Relativistic Aharonov-Bohm effect in the presence of planar Coulomb potentials

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Exact analytic solutions are found to the Dirac equation in 2+1 dimensions for a combination of an Aharonov-Bohm potential and the Lorentz three-vector and scalar Coulomb potentials. By means of the solutions obtained the relativistic quantum Aharonov-Bohm effect is studied for the free (in the presence of a Lorentz three-vector Coulomb potential) and bound fermion states. We obtain the total scattering amplitude in a combination of the Aharonov-Bohm and Lorentz three-vector Coulomb potentials as a sum of two scattering amplitudes. This modifies the expression for the standard Aharonov-Bohm cross section due to the interference of these two amplitudes with each other. We discuss that the observable quantities can be the phases of electron wave functions or the energies of bound states.

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I. INTRODUCTION

The quantum Aharonov-Bohm (AB) effect, first predicted by Aharonov and Bohm [1], was analyzed in many physical aspects in numerous works (see Ref. [2]). It occurs when an electron travels in a certain configuration of a vector potential A_μ in which the corresponding magnetic flux is confined to a finite-radius tube topologically equivalent to a cylinder. In the case of a cylindrical external field configuration, where a natural assumption is that the relevant quantum mechanical system is invariant along the symmetry (z) axis, the system then becomes essentially two dimensional in the xy plane [3]. When an electron travels in an Aharonov-Bohm potential the electron wave function acquires a (topological) phase which further influences the interference pattern. The Aharonov-Bohm vector potential can produce observable effects because the relative (gauge invariant) phase of the electron wave function, correlated with a nonvanishing gauge vector potential in the domain where the magnetic field vanishes, depends on the magnetic flux [3]. In a definite sense one can say that the Aharonov-Bohm effect is due to the topological properties of a space of electron wave functions in $2+1$ dimensions in a topologically nontrivial background. In [4,5] the contribution to the Aharonov-Bohm scattering amplitude which can arise from the inclusion of the spin-orbit interaction of the electron magnetic moment with the electric field oriented along the solenoid axis was theoretically studied in the nonrelativistic approximation. This effect has been recently confirmed in experiment [6]. We note that this quantum system also has axial symmetry.

There are two more questions of how the effect of other physical fields modifies the usual AB phenomenon as well as of how the AB effect will manifest itself when an electron is in the bound state. In order to approach the solution of these problems the simplest physical models can be considered. For the usual scattering of nonrelativistic particles one needs to solve the Schrödinger equation in two spatial dimensions but when the particle spin is included one is concerned with the Dirac equation. As the relativistic AB effect seems to be likely to occur in cylindrically symmetric potentials the above problem can be considered by using the usual four-component Dirac equation in the absence of a third spatial coordinate. The latter equation can be easily written in terms of the two-component Dirac equation. Thus, the above models can be reduced to $(2+1)$-dimensional ones.

The results of Ref. [1] modified by using the Dirac equation in $2+1$ dimensions were applied to other problems. Solutions to the two-component Dirac equation in the Aharonov-Bohm potential were first obtained and applied by Alford and Wilczek in Ref. [7] to the study of the interaction of cosmic strings with matter. The above solutions coincide with solutions of the Dirac equation in $2+1$ dimensions for a massive neutral fermion with an anomalous magnetic moment in a planar field of a point electric charge placed at the origin $z=0$ (see, for example, Ref. [8]). In the three-dimensional space, this field corresponds to the electric field of a thin thread that is perpendicular to the plane $z=0$ and carries an electric charge with constant linear density. Thus, the solutions to the two-component Dirac equation in the Aharonov-Bohm potential can directly be applied to the planar scattering of a massive neutral fermion with anomalous magnetic moment interacting in the electric field of the thin thread, which was first predicted by Aharonov and Casher in Ref. [9]. The Aharonov-Casher effect is, however, different in many ways from the AB effect. In particular, the Aharonov-Casher effect is a phenomenon involving two spatial dimensions in an essential way [10].

A permanent interest in this topic also is stimulated by the studies of $(2+1)$-dimensional models in both superconductivity [11] and particle theory (including the quantum Hall effect [12] and degenerate planar semiconductors with low-energy electron dynamics [13]) in Refs. [7–18].

