above, when the quantum channels are not prepared in maximally entangled states, at least one of the A_m^u coefficients is less than $1/d$. Thus, without loss of generality, we can write

$$|A_{min}^u|^2=\frac{1}{d}(1-\lambda),\quad |A_{m'}^u|^2=\frac{1}{d}\left(1+\frac{\lambda}{d-1}+\varepsilon_{m'}\right),$$

where $1\leqslant\lambda\leqslant 0$; $\Sigma_{m'}\ \varepsilon_{m'}=0$ and m' means $m\neq\min$. Hence, the entanglement difference between the case without quantum state discrimination Eq. (28) , and that with discrimination in case of failure, Eq. (25) , is given by

$$\tilde{E}-E=\frac{(1-\lambda^2)}{d(d-1)^2\lambda^2}\left((d-1)\sum_{m'}\varepsilon_{m'}^2+\lambda^2 d\right). \tag{29}$$

We found bounds for this expression by using the extreme values of coefficients $\varepsilon_{m'}$. A lower bound for the above expression corresponds to the case where all the coefficients are $A_{m'}^u=[1+\lambda/(d-1)]/d$ except one, which is A_{min}^u—i.e., $\varepsilon_{m'}=0$. In this case the difference of entanglement is given by

$$\tilde{E}-E=\frac{(1-\lambda^2)}{(d-1)^2}. \tag{30}$$

The upper bound is obtained when all the coefficients are $A_{m'}^u=(1+\lambda/d)/d$ except one, which is A_{max}^u, with $\varepsilon_{max}=\lambda d(d-2)/(d-1)$, so that $(d-1)\varepsilon_{max}^2\geqslant\Sigma_{m'}\ \varepsilon_{m'}^2$. Thus, for an arbitrary state, the entanglement difference satisfies

$$\frac{(1-\lambda^2)}{(d-1)^3}[d(d-2)^2+(d-1)]\geqslant\tilde{E}-E\geqslant\frac{(1-\lambda^2)}{(d-1)^2}. \tag{31}$$

Here, we can appreciate that, in case of failure, the entanglement of the residual pair is always less than in the case of nonimplementing quantum state discrimination. However, the main difference between both processes arises when the discrimination is successful.

Comparing with state, Eq. (20) , we obtain that the protocol for conclusive entanglement swapping can be described in terms of a channel of the form

$$\sum_{l=0}^{d-1}\frac{1}{\sqrt{d}}|e_l\rangle_1\otimes|e_l\rangle_2\otimes\sum_{k=0}^{d-1}\frac{1}{\sqrt{d}}|e_k\rangle_3\otimes|e_k\rangle_4 \\ +\sum_{l=0}^{d-1}\frac{1}{\sqrt{d}}|a_l\rangle_1\otimes|a_l\rangle_2\otimes\sum_{k=0}^{d-1}\sqrt{|A_m|^2-|A_{\min}|^2}|a_k\rangle_3\otimes|a_k\rangle_4, \tag{32}$$

which corresponds to the superposition of the product of two maximally entangled states in subspace $\mathcal{U}_u$ and the product of a maximally entangled state and a nonmaximally entangled state in subspace $\mathcal{A}_u$.

IV. PHYSICAL IMPLEMENTATIONS USING COLD TRAPPED IONS

For a physical implementation of the above described protocol, in the case of using qubits, we consider cold trapped ions inside optical cavities. Let us consider a set of three cavities C_1, $C_{2,3}$, and C_4. Each one of these cavities contains a set of ions in a linear array. The ions have been cooled down to the ground state of their center-of-mass motion. Let us select one ion in cavity C_1, two ions in $C_{2,3}$, and one ion in C_4 labeled as A_1, A_2 and A_3, and A_4, respectively.

