Quantum transitions in the center-of-mass tomographic probability representation

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Propagators for quantum evolution equation of multipartite systems in the center-of-mass probability representation are introduced. Properties of these propagators and their relation to the Green function of Schrödinger equation and propagator of Moyal evolution equation are studied. Examples of quadratic systems are considered and explicit expressions for the center of mass tomographic propagator are obtained.

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I. INTRODUCTION

Quantum mechanics, usually formulated in terms of the density matrix, can be as well formulated in many other ways. One such alternative way is the probability representation of quantum mechanics, or quantum tomography, recently proposed in Ref. [1] (see also [2–6] for reference about the analogous quasidistribution functions). The probability representation is remarkable for it describes the quantum state in terms of *non-negative distribution function* (the tomogram, or marginal distribution), *directly measurable* in experiments on the state reconstruction [7–18]. The tomogram is a *probability distribution* completely describing the quantum state. In addition, the tomography can be applied to describe classical systems. As a result one can compare the classical and quantum behavior of the same system in the framework of one formalism. Sign conservation of the tomogram can be valuable in the computer simulation [19,20], for example in the overcoming of the "sign problem" in simulation of the Fermi systems (see [21,22] and references therein). Another advantage of the probability representation is the fact that not only quantum states are described by probability distributions but also quantum transitions between the quantum states are described by nonnegative probabilities (propagators) instead of complex probability amplitudes. The latter ones can be expressed in terms of the propagators. One of the aims of the present work is to make an explicit presentation of such expressions.

Besides the above-mentioned advantages of the tomographic representation, there is a disadvantage connected with the use of this formalism. Consider a quantum system with N degrees of freedom. Density matrix [23,24] describing the state of such system is the function of $2N$ independent variables (time, temperature, etc., being the external parameters). The so-called symplectic tomography [25–27] describes the same state by the symplectic tomogram, which is the function of $3N$ variables. The idea of tomography representation is to *rotate and scale* the reference frame in the phase space and work with the distribution of *position* in the new reference frames: $X=\mu q+\nu p$, where q and p are coordinate and momentum; μ and ν are the parameters of scaling and rotation. Symplectic tomogram's $3N$ variables are two N-component vectors of scaling and rotation ($\vec{\mu}$ and $\vec{\nu}$) and N-component vector $\vec{X}$ of positions, measured in all possible rotated and scaled reference frames (i.e., for all possible $\vec{\mu}$ and $\vec{\nu}$). In fact, the symplectic tomogram is the homogeneous function [28,29]. Effectively it has only $2N$ independent variables and contains the same amount of information about the system as the density matrix does. But straightforward description of the quantum evolution in symplectic tomography is possible only using the full set of $3N$ variables. Of course, for the investigation of the multipartite systems (large N), it is quite inconvenient to operate with $3N$ variables instead of $2N$, even if it implies the description of the state in terms of the non-negative function.

Fortunately, it appears that one can avoid the half as much increase of the number of variables in replacing the density matrix formalism by the tomographic one. Recently [30,31] we introduced the *center of mass* tomography, that operates with only $2N+1$ variables, describes the state by the non-negative function (center of mass tomogram) and contains the same amount of information about the system as the density matrix and symplectic tomogram do (the density matrix, symplectic tomogram, and center of mass tomogram are connected by the invertable maps). The center of mass tomography also uses two N-component vectors ($\vec{\mu}$ and $\vec{\nu}$) of scaling and rotation of reference frame in the phase space. But, contrary to the symplectic tomogram, the center of mass tomogram is the distribution of one variable X: the sum of positions over all degrees of freedom, or position of the center of mass of the system, measured in the scaled and rotated reference frame. This is the origin of the name "center of mass tomography." Amazingly, the function of $2N+1$ variable contains the same amount of information about the system as the function of $3N$ variables (symplectic tomogram) does. This is the consequence of overcompleteness of the symplectic tomography description: the symplectic tomogram operates with some additional variables that do not give more information about the system (see [30,31]). It also should be noted that the proposed schemes of state-reconstruction experiments for several degrees of freedom [13,14] used a particular variant of what we later called the center of mass tomogram.