The main purpose of the paper is to study the relativistic quantum Aharonov-Bohm effect in the presence of other physical fields. This study is possible only in $2+1$ dimensions. In $3+1$ dimensions, analytic solutions even to the Schrödinger equation in the Coulomb and Aharonov-Bohm potentials still are not found.

This paper is organized as follows. In Sec. II we study the electron states in the Aharonov-Bohm potential and briefly

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discuss the topological properties of a space of electron wave functions in $2+1$ dimensions. In Sec. III we find exact analytic solutions to the Dirac equation in $2+1$ dimensions for a combination of the Lorentz three-vector and scalar Coulomb as well as Aharonov-Bohm potentials. In Sec. IV the relativistic Aharonov-Bohm scattering in the presence of the Lorentz three-vector Coulomb potential is studied.

We use the units where $c=\hbar=1$.

II. DIRAC ELECTRON IN AN AHARONOV-BOHM POTENTIAL

The most general combination of external potentials in which exact analytic solutions to the Dirac equation in $2+1$ dimensions can be found in the form of special functions is an Aharonov-Bohm potential

$$A^0=0, \quad A_x=-\frac{By}{r^2}, \quad A_y=\frac{Bx}{r^2}, \quad A^0=0, \quad A_r=0,$$

$$A_\varphi=\frac{B}{r}, \quad B=\frac{\Phi}{2\pi}, \quad r=\sqrt{x^2+y^2}, \quad \varphi=\arctan(y/x), \quad (1)$$

and the Lorentz three-vector $[A^\mu(r)]$ and scalar $[U(r)]$

$$A^0(r)=-\frac{a}{er}, \quad A_r=0, \quad A_\varphi=0, \quad U(r)=-\frac{b}{r} \qquad (2)$$

Coulomb potentials. Here e is the electrical charge of a fermion. The interaction with a scalar field can be introduced in theory by means of the replacement $m\to m+U$, where m is the fermion mass.

In $2+1$ dimensions, the Dirac γ^μ-matrix algebra is known [16] to be represented in terms of the two-dimensional Pauli matrices σ_j. In addition, two kinds of fermions can be introduced in accordance with the signature of the two-dimensional Dirac matrices [16]

$$\eta=\frac{i}{2}\mathrm{Tr}(\gamma^0\gamma^1\gamma^2)=\pm 1,$$

where the two signs of η correspond to two nonequivalent representations of the Dirac matrices. We choose

$$\gamma^0=\eta\sigma_3, \quad \gamma^1=i\sigma_1, \quad \gamma^2=i\sigma_2. \qquad (3)$$

It will be noted that the model with charged fermions is invariant under the charge conjugation operation and the transformation $m\to -m$, which is equivalent to the transformation $\gamma^\mu\to-\gamma^\mu$ or $\eta\to-\eta$. Hence, we can fix the signs of e and m.

First let us consider an electron of mass $m>0$ and charge e in the xy plane in potential (1). In the three-dimensional space this potential describes the magnetic field of an infinitely thin solenoid creating a finite magnetic flux Φ in the z direction [the magnetic field $B_z=\Phi\delta(\mathbf{r})$]. The Dirac equation in $2+1$ dimensions in the potential A_μ is

$$(\gamma^\mu\hat{P}_\mu-m)\Psi=0. \qquad (4)$$

Here $\hat{P}_\mu=-i\partial_\mu-eA_\mu$ is the generalized electron momentum operator. Note that the parameter $\eta=\pm1$ in Eq. (4) can be applied for spin "up" and spin "down," respectively (see Ref. [20]).

We seek solutions of Eq. (4) in potential (1) in the form [17]

$$\Psi(t,\mathbf{x})=\frac{1}{\sqrt{2\pi}}\exp(-iEt+il\varphi)\psi(r,\varphi), \qquad (5)$$

where E is the electron energy, l is an integer, and $\psi(r,\varphi)$ is a two-component function (i.e., a two-spinor)

$$\psi(r,\varphi)=\begin{pmatrix} f(r) \\ g(r)e^{i\varphi}\end{pmatrix}. \qquad (6)$$

The upper ("large") and lower ("small") components of the two-spinor can be interpreted in the sense of positive- and negative-energy solutions of the Dirac equation. It should be noted that an electron in the two-dimensional space corresponds to one in the three-dimensional space with only one spin projection on the z axis [18]. The upper component only remains in the nonrelativistic approximation (see, for example, [19]). Therefore, there is a conventional relationship to the corresponding nonrelativistic Schrödinger limit.