The first step of the implementation is to establish a pair of nonmaximally entangled states between the pairs of ions (A_1,A_3) and (A_2,A_4), which is accomplished by the following procedure: We assume that each ion in the linear trap in cavity C_{12} is individually addressed as is proposed in the original scheme of Ref. [19]. In what follows, we assume the ions to be in a five-level configuration as depicted in Fig. 1. The two paths $|g\rangle_i\rightarrow|c_1\rangle_i\rightarrow|e_1\rangle_i$ and $|g\rangle_i\rightarrow|c_2\rangle_i\rightarrow|e_2\rangle_i$ are addressed using fields with different polarizations. A classical field Ω_c detuned in Δ_1 with respect to the allowed transition $|c_q\rangle_i\rightarrow|g\rangle_i$ of the ith ion in the central cavity $C_{2,3}$, is considered. The $|c_q\rangle_i\rightarrow|e_q\rangle_i$ dipole-allowed transition is quantum mechanically described by creation and annihilation operators a and $a^\dagger$, respectively. This quantized field mode is assumed to be initially in the vacuum state. In the high detuning limit the upper level $|c_q\rangle_i$ is adiabatically

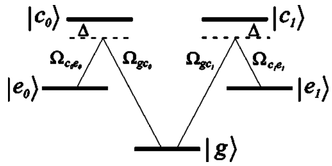

FIG. 1. Electronic level structure of trapped ions. Partially entangled state $|\psi\rangle_4$ of ions i and j are prepared using levels $|e_q\rangle_k$ and $|g\rangle_k$, with $k=i,j$ and $q=0,1$. The transitions involving effective interactions between levels $|g\rangle\rightarrow|c_q\rangle$ and $|c_q\rangle\rightarrow|e_q\rangle$ are driven by classical fields with different polarizations.

eliminated, such that a Raman configuration leads to an effective interaction term:

$$V^I(t) = \frac{r^2(t)}{\Delta_s}|g\rangle_{ii}\langle g| + \Delta_s a^\dagger a|e_q\rangle_{ii}\langle e_q| + ir(t)(e^{-i\phi(t)}a^\dagger|e_q\rangle_{ii} \times\langle g| - e^{i\phi(t)}|g\rangle_{ii}\langle e_q|a), \tag{33}$$

where $q=0,1$ denotes orthogonal polarizations of the created photon, $r(t)=g\Omega_c(t)/2\delta$ (Rabi frequency of Raman transition), and $\Delta_s=g^2/\delta$ (cavity-induced Stark shift). For simplicity, we have assumed a common detuning parameter between the pump fields and the cavity modes, and real coupling constants. Any additional frequency shifts can be included in the phase of the classical field $\phi(t)$. The time- and intensity-dependent terms correspond to dynamical shifts arising from the adiabatic elimination of the upper atomic level.

Thus, we have obtained an effective anti-Jaynes-Cummings interaction. An important element in this process is the production of polarized photons, provided that we drive the transition $|c_q\rangle_i\rightarrow|g\rangle_i$ using classical fields of a given polarization. Assume that the quantum field is initially in the vacuum state and the ion in the ground state, such that

$$|\psi\rangle_0 = |g\rangle_i|0\rangle_{C_{2,3}}|0\rangle_E, \tag{34}$$

where $|0\rangle_E$ represents the state of the environment, which initially is the vacuum connecting cavities $C_{2,3}$ and C_j, with $j=1,4$. Driving the transition $|c_q\rangle_i\rightarrow|g\rangle_i$ during an appropriate time Δt, we have

$$|\psi\rangle_1 = (c_{ei}|e_q\rangle_i|1_q\rangle_{C_{2,3}} + c_{gi}|g\rangle_i|0\rangle_{C_{2,3}})|0\rangle_E, \tag{35}$$

where $c_{ei}=\cos\varphi_i$ and $c_{gi}=\sin\varphi_i$. The photon is emitted by the cavity in a short time compared to the relaxation scales of the ion because of a sufficiently high cavity decay rate. The emission process leads to an emitted one-photon wave packet such that

$$|\psi\rangle_2 = (c_{ei}|e_q\rangle_i|1\rangle_E + c_{gi}|g\rangle_i|0\rangle_E)|0\rangle_{C_{2,3}}, \tag{36}$$

where $|1\rangle_E$ denotes a one-photon wave packet emitted by cavity $C_{2,3}$.

