Photon-echo quantum memory with efficient multipulse readings

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(Received 8 May 2003; revised manuscript received 8 July 2004; published 15 December 2004)

We present a *photon-echo quantum memory technique* for manipulating quantum states of photons interacting with an atomic gas medium. The technique offers vast potential for the complete nonlocal-in-time multipulse reconstruction of a stored light as a superposition of echo fields, irradiated from the medium at different moments of time. Using this technique the dynamic control of quantum states is effectively possible by simply varying the laser parameters.

DOI: 10.1103/PhysRevA.70.063809 PACS number(s): 42.50.Gy, 03.65.Wj, 03.67.−a

I. INTRODUCTION

There has been significant interest in the development of an optical *quantum memory* (QM) technique motivated by topical subjects of quantum information and communications [1–3], particularly in investigations of fundamental aspects of quantum entanglement and nonlocality based on macroscopic objects [4–7]. In comparison with a classical optical memory the QM technique should preserve very fragile quantum correlations of light from the irreversible disappearance both for the weak photon fields and for the relatively intensive light pulses. Optical QM processes have already been demonstrated with quantum fields interacting with single atoms (molecules) [8,9] and macroscopic coherent media [10–20], where the unitary evolution can be used for the reconstruction of the stored light.

Manipulation of quantum states of arbitrary fields with either multiple photons or a single photon is one of the principal topics of research in the development of QM techniques. A macroscopic coherent system may play a key role in QM research of lights with arbitrary spectral and temporal profiles. A key interest of the QM technique is related to quantum information science in which entangled quantum states of light can be stored and retrieved on demand. Recently, QM techniques based on *electromagnetically induced transparency* (EIT) [14] have been demonstrated using both fast light [15] and slow light properties [16–19]. The fundamental concept behind the EIT QM techniques is based on the storage of optical information in dark atomic states excited by the resonant Raman fields whose photons are trapped in a small volume of the medium. The EIT technique utilizing quantum properties of the medium such as absorption and dispersion allows one to achieve reconstruction of the stored field with efficiency near 100%, as confirmed by the proof-of-principle experiments with intensive laser pulses in atomic vapors [16,17], Bose-Einstein condensate (BEC) atoms [18], and solid-state medium [19].

Currently, the main properties of the QM techniques are being intensively investigated (see Ref. [20] for recent review). Using traditional properties of the photon echoes [21,22] for storage of optical information in the spectral profile of atomic excitation [23,24], new ideas have been proposed for the implementation of QM techniques by modifying the photon-echo techniques [11–13,25]. These ideas rely on modified schemes of photon echoes in gaseous and solid-state resonant media [11,13] as well as a combination of photon echo with EIT effects [12,15]. In comparison with the EIT technique, photon-echo QM utilizes full absorption in an optically dense medium and a completely reversible reconstruction process between the light and medium. Such reversibility is realized by the control of atomic coherence dephasing in an inhomogeneously broadened system. Although currently there are no immediately relevant experiments based on these ideas, some indirect experiments on trilevel photon echoes [26,27] give a degree of credibility to the potential applications of the photon-echo QM technique.

Recently it has been shown that the EIT technique may effectively change the phase, polarization, frequency, and propagation direction of the reconstructed field with respect to that stored in the QM medium, where additional magnetic fields and reading laser fields with variable parameters are used [20,28]. Such manipulations do not involve any quantum specificity and can be performed with both classical and quantum fields, while some manipulations with quantum states of light, which are of interest for quantum communications, can be realized using the EIT technique [3,20].

In this paper we demonstrate how to manipulate quantum states of reconstructed light using photon-echo QM techniques. We also discuss weak field (or a single photon) QM using Doppler broadening in a gas medium. The weak field QM is important because interactions of a single photon with a macroscopic system should determine basic quantum optic phenomena such as quantum information processing and communications. It should be noted that investigations of these problems give rise to the development of new experimental techniques for single-photon wave-packet generation using various single atomic and molecular emitters [29].