This paper is devoted to further development of the center of mass tomographic map. In previous work [30,31] we de-

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rived the time evolution equation for this representation, investigated the properties of the map and its relation to the star product formalism, and symmetry properties in respect to identical particles permutations. In this work we present the new results concerning the quantum evolution in the center of mass representation. Namely, we investigate the form and properties of the propagators in this formalism, and their connection with the propagators in symplectic tomography, Wigner-Moyal and Schrödinger-Heisenberg representations of quantum mechanics. As an example, we consider the systems with N degrees of freedom and Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hat{\vec{Q}}B(t)\hat{\vec{Q}} + \vec{c}(t), \tag{1}$$

where $\hat{\vec{Q}}$ is the $2N$-component vector of momentum and coordinate operators:

$$\hat{\vec{Q}} = \{\hat{\vec{p}},\hat{\vec{q}}\}, \tag{2}$$

and $B(t)$ and $\vec{c}(t)$ are the time-dependent $2N\times 2N$ matrix and $2N$-component vector. The Hamiltonians of type (1) describe such physically valuable cases as a motion of free particles, time evolution of multimode oscillators, and motion of noninteracting particles in the magnetic and electric fields.

Below, for the system described by Hamiltonian (1), we investigate the evolution equation and integrals of motion in the center of mass tomography representation (Sec. II). In Sec. III the general properties of propagators for center of mass tomography are elucidated, as well as their connection to the propagators in other representations is explained. In Sec. III we also obtain the expression for the center of mass tomography propagator for quadratic Hamiltonians (1). The work is summarized in Sec. IV.

II. QUANTUM EVOLUTION AND INTEGRALS OF MOTION

A. Connection between different state-describing functions

The center of mass tomogram w is connected with the Wigner function W [32], density matrix ρ and symplectic tomogram w_s through the invertable maps [30,31]. For example, the Wigner function and center of mass tomogram are connected in the following way:

$$w(X,\vec{\mu},\vec{\nu}) = \int W(\vec{q},\vec{p})e^{-ik(X-\vec{\mu}\cdot\vec{q}-\vec{\nu}\cdot\vec{p})}\frac{dkd\vec{q}d\vec{p}}{(2\pi)}, \tag{3}$$

$$W(\vec{q},\vec{p}) = \int e^{-i(\vec{\mu}\cdot\vec{q}+\vec{\nu}\cdot\vec{p}-X)}w(X,\vec{\mu},\vec{\nu})\frac{dXd\vec{\mu}d\vec{\nu}}{(2\pi)^{2N}}. \tag{4}$$

Here $\vec{q},\vec{p}$ are the N-component vectors of coordinates and momenta, $\vec{\mu},\vec{\nu}$ are also the N-component vectors, and X is a real number. The meaning of the arguments of the center of mass tomogram is given by the following relation:

$$w(X,\vec{\mu},\vec{\nu}) = \int W(\vec{q},\vec{p})\,\delta(X-(\vec{\mu}\cdot\vec{q}+\vec{\nu}\cdot\vec{p}))d\vec{q}d\vec{p}, \tag{5}$$

which means that $X=\vec{\mu}\cdot\vec{q}+\vec{\nu}\cdot\vec{p}$ is the sum of positions (or the center of mass position) measured in *scaled and rotated reference frame* in the phase space.

Connection between w and the density matrix ρ can be easily obtained from Eqs. (3) and (4) and the expression connecting ρ and the Wigner function:

$$W(\vec{q},\vec{p}) = \int \rho\left(\vec{q}+\frac{\vec{u}}{2},\vec{q}-\frac{\vec{u}}{2}\right)e^{-i\vec{p}\cdot\vec{u}}\frac{d\vec{u}}{(2\pi)^N}, \tag{6}$$

$$\rho(\vec{q}\,',\vec{q}\,'') = \int W\left(\frac{\vec{q}\,'+\vec{q}\,''}{2},\vec{p}\right)e^{i\vec{p}(\vec{q}\,'-\vec{q}\,'')}d\vec{p}. \tag{7}$$