In cavity C_j one ion initially is in the $|e_q\rangle_j$ state, so that the state of the whole system is given by

$$|\psi\rangle_2 = (c_{ei}|e_q\rangle_i|1_q\rangle_E + c_{gi}|g\rangle_i|0\rangle_E)|e_q\rangle_j|0\rangle_{C_j}, \tag{37}$$

where we have omitted the vacuum state of the $C_{2,3}$ cavity, because it is factorized to the state of ions. The same is done in the following steps with contributions from factorized states. If the photon is absorbed by C_j cavity, the state of the system evolves to

$$|\psi\rangle_3 = (c_{ei}|e_q\rangle_i|1_q\rangle_j + c_{gi}|g\rangle_i|0\rangle_j)|e_q\rangle_j. \tag{38}$$

Now, the classical field for driving the ion Raman transition in cavity C_j is turned on. Thus, after a time Δt, we get

$$|\psi\rangle_4 = (c_{ei}|e_q\rangle_i|g\rangle_j + c_{gi}|g\rangle_i|e_q\rangle_j). \tag{39}$$

Following the above procedure, a nonmaximal entangled pair between ions (A_1,A_3) using the polarized path $q=0$ has been prepared. In the same way, the (A_2,A_4) pair is prepared in the same state, through the polarized path $q=1$. In the following, we describe the set of logical operations leading to the conditional operation between the ion pair (A_2,A_3).

Here, we consider that ions A_2 and A_3 in cavity $C_{2,3}$ are individually addressed using a pair of Raman fields Ω_{1i}^q detuned in $\Delta_1+\Delta_2$ and Ω_{2i}^q detuned in $\Delta_1+\Delta_2+\delta_i$, respectively, to the lower ionic transitions $|c_q\rangle_i\rightarrow|g\rangle_i$ and $|c_q\rangle_i\rightarrow|e_q\rangle_i$. Eliminating adiabatically the excited level $|c_q\rangle_i$, and properly adjusting the detuning δ_i, it is possible to generate general dynamic evolution between the quantized center-of-mass motion and the electronic transition $|g\rangle_i\rightarrow|e_q\rangle_i$. It can be shown that, by adjusting $\delta_i=-\nu_x$, the center-of-mass collective motion and the internal ionic levels evolve under the effective Hamiltonian

$$H_{ql} = \hbar\frac{\Omega_i^q\eta}{2\sqrt{N}}(be^{i\phi}|e_q\rangle_{ll}\langle g| + |g\rangle_{ll}\langle e_q|b^\dagger e^{-i\phi}), \tag{40}$$

with η being the Lamb-Dicke parameter, b and $b^\dagger$ are the center-of-mass motion operators, and $\Omega_i^q=\Omega_{1i}^q\Omega_{2i}^q/(\Delta_1+\Delta_2)^2$. For $\delta_i=0$ (carrier transition) we obtain a single Hamiltonian

$$H_R = \hbar\frac{\Omega_i^q}{2}(e^{i\phi}|e_q\rangle_{ll}\langle g| + |g\rangle_{ll}\langle e_q|e^{-i\phi}). \tag{41}$$

Choosing an interaction time between the laser pulses and the lth ion such that $t=k\pi\sqrt{N}/\Omega_i^q\eta$, the Hamiltonian structure H_{qi} leads to the evolution operator

$$U_l^{kq}(\phi) = e^{-ik\pi/2(be^{i\phi}|e_q\rangle_{ll}\langle g|+|g\rangle_{ll}\langle e_q|b^\dagger e^{-i\phi})}. \tag{42}$$

The Hamiltonian H_R allows us to implement unitary operations on one qubit through the following rotation operator:

$$R_l^k(\phi) = \exp\left[-ik\frac{\pi}{2}(|e_q\rangle_{ll}\langle g|e^{i\phi} + e^{-i\phi}|g\rangle_{ll}\langle e_q|)\right]. \tag{43}$$

According to Ref. [19], this physical picture allows implementing a controlled-NOT gate as

$$XOR_{23} = R_3^{1/2}(-\Upsilon)U_2^{1,0}U_3^{2,1}U_2^{1,0}R_3^{1/2}(\Upsilon), \tag{44}$$

where $\Upsilon=\pi/2$ and ions A_2 and A_3 are the control and target qubits, respectively.

Before applying the control-NOT gate, using the carrier transition we apply a π pulse to ions (A_1 and A_2); i.e., we apply the operator $R_1^1(\pi)$ and $R_2^1(\pi)$. Thus, the partially entangled pairs in Eq. (3) in this case are given by

$$|\psi\rangle = |\psi\rangle_{13}\otimes|\psi\rangle_{24},$$

$$|\psi\rangle_{13} = (c_{e_1}|g\rangle_1|g\rangle_3 + c_{g_1}|e_0\rangle_1|e_0\rangle_3),$$

$$|\psi\rangle_{24} = (c_{e_2}|g\rangle_2|g\rangle_4 + c_{g_2}|e_1\rangle_2|e_1\rangle_4). \quad (45)$$