This paper is organized as follows. In Sec. II, we introduce a physical model of the photon-echo QM technique. In Sec. III, we present a quantum mapping where a single-photon field interacts with a three-level gas medium. In Sec. IV, we analyze a complete on-demand reconstruction of the absorbed single-photon wave packet as a quantum superpo-

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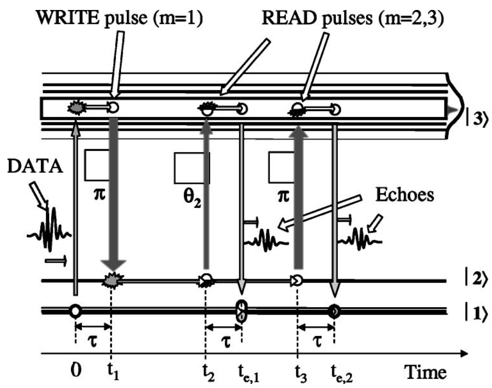

FIG. 1. A temporal diagram of quantum mapping, storage, and readout process. All atoms are initially on the ground level $|1\rangle$. The photon is absorbed by the inhomogeneously broadened atomic system. The READ pulse whose pulse area is π completely transfers the atomic excitations onto the level $|2\rangle$ after the time delay $\tau = t_1$. Two echo signals are irradiated after the action of the consecutive two READ pulses with time delays τ, while the atoms are transferred back onto the ground level $|1\rangle$. These two echo signals correspond to a single-photon state in the form of two wave packets. The temporal profile of the echo signals is reversed in time with respect to the initial photon.

sition of time separated multiple wave packets irradiated by the medium at different moments. In Sec. V, we discuss generalization of the proposed photon-echo QM technique and its connection to the slow-light-based EIT QM technique.

II. PHYSICAL MODEL

The presently conceptualized photon-echo QM technique can be realized in an inhomogeneously broadened atomic gas medium composed of Λ-type three energy levels [11]. Temporal and spatial schemes of the proposed photon-echo QM technique are shown in Figs. 1 and 2, respectively. As an initial condition, we assume that all atoms stay on the ground level $|1\rangle$. A single-photon wave packet resonant to the transition of $|1\rangle$-$|3\rangle$ as a DATA signal entering an optically dense atomic gas medium at $t=0$ will be fully absorbed. The photoexcited atoms to level $|3\rangle$ start to diphase due to Doppler broadening. This phenomenon is well known as free induction decay [30]. A WRITE pulse followed by the DATA signal with time delay t_1, where the WRITE pulse is resonant to the transition of $|2\rangle$ and $|3\rangle$, transfers the atomic excitations to the long-lived level $|2\rangle$. Here, the time interval t_1 should be comparable or longer than the inverse of inhomogeneous broadening Δ but less than the inverse of the optical coherence decay rate γ. Thus, quantum information of a single photon is mapped into the macroscopic coherence between the levels $|1\rangle$ and $|2\rangle$. Owing to the very slow decay process on level $|2\rangle$, the optical quantum information can be retained for a long period of time, $t \leqslant \gamma_2^{-1}$, where γ_2 is coherence decay rate of the state $|2\rangle$.

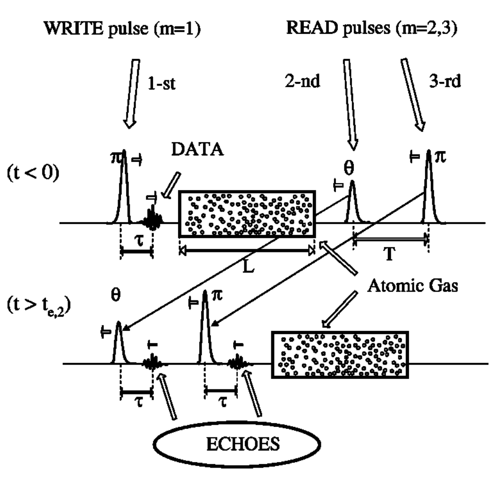

FIG. 2. A spatial diagram of the two-pulse quantum recovery. The first and second READ pulses propagate in the backward direction with respect to the direction of the DATA photon.