The electron wave function in potential (1) (limited as $r=0$) has the form

$$\Psi_p(r,\varphi)=e^{-iEt+il\varphi}\sqrt{\frac{\pi p}{2E}}\begin{pmatrix}\sqrt{E+\eta m}J_{|\nu|}(pr) \\ -i\sqrt{E-\eta m}e^{i\varphi}J_{|\nu+1|}(pr)\end{pmatrix}. \qquad (7)$$

Here $p=\sqrt{E^2-m^2}$, and $J_\nu(pr)$ is the Bessel function of order

$$\nu=|l+eB|.$$

In order that the irregular [Neumann function $N_{|\nu|}(pr)$] solution can be eliminated we need to allot it on the "background" of the regular solution $J_{|\nu|}(pr)$ as $r\to0$ which leads to the condition $|l+eB|>0$.

The wave functions are normalized by the condition

$$\int \psi^*_{p,l,\eta}\psi_{p',l',\eta}d^2x=2\pi\delta_{l,l'}\delta(p-p'). \qquad (8)$$

As $B=0$ one recover the free electron solutions in $2+1$ dimensions from Eq. (7).

When ν is an integer, for example, $l+s$, the magnetic field flux is quanitized as

$$\Phi=2\pi\hbar cs/e\equiv\Phi_0 s,$$

where Φ_0 is the elementary magnetic flux, and $eB=s$.

One can define the scattering amplitude in a conventional manner. We assume that the incident electron wave is from the left and the wave function is normalized in the standard manner, i.e., the upper component of the incident wave is $\psi=e^{ipx}$. In fact, the electron wave function in potential (1) must have the asymptotic form

$$\psi_p(r,\varphi) = \begin{pmatrix} 1 \\ -ip/(E+\eta m) \end{pmatrix} e^{ipx+ieB\varphi} + \begin{pmatrix} 1 \\ ip/(E+\eta m) \end{pmatrix} \frac{f(\varphi)}{\sqrt{r}} e^{ipr} \tag{9}$$

as $r \to \infty$. Here $f(\varphi)$ is the scattering amplitude.

Writing $\psi(r,\varphi)$ in the form

$$\psi(r,\varphi) = \sum_{l=-\infty}^{\infty} A_l J_{|\nu|}(pr) e^{il\varphi}, \tag{10}$$

it is easy to show that

$$A_l = e^{-i(\pi/2)|l+eB|}. \tag{11}$$

The scattering amplitude is proportional to $S_l - 1 \equiv e^{2i\delta_l} - 1$, where $\delta_l = (\nu - l)\pi = eB\pi \equiv e\Phi/2\hbar c$ are the partial phase shifts. They depend upon only the total magnetic flux Φ.

The coefficient before the term $e^{ipr}/\sqrt{r}$ is the standard Aharonov-Bohm amplitude for the scattering of nonrelativistic particles,

$$f_A(\varphi) = \frac{1}{\sqrt{2\pi p i}} \frac{e^{-i\varphi(s-1/2)} \sin(e\Phi/2)}{\sin(\varphi/2)}. \tag{12}$$

Here $e\Phi = 2\pi s + 2\pi\Delta$, $-1/2 \leqslant \Delta \leqslant 1/2$. First, the amplitude (12) was calculated in Ref. [7]. Thus, the scattering amplitude is unaffected by the spin.

One can define the so-called "topological" current

$$J^\mu = \psi^* e^{\mu\nu\rho}[-ix_\nu \partial_\rho + e\partial_\nu A_\rho]\psi = -i\psi^* e^{\mu\nu\rho} x_\nu \partial_\rho \psi + j^\mu, \tag{13}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$, and ψ is the free electron wave function (at $B=0$). The currents J^μ and j^μ satisfy the continuity equation $\partial_\mu J^\mu = \partial_\mu j^\mu = 0$ and, therefore,

$$Q = \frac{1}{2\pi}\int J^0 d\mathbf{r} = \int \psi^* \left(-i\frac{\partial}{\partial\varphi} + 2\pi eB\delta(\mathbf{r})\right)\psi\, d\mathbf{r}, \tag{14}$$

and

$$q = \frac{1}{2\pi}\int j^0 d\mathbf{r} = eB\int \psi^* \delta(\mathbf{r})\psi\, d\mathbf{r} = \frac{eB}{\hbar c} = \frac{\Phi}{\Phi_0} = \left[\frac{\Phi}{\Phi_0}\right] + \left(\frac{\Phi}{\Phi_0}\right)_d \tag{15}$$

(where $[\Phi/\Phi_0]$ is the integer part of Φ/Φ_0) are conserved. The quantity $[\Phi/\Phi_0]$ is the "topological number," and the quantity $q - [\Phi/\Phi_0]$ can be called the "topological defect."