Now, we apply the controlled-NOT gate to the above state (45), so that we arrive at the state given by Eq. (5). In this case, the $|\nu_{0,u}\rangle_2$ states are explicitly given by

$$|\tilde{\nu}_{0,g}\rangle_2 = \frac{1}{\sqrt{N_g}}(c_{e_1}c_{e_2}|g\rangle_2 + c_{g_1}c_{g_2}|e_1\rangle_2),$$

$$|\tilde{\nu}_{0,e_1}\rangle_2 = \frac{1}{\sqrt{N_{e_1}}}(c_{g_1}c_{e_2}|g\rangle_2 + c_{e_1}c_{g_2}|e_1\rangle_2), \quad (46)$$

where N_g and N_{e_1} are the obvious normalization constants for each state. For the sake of simplicity, from now on we shall omit the label referring to particle "2."

The most remarkable feature in this physical implementation of the protocol is that we can make use of the additional electronic levels for discriminating between states $\{|\nu_{0,u}\rangle, |\nu_{1,u}\rangle\}$. We have already developed this discrimination protocol [20] for two nonorthogonal states. Here, we apply previous results to this particular case. In this case the Raman transitions through the independent channels associated with orthogonal polarizations are driven by classical fields Ω_{c_jg} and $\Omega_{e_jc_j}$ with frequencies $\tilde{\nu}_{c_jg}$ and $\tilde{\nu}_{c_je_j}$, respectively. Here, $j=0,1$ denote the orthogonal polarization channels. Only the carrier transition in the ion is considered, so that no explicit effects on the center of mass of the ion are included.

The following conditions hold: $\Delta = \omega_{c_jg} - \nu_{c_jg} = \omega_{c_je_j} - \nu_{c_je_j}$ and $\Delta \gg \Omega_{c_jg}, \Omega_{e_jc_j}$, where ω_{c_jg} and $\omega_{c_je_j}$ are the transition frequencies of the involved electronic levels, so that electronic levels $|c_0\rangle$ and $|c_1\rangle$ are adiabatically eliminated. Thus the evolution operator in the $\{|e_0\rangle, |e_1\rangle, |g\rangle\}$ basis is given by

$$U(\varphi) = \begin{pmatrix} 1 + |g_0|^2 C(\varphi) & g_0 g_1^* C(\varphi) & -ig_0 \sin\varphi \\ g_1 g_0^* C(\varphi) & 1 + |g_1|^2 C(\varphi) & -ig_1^* \sin\varphi \\ -ig_0^* \sin\varphi & -ig_1^* \sin\varphi & \cos\varphi \end{pmatrix}, \quad (47)$$

where $\varphi = \Omega t$ is an dimensionless interaction time, $C(\varphi) = (\cos\varphi - 1)$, and $\Omega^2 = |g_{2_1}|^2 + |g_{2_0}|^2$. The coupling constants are defined as $g_j = g_{2_j}/\Omega$, and $g_{2_j} = \Omega_{c_jg}\Omega^*_{e_jc_j}/\Delta$. This evolution allows implementing on the ion all the required coherent operations for discriminating nonorthogonal states. The states to be recognized are coherent superpositions of states $|e_1\rangle$ and $|g\rangle$. Using the carrier transition in levels $|e_0\rangle$ and $|g\rangle$, the superpositions to be recognized can change into two nonorthogonal states $|P\rangle$ or $|Q\rangle$ in such a way that

$$|Q\rangle = \frac{1}{\sqrt{N_g}}(c_{e_1}c_{e_2}|e_0\rangle + c_{g_1}c_{g_2}|e_1\rangle),$$

$$|P\rangle = \frac{1}{\sqrt{N_{e_1}}}(c_{e_1}c_{e_2}|e_0\rangle - c_{g_1}c_{g_2}|e_1\rangle). \quad (48)$$