To retrieve the stored quantum information on level $|2\rangle$, a READ pulse resonant to the transition of $|2\rangle$-$|3\rangle$ with an opposite propagation direction to the WRITE pulse follows at $t = t_2$, where t_2 must be shorter than the decoherence time γ_2^{-1} of the level $|2\rangle$. At this point the reexcited atomic dipoles start to rephase, and the medium can then irradiate the photon of the data signal as an echo pulse in the opposite direction with respect to the DATA at $t = t_2 + t_1$. In an optically dense medium the photon emission probability can be close to unity if the time delay of the echo irradiation is small enough with respect to the typical time scale of the decoherent process of the system. In the read-out (reconstruction) process the interaction system gradually evolves to the final separable state through the entangled states of the atoms and field. Therefore, we can develop a reading process in this scheme by using multipulse laser excitations. We are able to show that the complete reconstruction of the stored information is also possible with the multipulse technique, which opens a door for manipulation with quantum state of a photon using classical fields.

In Fig. 1, the initial atomic state is represented as

$$|A(t)\rangle = \prod_{j=1}^{N} |\phi(t)\rangle_j, \tag{1}$$

where $|\phi(t)\rangle_j = \delta^{1/2}(r_j - r_j(t))|1\rangle_j$, $\delta(r_j - r_j(t))$ describes the classical movement of a jth pointlike atom in the gas with the spatial coordinate $r_j(t) = r_j^0 + v^j t$, v^j is the velocity of the jth atom, and N is the macroscopic number of the atoms. The optical dipole transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ are allowed to transit to the excited state $|3\rangle$, where the level $|2\rangle$ is initially unpopulated but energetically close to $|1\rangle$. The initial atomic state in Eq. (1) can be prepared by an optical pumping with a cw laser light resonant from the level $|2\rangle$ to the level $|3\rangle$.

We assume that a photon wave packet enters the gas tube at $t=0$ (see Figs. 1 and 2). The state of the photon and the atoms $|\psi_{\text{in}}(t)\rangle$ before interaction is introduced formally for time $t \to -\infty$ that closely corresponds to the physical situation for $t \ll \delta t_{\text{ph}}$ (δt_{ph} is the duration of the single photon wave packet), where the interaction of the photon with the medium is negligibly weak:

$$|\psi_{\text{in}}(t)\rangle\big|_{t\ll\delta t_{\text{ph}}} = |\psi_f(t)\rangle = |\varphi_{\text{in}}(t)\rangle|A(t)\rangle, \tag{2}$$

where $|\varphi_{\text{in}}(t)\rangle = \int dk f_k(t) a_k^\dagger|0\rangle$ is the well-known state of a photon wave packet, $dk|f_k(t)|^2$ is the probability to find a photon with a wave vector in the interval k and $k+dk$ at the moment of time t, $f_k(t\to-\infty)$ is a sharp function of k, peaking at $k=k_{\text{ph}}=\vec{e}_z\omega_{\text{ph}}/c$, and the spectral width $\delta\omega_{\text{ph}}\approx\delta t_{\text{ph}}^{-1}$. The function $f_k(t\to-\infty)$ is normalized to satisfy the condition $\int dk|f_k(t\to-\infty)|^2=1$; $|0\rangle$ is the vacuum state of light, and a_k and $a_k^\dagger$ are annihilation and creation field operators, respectively. For simplicity, we do not consider photon polarization dependence in the interactions with the atoms.

We describe the quantum evolution of the system with Hamiltonian interaction of the atoms and the quantum fields, in which the laser field is treated as classically:

$$H = H_a + H_f + V_{af} + \sum_{m=1}^{n} V_m(t), \tag{3}$$

$$H_a = \hbar\sum_{j=1}^{N}[\omega_{31}+\delta\omega_j(t)]P_{33}^j + \hbar\omega_{21}\sum_{j=1}^{N}P_{22}^j,$$

$$H_f = \int dk\hbar\omega_k a_k^\dagger a_k, \tag{4}$$

$$V_{af} = \hbar g\int dk\sum_{j=1}^{N}\{a_k P_{31}^j \exp(ikz_j) + \text{H.c.}\} \tag{5}$$