The relation between the center of mass and symplectic tomograms is given by the following formulas:

$$w(X,\vec{\mu},\vec{\nu}) = \int w_s(\vec{Y},\vec{\mu},\vec{\nu})\,\delta\left(X-\sum_{j=1}^{N}Y_j\right)d\vec{Y}, \tag{8}$$

$$w_s(\vec{X},\vec{\mu},\vec{\nu}) = \int w(Y,\vec{k}\circ\vec{\mu},\vec{k}\circ\vec{\nu})e^{i(Y-\vec{k}\cdot\vec{X})}d\vec{k}dY. \tag{9}$$

Here and throughout the paper the designation $\vec{c}=\vec{a}\circ\vec{b}$ means the componentwise product of vectors: c_j, the jth component of vector $\vec{c}$, is given by a_jb_j; $\vec{a}\cdot\vec{b}$ means usual scalar product of vectors.

B. Evolution equations

Now, when we know how to obtain one state-describing function from another, let us consider the evolution equation for the center of mass tomography. In Refs. [30,31] one can find the evolution equations for the center of mass tomogram w, derived for the motion in potential $V(\vec{q})$ (taking into account both an external potential and an interaction between the particles). Such equation is obtained applying the formula connecting w with the density matrix ρ to the evolution equation for ρ:

$$i\frac{\partial\rho(\vec{q}\,',\vec{q}\,'')}{\partial t} = [\hat{H},\rho(\vec{q}\,',\vec{q}\,'')]. \tag{10}$$

Note that the Hamiltonian (1) is not a special case of the Hamiltonian describing the motion in the potential V, because Eq. (1) contains the terms with $\hat{q}_i\hat{p}_j$. Therefore the evolution equations in Refs. [30,31] do not cover all possible forms of the Hamiltonian (1).

Before presenting the evolution equation for w, derived from Eq. (10), let us introduce some useful designations for B and $\vec{c}$ in Eq. (1):

$$\vec{c} = \{\vec{c}^{\,p},\vec{c}^{\,q}\}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \tag{11}$$

where $\vec{c}^{\,p}$ and $\vec{c}^{\,q}$ are the N-component vectors and b_1,b_2,b_3,b_4 are the $N\times N$ matrices. Using these designations, one gets the evolution equation for the system with the Hamiltonian (1) in the center of mass tomography representation:

$$\frac{\partial w}{\partial t}+\left(\frac{1}{2}\sum_{j,k=1}^{N}\left\{b_4^{jk}(t)\left(\nu_j\frac{\partial}{\partial\mu_k}+\nu_k\frac{\partial}{\partial\mu_j}\right)-b_1^{jk}(t)\left(\mu_j\frac{\partial}{\partial\nu_k}+\mu_k\frac{\partial}{\partial\nu_j}\right)+i[b_2^{jk}(t)+b_3^{jk}(t)]\left[\frac{1}{2}\nu_j\mu_k\frac{\partial^2}{\partial X^2}+2\frac{\partial^2}{\partial\mu_j\partial\nu_k}\left(\frac{\partial}{\partial X}\right)^{-2}\right]\right\}-\vec{c}^{q}\vec{\nu}\frac{\partial}{\partial X}-2i\vec{c}^{p}\frac{\partial}{\partial\vec{\nu}}\left(\frac{\partial}{\partial X}\right)^{-1}\right)w=0. \tag{12}$$

Evolution of the same system in the Wigner-Moyal representation is described by the following equation (see also [33]):

$$\frac{\partial W}{\partial t}+\left\{\frac{1}{2}\sum_{j,k=1}^{N}\left[b_1^{jk}(t)\left(p_j\frac{\partial}{\partial q_k}+p_k\frac{\partial}{\partial q_j}\right)-b_4^{jk}(t)\left(q_j\frac{\partial}{\partial p_k}+q_k\frac{\partial}{\partial p_j}\right)+i[b_2^{jk}(t)+b_3^{jk}(t)]\left(\frac{1}{2}\frac{\partial^2}{\partial p_j\partial q_k}+2q_jp_k\right)\right]-\vec{c}^{q}\frac{\partial}{\partial\vec{p}}+2i\vec{c}^{p}\vec{p}\right\}W=0. \tag{13}$$