Note that the topological quantities introduced here characterize such properties of the Dirac equation solutions in $2+1$ dimensions in a topologically nontrivial background as the limit, the continuity, and the uniqueness, in contrast to the usual topological numbers which are determined by the boundary conditions for the corresponding solutions at infinity. The latter is conserved due to the finiteness of energy. In our case the "topological defect" is of importance. It characterizes the branching of solutions at the point $x,y=0$.

III. DIRAC FERMION IN A COMBINATION OF COULOMB AND AHARONOV-BOHM POTENTIALS

Let us find the exact solutions of Dirac equation (4) in potentials (1) and (2). After simple standard rearrangement, we obtain for the functions $f(r)$ and $g(r)$

$$\frac{df}{dr} - \frac{l+eB}{r} f + \left(E + m + \frac{a-b}{r}\right) g = 0,$$

$$\frac{dg}{dr} + \frac{1+l+eB}{r} g - \left(E - m + \frac{a+b}{r}\right) f = 0. \tag{16}$$

Here and below we put $\eta=1$. The functions $f(r)$ and $g(r)$ are normalized

$$\int_0^\infty (|f^2| + |g^2|) dr = 1. \tag{17}$$

Further, following Ref. [21] for the functions $f(r)$ and $g(r)$, we obtain

$$f(r) = \sqrt{m+E}\, e^{-\rho/2} \rho^{\gamma-1} (Q_1 + Q_2),$$

$$g(r) = -\sqrt{m-E}\, e^{-\rho/2} \rho^{\gamma-1} (Q_1 - Q_2), \tag{18}$$

where

$$\rho = 2\lambda r, \quad \lambda = \sqrt{m^2 - E^2}, \tag{19}$$

γ is determined by

$$\gamma = \frac{1}{2} \pm \sqrt{\left(l + eB + \frac{1}{2}\right)^2 - a^2 + b^2}, \tag{20}$$

and the solutions (finite at $\rho=0$) are expressed in terms of the confluent hypergeometric function $F(b,c;z)$:

$$Q_1 = AF\left(\gamma - \frac{1}{2} - \frac{aE - mb}{\lambda}, 2\gamma; \rho\right),$$

$$Q_2 = CF\left(\gamma + \frac{1}{2} - \frac{aE - mb}{\lambda}, 2\gamma; \rho\right). \tag{21}$$

The constants A and C are related by

$$C = \frac{\gamma - 1/2 - (Ea - mb)/\lambda}{l + eB + 1/2 + (ma - bE)/\lambda} A. \tag{22}$$

As $a^2 > (l+eB+1/2)^2 + b^2$ the quantity γ is real, and must be chosen positive. If $a^2 > (l+eB+1/2)^2 + b^2$ then two roots of γ are imaginary and the corresponding wave functions oscillate as $r \to 0$, which means the occurrence of Klein's paradox [22].

In order to normalize Q_1 and Q_2 they must reduce to polynomials. For $F(b,c;z)$ it means that the parameter b must be equal to a negative integer or zero; therefore

$$\gamma - \frac{1}{2} - \frac{Ea - mb}{\lambda} = -n_r, \quad \frac{Ea - mb}{\lambda} = n_r + \gamma - \frac{1}{2} \equiv n. \tag{23}$$

It is easy to show that the admitted values of the quantum number n_r are $0,1,2,\ldots$ for $l+eB+1/2>0$ and $1,2,3,\ldots$

for $l+eB+1/2<0$, and the discrete fermion energy levels are given by

$$E_n = -m\left[\frac{ab}{(n+\gamma)^2+b^2} + \sqrt{\left(\frac{ab}{(n+\gamma)^2+b^2}\right)^2 + \frac{(n+\gamma)^2-a^2}{(n+\gamma)^2+b^2}}\right]. \quad (24)$$