A rotation $U(\varphi=\pi)$ in Eq. (47) is applied around the $|g\rangle$ state such that $|Q\rangle \to |Q_2\rangle = |e_0\rangle$. This is obtained by assuming that $(1-2|g_0|^2) = c_{e_1}c_{e_2}/\sqrt{N_g}$ and that the phase difference of $g_0 g_1^*$ is equal to π. Thus, $|P\rangle \to |P_2\rangle$, with

$$|P_2\rangle = \cos\delta|e_0\rangle - \sin\delta|e_1\rangle, \quad (49)$$

and $\sin\delta = 2c_{g_1}c_{g_2}c_{e_1}c_{e_2}/\sqrt{N_g N_{e_1}}$. We shall remove the $|e_0\rangle_2$ component from state $|P_2\rangle$ without adding an additional component along $|e_1\rangle_2$ in state $|Q_2\rangle$. This is achieved through extending state $|P_2\rangle$ to a third dimension $|g\rangle_2$ by applying a rotation $U(\varphi_1)$ in the $|e_1\rangle - |g\rangle$ plane. Hence, a subsequent rotation $U(\varphi_2)$ in the $|e_0\rangle - |g\rangle$ plane permits canceling the probability amplitude in direction $|e_0\rangle$ of state $|P_2\rangle$. The rotation $U(\varphi_1)$ is obtained from Eq. (47) by imposing $g_0 = 0$. States $|e_1\rangle$ and $|g\rangle$ transform according to

$$|e_1\rangle \to \cos\varphi_1|e_1\rangle - ie^{-i\gamma_1}\sin\varphi_1|g\rangle,$$

$$|g\rangle \to -ie^{i\gamma_1}\sin\varphi_1|e_1\rangle + \cos\varphi_1|g\rangle, \quad (50)$$

where $e^{i\gamma_1} = g_1/|g_1|$. The additional rotation $U(\varphi_2)$ is implemented by choosing $g_1 = 0$ in Eq. (47) and defining $e^{i\gamma_2} = g_0/|g_0|$. As a result of these operations we obtain states $|Q_3\rangle$ and $|P_3\rangle$ in the $|e_0\rangle$-$|g\rangle$ plane and in the $|e_1\rangle - |g\rangle$ plane, respectively. The condition to eliminate the $|e_0\rangle$ component of state $|P_3\rangle$ leads to the following relation among the parameters of rotations:

$$\cos\delta\cos\varphi_2 = e^{i(-\gamma_1+\gamma_2)}\sin\delta\sin\varphi_1\sin\varphi_2.$$

In addition to this constraint, for an optimal conclusive discrimination one needs to have equal probabilities in the conclusive measurement states, which is equivalent to a constraint of equal components along the $|g\rangle$ direction—that is,

$$\sin\varphi_2 = \cos\delta\sin\varphi_2 + e^{i(-\gamma_1+\gamma_2)}\sin\delta\sin\varphi_1\cos\varphi_2. \quad (51)$$

By imposing $-\gamma_1 + \gamma_2 = 2\pi$, the result of the preparation process for discrimination is

$$|P_3\rangle = \cos\varphi_2|e_1\rangle - ie^{-i\gamma_2}\sin\varphi_2|g\rangle,$$

$$|Q_3\rangle = \cos\varphi_2|e_0\rangle - ie^{-i\gamma_2}\sin\varphi_2|g\rangle, \quad (52)$$

where $\varphi_1 = \arccos(\sqrt{1-\cos\delta}/\sin\delta)$ and $\varphi_2 = \arcsin(\sqrt{\cos\delta})$. Summarizing, the protocol $U(\varphi_2)U(\varphi_1)U(\pi)$ obtains a pair of states having equal independent components along a conclusive measurement $|e_0\rangle$-$|e_1\rangle$ plane as well in the nonconclusive measurement state $|g\rangle$.

V. SUMMARY

In this work the problem of entanglement swapping from nonmaximally entangled states has been studied. The protocol that we have described considers the implementation of a quantum discrimination process for a set of nonorthogonal quantum states. Maximally entangled states are distilled when a successful discrimination process takes place. Partial entanglement arises in case of a failure in the discrimination process, as occurs in the case when no discrimination process is implemented. We found an expression for comparing the entanglement among these cases. The amount of entanglement when no discrimination process is implemented is always greater than when an unsuccessful discrimination process takes place. However, a discrimination process leads to the possibility of generating maximally entangled states. A physical model where the protocol could be tested has been given for trapped ions located in separated cavities.

An interesting application of this protocol is the enhancement of entanglement between parties belonging to widely separated nodes of a quantum network. In one of these nodes there are at least three particles. Two of them are in a maximally entangled state, pair 1-2, and the other one is in a partially entangled state with the particle lying at the second node, pair 3-4. Thus, the degree of entanglement of the distributed pair could be improved at expenses of the localized maximally entangled pair.

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