C. Integrals of motion

Evolution of the system with N degrees of freedom can be characterized by $2N$ *integrals of motion* [33], linearly independent operators with constant average values. For the quadratic systems described by the Hamiltonian (1) one can write the $2N$-component vector $\hat{\vec{I}}$ of integrals of motion in the form [33]

$$\hat{\vec{I}}(t)=\Lambda(t)\hat{\vec{Q}}+\vec{\Delta}(t), \tag{14}$$

where matrix $\Lambda=\binom{\lambda_1\lambda_2}{\lambda_3\lambda_4}$ consists of $N\times N$ matrices $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, and vector $\vec{\Delta}=\{\vec{\delta}_1,\vec{\delta}_2\}$ consists of N-component vectors $\vec{\delta}_1$ and $\vec{\delta}_2$. These matrices and vectors obey the differential equations

$$\dot{\lambda}_1=\lambda_1b_3-\lambda_2b_1,\ \dot{\lambda}_2=\lambda_1b_4-\lambda_2b_2,$$
$$\dot{\lambda}_3=\lambda_3b_3-\lambda_4b_1,\ \dot{\lambda}_4=\lambda_3b_4-\lambda_5b_2, \tag{15}$$
$$\dot{\vec{\delta}}_1=\lambda_1\vec{c}^{p}-\lambda_2\vec{c}^{q},\ \dot{\vec{\delta}}_2=\lambda_3\vec{c}^{q}-\lambda_4\vec{c}^{p},$$

with initial conditions $\Lambda(0)=E_{2N},\vec{\Delta}(0)=0$, with E_{2N} being the $2N\times2N$ identity matrix.

In tomography representation any operator $\hat{A}$ is replaced by the tomographic symbol w_A [31]. The center of mass tomography symbols w_I that correspond to the integrals of motion are given by

$$\vec{w}_I(t)=\int\delta(X-\vec{M}\cdot\vec{Q})[\Lambda(t)\vec{Q}+\vec{\Delta}(t)]\frac{d\vec{Q}}{(2\pi)^N}, \tag{16}$$

where $\vec{M}=\{\vec{\nu},\vec{\mu}\}$, or the same in the symplectic tomography:

$$\vec{w}_I^{s}(t)=\int\delta(\vec{X}-\vec{M}\cdot\vec{Q})[\Lambda(t)\vec{Q}+\vec{\Delta}(t)]\frac{d\vec{Q}}{(2\pi)^N}. \tag{17}$$

In the center of mass tomography one can find the invariant average values of the integrals of motion

$$\langle\vec{I}(t)\rangle=\Lambda(t)\langle\vec{Q}\rangle+\vec{\Delta}(t) \tag{18}$$

from the formulas

$$\langle p_i\rangle=\int Xw(X,\vec{\mu}=0,\nu_i=1,\tilde{\vec{\nu}}=0)dX,$$
$$\langle q_i\rangle=\int Xw(X,\mu_i=1,\tilde{\vec{\mu}}=0,\vec{\nu}=0)dX, \tag{19}$$

where the designation $a_i=1,\tilde{\vec{a}}=0$ means that all components of vector $\vec{a}$ equal zero, except a_i. It means that the integrals of motion are connected with the first moments of X, obtained from the one-variable (X) distribution, taken at *particular points in the space* $\{\vec{\mu},\vec{\nu}\}$.

III. PROPAGATORS

A. General relations

Let us now turn to the propagators that describe the evolution of the system independently of the initial state. We start from the propagator for the wave function Ψ,

$$\Psi(\vec{x},t)=\int G(\vec{x},\vec{x}',t,t')\Psi(\vec{x}',t')d\vec{x}', \tag{20}$$

or for the density matrix,

$$\rho(\vec{x},\vec{x}',t)=\int G(\vec{x},\vec{y},t,t')G*(\vec{x}',\vec{z},t,t')\rho(\vec{y},\vec{z},t')d\vec{y}d\vec{z}. \tag{21}$$

Standard meaning of the above propagator in Eq. (20) is that the propagator is transition probability amplitude from initial to final position.