If the scalar Coulomb potential is absent, then

$$Q_1 = AF\left(\gamma - \frac{1}{2} - \frac{aE}{\lambda}, 2\gamma; \rho\right),$$

$$Q_2 = CF\left(\gamma + \frac{1}{2} - \frac{aE}{\lambda}, 2\gamma; \rho\right), \quad (25)$$

with

$$C = \frac{\gamma - 1/2 - Ea/\lambda}{l + eB + 1/2 + ma/\lambda}A \quad (26)$$

and

$$E_n = m\left[1 + \frac{a^2}{[n_r + \sqrt{(l+eB+1/2)^2 - a^2}]^2}\right]^{-1/2}. \quad (27)$$

In the nonrelativistic Schrödinger limit, the expression for the energy spectrum becomes

$$E_{\text{non}} = -\frac{a^2}{2(n_r + |l+eB+1/2|)^2}. \quad (28)$$

We see that the quantity eB in Eq. (28) has a full analogy with the Rydberg correction.

If the vector Coulomb potential is absent, then

$$E_n = m\sqrt{\frac{n^2-b^2}{n^2}}, \quad n = n_r + \sqrt{(l+eB+1/2)^2 + b^2}. \quad (29)$$

The lowest energy level is

$$E_0 = m\frac{|l+eB+1/2|}{\sqrt{(l+eB+1/2)^2 + b^2}}, \quad (30)$$

so that $E_0 \to 0$ for $b \to \infty$.

Note that the solutions of the Klein-Gordon equation contain the parameter

$$\gamma_s = \sqrt{(l+eB)^2 - a^2 + b^2}; \quad (31)$$

therefore the energy spectrum, for example, in the absence of the scalar Coulomb potential is

$$E_n^s = m\left[1 + \frac{a^2}{[n_r - 1/2 + \sqrt{(l+eB)^2 - a^2}]^2}\right]^{-1/2}. \quad (32)$$

This expression makes sense only when $|l+eB| > a^2$, a condition that forbids the existence of the $l=0$ energy levels at $B=0$. Thus, the relativistic spectra (27) and (32) lead one to conclude that the two-dimensional Dirac particle only connects to the physically reasonable nonrelativistic Schrödinger limit for the electron energy.

It is seen from Eq. (27) that the Aharonov-Bohm potential must influence the radiation spectrum of the electron.

IV. RELATIVISTIC AHARONOV-BOHM SCATTERING IN THE PRESENCE OF A LORENTZ THREE-VECTOR COULOMB POTENTIAL

As long as the scalar interaction is not actual for the AB scattering we shall put $b=0$ below. The wave functions of the continuous spectrum ($E>m$) can be obtained from Eq. (25) by means of the following replacements:

$$\sqrt{m-E} \to -i\sqrt{E-m}, \quad \lambda \to -ip,$$

$$-n_r \equiv -n \to \gamma - 1/2 - iaE/p. \quad (33)$$

These functions should also be normalized anew. Using Eq. (33), let us represent the functions f and g in the form

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{pmatrix} A' e^{ipr}(2pr)^{\gamma-1}[e^{i\xi}F(\gamma - 1/2 - i\mu, 2\gamma, -2ipr) \mp e^{-i\xi}F(\gamma + 1/2 - i\mu, 2\gamma, -2ipr)], \quad (34)$$

where A' is the normalization constant and

$$\mu = \frac{aE}{p}, \quad e^{-2i\xi} = \frac{\gamma - 1/2 + i\mu}{l + eB + 1/2 - i\mu'}, \quad \mu' = \frac{ma}{p} \equiv \mu\frac{m}{E}.$$

We note that the quantity ξ is real.

After simple transformations given for the three-dimensional case in Ref. [21], we obtain

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sqrt{\frac{E \pm m}{Ep}} \frac{(2pr)^\gamma}{r} \frac{|\Gamma(\gamma + 1/2 + i\mu)|}{\Gamma(2\gamma)} e^{\pi\mu/2} \times e^{-i[\pi/2 - \gamma + 1/2 + \mu \ln 2pr - \arg\Gamma(\gamma + 1/2 + i\mu)]} \times \begin{pmatrix} \text{Im} \\ \text{Re} \end{pmatrix} [e^{i(pr+\xi)}F(\gamma - 1/2 - i\mu, 2\gamma, -2ipr)]. \quad (35)$$

Here $\Gamma(z)$ is the Γ function.

Asymptotically, the wave function has the form

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sqrt{\frac{2(E \pm m)}{Er}} \begin{pmatrix} \sin \\ \cos \end{pmatrix} (pr + \delta_l + \mu \ln 2pr - \pi l/2), \quad (36)$$

where

$$\delta_l = \xi - \pi\gamma/2 - \arg\Gamma(\gamma + 1/2 + i\mu) + \pi/4 + \pi l/2 \quad (37)$$

and

$$e^{2i\delta_l} = \frac{l + eB + 1/2 - i\mu'}{\gamma - 1/2 + i\mu} \frac{\Gamma(\gamma + 1/2 - i\mu)}{\Gamma(\gamma + 1/2 + i\mu)} e^{i\pi(l - \gamma + 1/2)}. \quad (38)$$

The expression for the analytical continuation of Eq. (38) in the range $E<m$,

$$e^{2i\delta_l} = \frac{l + eB + 1/2 + (am)/\lambda}{\gamma - 1/2 - (aE)/\lambda}\frac{\Gamma(\gamma + 1/2 - (aE)/\lambda)}{\Gamma(\gamma + 1/2 + (aE)/\lambda)}e^{i\pi(l-\gamma+1/2)}, \tag{39}$$

has poles at the points where $\gamma+1/2-(aE)/\lambda=1-n$, $n=1,2\ldots$, as well as at the point $\gamma-1/2-(aE)/\lambda=-n=0$. At these points the energy levels are discrete. Near the poles with $n\neq 0$, it is easy to obtain

$$e^{2i\delta_l} \approx (-1)^{n+l}\frac{(l + eB + R + 1/2)\lambda^3}{\Gamma(n+1)\Gamma(2\gamma+n)m^2a(E-E_0)}e^{-i\pi(\gamma-1/2)}. \tag{40}$$

The residue of the function $\exp(2i\delta_l)$ in its pole is related to the coefficient in the asymptotic expression of the wave function of the corresponding bound state as follows:

$$f \approx A_0 e^{-\lambda r}, \quad g = \sqrt{\frac{m-E}{m+E}}f, \tag{41}$$

where

$$A_0 = \left[\sqrt{\frac{m+E}{m-E}}\frac{(l + eB + ma/\lambda + 1/2)\lambda^3}{2m^2a\Gamma(n+1)\Gamma(2\gamma+n)}\right]^{1/2}(2\lambda r)^{\gamma+n-1/2}. \tag{42}$$

Let us consider the scattering problem for a combination of the Aharonov-Bohm and Lorentz three-vector Coulomb potentials. The total phase shifts according to Eq. (36) are

$$\delta_l = -\pi\gamma/2 + \pi/4 + \pi l/2 + \xi - \arg\Gamma(\gamma + 1/2 + i\mu) \equiv \delta_{AB} + \delta_l^a, \tag{43}$$

where

$$\delta_{AB} = -\pi\gamma/2 + \pi/4 + \pi l/2 \tag{44}$$

and

$$\delta_l^a = \xi - \arg\Gamma(\gamma + 1/2 + i\mu) \tag{45}$$

are the phase shifts due to the effect of Aharonov-Bohm and Coulomb potentials, respectively.

The total scattering amplitude is

$$f_{tot}(\varphi) \sim \sum_{l=0}^{\infty}[\exp(2i\delta_A + 2i\delta_l^a) - 1] \tag{46}$$

and the difference in the square brackets we write in the form [23]

$$\exp(2i\delta_{AB} + 2i\delta_l^a) - 1 = [\exp(2i\delta_A) - 1] + [\exp(2i\delta_{AB})(\exp(2i\delta_l^a) - 1)]. \tag{47}$$

The Coulomb phases mainly contribute to the scattering amplitude for large l, so their contribution can be calculated in the quasiclassical approximation. After simple calculations and taking into account Eq. (12), we obtain

$$f_{tot}(\varphi) = \frac{1}{\sqrt{2\pi pi}}\frac{1}{\sin(\varphi/2)}\left[\sin(\pi eB)e^{-i\varphi(s-1/2)} + \frac{am}{p}e^{i\pi eB}\right] \equiv f_{AB}(\varphi) + f_a(\varphi). \tag{48}$$

From Eq. (48) it follows that these two amplitudes interfere with each other in the scattering cross section:

$$d\sigma = |f_{tot}(\varphi)|^2 d\varphi = \frac{d\varphi}{2\pi p \sin^2\varphi/2}\left[\sin^2\pi eB + \left(\frac{2am}{p}\right)\sin\pi eB\cos(s\varphi + \varphi/2 + \pi eB) + \left(\frac{am}{p}\right)^2\right]. \tag{49}$$

From Eq. (49) it is seen that the periodic dependence of the interference term in the cross section differs for forward ($\varphi=0$) and backward ($\varphi=\pi$) scattering.