The propagator $\Pi^f(z,z',t,t')$ for any state-describing function $f(z,t)$ allows us to obtain this function for any t from given $f(z,t')$:

$$f(z,t)=\int\Pi^f(z,z',t,t')f(z',t')dz'. \tag{22}$$

Below we designate the propagators in the center of mass tomography, symplectic tomography, and Wigner representation as Π, Π^s, and Π^W, respectively. Their connections with each other and relations to the function G are given by the following set of equations:

$$\Pi^W(\vec{q},\vec{p},\vec{q}',\vec{p}',t,t')=\int e^{i(\vec{p}'\cdot\vec{v}-\vec{p}\cdot\vec{u})}\times G\left(\vec{q}+\frac{\vec{u}}{2},\vec{q}'+\frac{\vec{v}}{2},t,t'\right)G*\left(\vec{q}-\frac{\vec{u}}{2},\vec{q}'-\frac{\vec{v}}{2},t,t'\right)\frac{d\vec{u}d\vec{v}}{(2\pi)^N}, \tag{23}$$

$$G(\vec{x},\vec{x}',t,t')G^*(\vec{y},\vec{y}',t,t') = \int e^{i[\vec{p}(\vec{x}-\vec{y})-\vec{p}'(\vec{x}'-\vec{y}')]}\Pi^W\left(\frac{\vec{x}+\vec{y}}{2},\vec{p},\frac{\vec{x}'+\vec{y}'}{2},\vec{p}',t,t'\right) \times \frac{d\vec{p}d\vec{p}'}{(2\pi)^N}, \tag{24}$$

$$\Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t') = \int \Pi^W(\vec{q},\vec{p},\vec{q}',\vec{p}',t,t')e^{-i[k(X-\vec{\mu}\cdot\vec{q}-\vec{\nu}\cdot\vec{p})+\vec{\mu}'\vec{q}'+\vec{\nu}'\vec{p}'-X']} \times \frac{d\vec{q}d\vec{q}'d\vec{p}d\vec{p}'dk}{(2\pi)^{2N+1}}, \tag{25}$$

$$\Pi^W(\vec{x},\vec{y},\vec{x}',\vec{y}',t,t') = \int \Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t')e^{-i[\vec{\mu}\cdot\vec{x}+\vec{\nu}\cdot\vec{y}-X+k'(X'-\vec{\mu}'\cdot\vec{x}'-\vec{\nu}'\cdot\vec{y}')]} \times \frac{dXdX'd\vec{\mu}d\vec{\nu}d\vec{\mu}'d\vec{\nu}'dk'}{(2\pi)^{2N+1}}, \tag{26}$$

$$\Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t') = \int e^{-i[k(X-\vec{\mu}\cdot\vec{q})+\vec{\mu}'\cdot\vec{q}'-X']}G\left(\vec{q}+\frac{k\vec{\nu}}{2},\vec{q}'+\frac{\vec{\nu}'}{2},t,t'\right)G^* \times\left(\vec{q}-\frac{k\vec{\nu}}{2},\vec{q}'-\frac{\vec{\nu}'}{2},t,t'\right)\frac{d\vec{q}d\vec{q}'dk}{(2\pi)^{N+1}}, \tag{27}$$

$$G(\vec{x},\vec{x}',t,t')G^*(\vec{y},\vec{y}',t,t') = \int e^{i[X-\vec{\mu}[(\vec{x}+\vec{y})/2]-k'(X'-\vec{\mu}'[(\vec{x}'+\vec{y}')/2])]} \times\Pi\left(X,\vec{\mu},\vec{x}-\vec{y},X',\vec{\mu}',\frac{\vec{x}'-\vec{y}'}{k'},t,t'\right)\frac{dXdX'd\vec{\mu}d\vec{\mu}'dk'}{|k'|^N(2\pi)^N}, \tag{28}$$

$$\Pi^s(\vec{X},\vec{\mu},\vec{\nu},\vec{X}',\vec{\mu}',\vec{\nu}',t,t') = \int e^{-i[\vec{k}\cdot\vec{X}-\vec{X}'\cdot\vec{e}-Y+k'Y']}\Pi\left(Y,\vec{k}\circ\vec{\mu},\vec{k}\circ\vec{\nu},Y',\frac{\vec{\mu}'}{k'},\frac{\vec{\nu}'}{k'},t,t'\right) \times\frac{dYdY'dk'd\vec{k}}{|k'|^{2N}(2\pi)^{N+1}}, \tag{29}$$