V. RELATIVISTIC CLASSICAL ENERGY

The classical motion of the electron in the Aharonov-Bohm and Lorentz three-vector Coulomb potentials is described by the Hamilton-Jacobi equation

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial r} + \frac{a}{r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial\varphi} + \frac{eB}{c}\right)^2 - m^2c^2 = 0. \tag{50}$$

The classical trajectory of the electron in the pure Aharonov-Bohm potential is linear:

$$r = \frac{r_{min}}{\cos(\varphi - \varphi_0)}, \quad r_{min} = \frac{Lc + eB}{\sqrt{E^2 - m^2c^4}},$$

where E is the electron energy and L is the z projection of the electron angular momentum and the scattering angle is zero:

$$\theta = \pi - 2\int_{r_{min}}^{\infty}\frac{dr}{r^2\sqrt{1/r_{min}^2 - 1/r^2}} = 0. \tag{51}$$

The solution of Eq. (50) (i.e., the classical action) is

$$S = -Et + L\varphi + \int dr\sqrt{\frac{1}{c}\left(E + \frac{a}{r}\right)^2 - \frac{L^2}{r^2} - m^2c^2}. \tag{52}$$

The classical trajectory of the electron, which is in the finite region of the x,y plane for $Lc > a$, $E < mc^2$ is

$$r = \frac{u}{c\sqrt{L^2E^2 - m^2c^2u}\cos\sqrt{u}(\varphi - \varphi_0)/Lc - aE}, \quad u = L^2c^2 - a^2, \tag{53}$$

and the action variables are

$$J_\varphi = \frac{1}{2\pi}\int_0^{2\pi}d\varphi\frac{\partial S}{\partial\varphi} = L, \tag{54}$$

$$J_r = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} dr\sqrt{\frac{1}{c}\left(E+\frac{a}{r}\right)^2 - \frac{L^2}{r^2} - m^2c^2}$$
$$= -\frac{\sqrt{u}}{c} + \frac{aEc}{\sqrt{m^2c^4 - E^2}}. \tag{55}$$

Here $r_{\min}$ and $r_{\max}$ are the classical turning points.

From Eq. (55), we easily find the electron energy expressed through J_r and J_φ,

$$E = mc^2\left[1 + \frac{a^2}{[cJ_r + \sqrt{(cJ_\varphi + eB)^2 - a^2}]^2}\right]^{-1/2}.$$

If we require the classical energy expression after the semiclassical quantization to reduce to Eq. (27), we must equate the right-hand side of Eq. (55) to $\hbar n_r$ and put $L=\hbar(l+1/2)$. The latter is necessary to obtain the correct value of the semiclassical wave function phase for large r (see Ref. [18]). Therefore, the semiclassical quantization of the action variables in the form $J_r=\hbar n_r$, $J_\varphi=\hbar(l+1/2)$ results in the same energy spectrum (27) as obtained from the eigenvalue problem.

VI. DISCUSSION

It is shown that the gauge-invariant (observable) quantities are the phases of electron wave functions or the energies of bound states. In experiments the quantum wave associated with each electron in the entrance splits into two wave packets that go around the solenoid on different sides. The paths of these wave packets intersect in the exit to result in a closed contour. So, though the Aharonov-Bohm potential satisfies the equation $[\boldsymbol{\nabla}\times\mathbf{A}]=0$ everywhere in the plane except the point $x=y=0$, the integral that gives the magnetic flux Φ through a closed contour C encircled by the wave packets

$$\oint \mathbf{A}\, d\mathbf{s} = \boldsymbol{\Phi}$$

is defined unambiguously. It is curious that for $eB\neq 0$ the electron wave function (7) is exactly equal to zero at $x=y=0$.

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