$$\Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t') = \int \frac{e^{-i(\vec{Y}'\cdot\vec{e}-X')}\delta(X-\vec{Y}\cdot\vec{e})}{\prod_{j=1}^N|k_j'|^3}\Pi^s\left(\vec{Y},\vec{\mu},\vec{\nu},\frac{\vec{Y}'}{\vec{k}'},\frac{\vec{\mu}'}{\vec{k}'},\frac{\vec{\nu}'}{\vec{k}'},t,t'\right) \times\frac{d\vec{Y}d\vec{Y}'d\vec{k}'}{(2\pi)^N}, \tag{30}$$

where $\vec{e}$ is the unit vector ($e_j=1$ for any j), and the components of vector $\vec{c}=\vec{a}/\vec{b}$ are given by $c_j=a_j/b_j$.

Physical meaning of tomographic propagators both in symplectic and center of mass tomography representations is as follows. The propagators are nonnegative transition probabilities from initial position in the symplectic tomography case and from initial center of mass position in the second case to the corresponding final position (final center of mass position in the second case). But the initial and final positions (centers of mass) are measured in ensembles of reference frames in the phase spaces of the system labeled by extra real parameters. The obtained formulas relate transition probability amplitudes describing quantum transitions (quantum evolution) with tomographic transition probabilities which also describe the quantum transitions in the probability representation.

Symplectic and center of mass tomograms are the homogeneous functions [30,31]. The same is true about the corresponding propagators, but only concerning the variables at "time slice" t:

$$\Pi^s(\vec{\lambda}\circ\vec{X},\vec{\lambda}\circ\vec{\mu},\vec{\lambda}\circ\vec{\nu},\vec{X}',\vec{\mu}',\vec{\nu}',t,t') = \frac{\Pi^s(\vec{X},\vec{\mu},\vec{\nu},\vec{X}',\vec{\mu}',\vec{\nu}',t,t')}{\prod_{j=1}^N|\lambda_j|}, \tag{31}$$

or, in particular,

$$\Pi^s(\alpha\vec{X},\alpha\vec{\mu},\alpha\vec{\nu},\vec{X}',\vec{\mu}',\vec{\nu}',t,t') = \frac{\Pi^s(\vec{X},\vec{\mu},\vec{\nu},\vec{X}',\vec{\mu}',\vec{\nu}',t,t')}{|\alpha|^N}. \tag{32}$$

The center of mass tomography propagator obeys the following relation:

$$\Pi(\alpha X,\alpha\vec{\mu},\alpha\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t') = \frac{\Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t')}{|\alpha|}. \tag{33}$$

These equations allow us to obtain the properties

$$\prod_{j=1}^N|X_j|^{-1}\Pi^s\left(\vec{e},\frac{\vec{\mu}}{\vec{X}},\frac{\vec{\nu}}{\vec{X}},\vec{X}',\vec{\mu}',\vec{\nu}',t,t'\right) = \prod_{j=1}^N|\mu_j|^{-1}\Pi^s\left(\frac{\vec{X}}{\vec{\mu}},\vec{e},\frac{\vec{\nu}}{\vec{\mu}},\vec{X}',\vec{\mu}',\vec{\nu}',t,t'\right) = \prod_{j=1}^N|\nu_j|^{-1}\Pi^s\left(\frac{\vec{X}}{\vec{\nu}},\frac{\vec{\mu}}{\vec{\nu}},\vec{e},\vec{X}',\vec{\mu}',\vec{\nu}',t,t'\right), \tag{34}$$

and

$$\frac{\Pi\left(1,\frac{\vec{\mu}}{X},\frac{\vec{\nu}}{X},X',\vec{\mu}',\vec{\nu}',t,t'\right)}{|X|} = \Pi(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t'), \tag{35}$$

that show that the quantum evolution even in the tomography representation is described by the functions of $2N$ variables.

B. Propagators for the system with quadratic Hamiltonians

It is convenient to begin the analysis of the center of mass tomography propagator for the system with quadratic Hamiltonian (1) from the propagator in the Wigner-Moyal representation [33]:

$$\Pi^W(\vec{Q},\vec{Q}',t,t') = \delta(\vec{Q}' - \vec{Q}_0(t,t')),$$

$$\vec{Q}_0(t,t') = \Lambda(t,t')\vec{Q} + \vec{\Delta}(t,t'), \tag{36}$$

where $\Lambda(t,t')$ and $\vec{\Delta}(t,t')$ are the matrix and vector given by Eqs. (15), as functions of t, with initial conditions $\Lambda(t=t',t')=E_{2N}, \vec{\Delta}(t=t',t')=0$.

Applying the transform (25) to the expression (36), one obtains the center of mass tomography propagator:

$$\Pi(X,\vec{M},X',\vec{M}',t,t') = \int e^{-i[kX+\vec{M}'\cdot\vec{\Delta}(t,t')-X']}\delta(k\vec{M} - \vec{M}'\Lambda(t,t'))\frac{dk}{2\pi},$$
$$\vec{M} = \{\vec{\nu},\vec{\mu}\}, \vec{M}' = \{\vec{\nu}',\vec{\mu}'\}. \tag{37}$$

From the first sight this expression seems quite inconvenient. But it becomes amazingly simple if we recall that the propagator Π does not have any meaning by itself: it must be utilized only in the context of the integral expressions, connecting the center of mass tomograms at times t' and t:

$$w(X,\vec{M},t) = \int \Pi(X,\vec{M},X',\vec{M}',t,t')w(X',\vec{M}',t')dX'd\vec{M}'$$
$$= \int e^{-i[kX+\vec{M}'\cdot\vec{\Delta}(t,t')-X']}\delta(k\vec{M} - \vec{M}'\Lambda(t,t'))\times w(X',\vec{M}',t')\frac{dX'd\vec{M}'dk}{2\pi}$$
$$= \int e^{-ik[X+\vec{M}'\cdot\vec{\Delta}(t,t')-X']}\delta(\vec{M} - \vec{M}'\Lambda(t,t'))\times w(X',\vec{M}',t')\frac{dX'd\vec{M}'dk}{2\pi}$$
$$= \int \delta(X'-X-\vec{M}'\cdot\vec{\Delta}(t,t'))\,\delta(\vec{M} - \vec{M}'\Lambda(t,t'))\times w(X',\vec{M}',t')dX'd\vec{M}'. \tag{38}$$

Here we used the homogeneity properties of the δ function and center of mass tomogram. Equation (38) means that for the integral expressions where the center of mass propagators are used, and for the functions such as center of mass tomograms, the propagator given by Eq. (37) is the same as the expression in the integral (38):

$$\Pi(X,\vec{M},X',\vec{M}',t,t') = \delta(X'-X-\vec{M}'\cdot\vec{\Delta}(t,t'))\,\delta(\vec{M} - \vec{M}'\Lambda(t,t')). \tag{39}$$

Let us consider the free motion as an example. Such a situation is described by the Hamiltonian (1) with $\vec{c}=\vec{0}$, $b_1=b_2=b_3=0$ and b_4 being the diagonal matrix with the diagonal elements equal to $\hbar^2/m$ (we consider the particles with equal masses m). Taking $\hbar=m=1$, we have $b_4=E$, the $N\times N$ identity matrix. In this case Eqs. (15) give the vector $\vec{\Delta}(t,t')$ and matrix $\Lambda(t,t')$ in the following form:

$$\vec{\Delta}(t,t') = 0,$$

$$\Lambda(t,t') = E_{2N\times 2N} - \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}(t-t').$$

Inserting these results in the expression (39), we obtain the center of mass tomography propagator for free motion:

$$\Pi_{Free}(X,\vec{\mu},\vec{\nu},X',\vec{\mu}',\vec{\nu}',t,t') = \delta(X'-X)\,\delta(\vec{\mu}'-\vec{\mu})\,\delta(\vec{\nu}'-\vec{\nu}-\vec{\mu}(t-t')), \tag{40}$$

and the same for the symplectic tomography:

$$\Pi^s_{Free}(\vec{X},\vec{\mu},\vec{\nu},\vec{X}',\vec{\mu},\vec{\nu},t,t') = \delta(\vec{X}'-\vec{X})\,\delta(\vec{\mu}'-\vec{\mu})\,\delta(\vec{\nu}'-\vec{\nu}-\vec{\mu}(t-t')). \tag{41}$$

C. Integral properties of the propagators

For given time moments t_1 and t_2 a propagator $\Pi_f(z_2,z_1,t_2,t_1)$ connects the state-describing function $f(z_1,t_1)$ at time t_1 with the same function at time t_2. For the coordinate, momentum, Wigner-Moyal, and many other representations there exists a fundamental property of the propagators:

$$\Pi_f(z_3,z_1,t_3,t_1) = \int \Pi_f(z_3,z_2,t_3,t_2)\Pi_f(z_2,z_1,t_2,t_1)dz_2. \tag{42}$$

Here we are going to show that the center of mass tomography propagators obey the same property. The argumentation is straightforward. Property (42) is known for the Wigner propagators Π^W. Applying the transform (25) to this expression, we have

$$\Pi(X_3,\vec{M}_3,X_1,\vec{M}_1,t_3,t_1) = \int e^{-i(kX'-X_1)}\times\delta(\vec{M}_1-k\vec{M}')\Pi(X_3,\vec{M}_3,X_2,\vec{M}_2,t_3,t_2)\times\Pi(X_2,\vec{M}_2,X',\vec{M}',t_2,t_1)\frac{dX_2d\vec{M}_2dX'd\vec{M}'dk}{2\pi}. \tag{43}$$

This integral expression again can be converted into a more simple form. This can be done in the same way as it was with obtaining expression (39) from Eq. (37). Considering Eq. (43) in the framework of the formula connecting $w(t_1)$ with $w(t_2)$, one has

$$\Pi(X_3,\vec{M}_3,X_1,\vec{M}_1,t_3,t_1) = \int \Pi(X_3,\vec{M}_3,X_2,\vec{M}_2,t_3,t_2) \times \Pi(X_2,\vec{M}_2,X_1,\vec{M}_1,t_2,t_1)dX_2 d\vec{M}_2. \tag{44}$$

The same is true for the symplectic tomography: the derivation of this property for the symplectic tomography propagators is the same as for the center of mass tomography ones.

Property (44) can be elucidated using the example of the system with quadratic Hamiltonian (1). In this case, analysis of Eq. (44) with the propagators given by Eq. (39) shows that the property (44) is analogous to the following set of equations:

$$\Lambda(t_2,t_1)\Lambda(t_3,t_2) = \Lambda(t_3,t_1), \tag{45}$$

$$\vec{\Delta}(t_3,t_1) = \Lambda(t_2,t_1)\vec{\Delta}(t_3,t_2) + \vec{\Delta}(t_2,t_1). \tag{46}$$

From Eq. (15) one can see that the requirement (46) is satisfied automatically if Eq. (45) is true. But requirement (45) is also satisfied because Eq. (15) for Λ are the first-order *linear* differential equations. This example shows that the evolution in the framework of quantum mechanics is closely related to the properties of propagators, and properties of the propagators and of the Hamiltonian are connected to each other.

IV. CONCLUSION

We have presented the new results concerning the description of quantum evolution of a multipartite system (transition from one state to another) in the framework of the center of mass tomography. These new results include the derivation of the center of mass tomography time evolution equation for a system with quadratic Hamiltonian (1), and detailed investigation of the integrals of motion, as they appear in the center of mass tomography formalism. We also studied the center of mass tomography propagator, its homogeneity properties, and the integral expressions connecting the propagators at successive time moments. We point out that in the center of mass tomography the quantum transitions are described by the transition probabilities and the complex transition probability amplitudes are connected with the transition probabilities by Eqs. (27) and (28). As an example, we considered in detail a system described by the quadratic Hamiltonian (1), and particularly the case of a free motion in a multipartite system. Expressions connecting the propagators in the center of mass and symplectic tomography, Wigner-Moyal, and Heisenberg-Schrödinger representations are given in the explicit form.

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