$$V_m(t) = -\frac{1}{2}\hbar\sum_{j=1}^{N}\Omega_m((t-t_m-n_m z_j/c)/T_m)\{P_{32}^j \times\exp[-i(\omega_m t - k_m z_j) + i\varphi_m] + \text{H.c.}\}, \tag{6}$$

where V_{af} is the interaction energy of the quantum field with atoms for the atomic transitions $|1\rangle_j\to|3\rangle_j$, g is the interaction constant of the photon with the jth atom [30]; $V_m(t)$ is the interaction energy of atoms with intensive WRITE-READ pulses, which are resonant to the transition $|2\rangle_j\to|3\rangle_j$; $\omega_{\nu\mu}$ is the central frequency of the atomic transitions $|\nu\rangle_j\to|\mu\rangle_j$; $\delta\omega_j(t)$ is the fluctuation on the optical transition frequency, which is determined by the interatomic collisions and independent from each jth atom in our consideration; $P_{\nu\mu}^j = |\nu\rangle_{jj}\langle\mu|$ is a jth atom operator coupling the states $|\nu\rangle_j$ to $|\mu\rangle_j$; $\Omega_m(t/T_m)=dE_m(t/T_m)/\hbar$ are Rabi frequencies; and T_m is the temporal duration of the mth laser pulse ($m=1$, WRITE pulse; $m>1$, READ pulse); $d=d_{32}=d_{23}$ is the dipole moment of the atomic transition $|2\rangle$-$|3\rangle$; $E_m(t)$, ω_m, and φ_m are the amplitude of the electric field, the carrier frequency, and the phase of the m-laser pulse, respectively; t_m is the time of the mth laser pulse acting on the medium; $n_m=k_m/|k_m|$; and $|k_m|=\omega_m/c$. Here we set the energy of the ground level, $E_1^j=0$. We assume $\hbar=1$ unless specifically stated otherwise.

A typical analysis of the three-dimensional spatial and temporal (r,t) dynamics [31] shows that the two-dimensional (z,t) model can be used for the time $t\ll a/v_n$ (v_n is average thermal velocity in the atomic gas, and a is the photon beam cross-section diameter in the xy plane). Consequently the atoms do not have enough time to leave the volume, while the probe quantum field (a single-photon wave packet) is being absorbed. Initial quantum state [Eq. (2)] and Hamiltonians [Eqs. (3)–(6)] determine the following structure of the wave function $|\psi(t)\rangle$ in the evolution:

$$|\psi(t)\rangle = |\psi_m(t)\rangle + |\psi_f(t)\rangle, \tag{7}$$

$$|\psi_m(t)\rangle = |\psi_m^{(2)}(t)\rangle + |\psi_m^{(3)}(t)\rangle, \quad |\psi_f(t)\rangle = \int dk f_k(t)a_k^\dagger|0\rangle|A(t)\rangle, \tag{8}$$

$$|\psi_m^{(2)}(t)\rangle = \sum_{j=1}^{N}\xi_j(t)P_{21}^j|0\rangle|A(t)\rangle,$$

$$|\psi_m^{(3)}(t)\rangle = \sum_{j=1}^{N}b_j(t)P_{31}^j|0\rangle|A(t)\rangle, \tag{9}$$

where the new values $b_j(t)$ and $\xi_j(t)$ are probability amplitudes of finding the jth atom in the state $|3\rangle_j$ or in the state $|2\rangle_j$.

Substituting Eqs. (7)–(9) into the Schrödinger equations leads to the following for $f_k(t)$, $b_j(t)$, and $\xi_j(t)$:

$$\frac{\partial}{\partial t}f_k(t) = -i\omega_k f_k(t) - ig\sum_{j=1}^{N}b_j(t)\exp\{-ikz_j(t)\} \tag{10}$$

$$\frac{\partial}{\partial t}b_j(t) = -i[\omega_{31}+\delta\omega_j(t)]b_j(t) - ig\int dk f_k(t)\exp\{ikz_j(t)\} + i\frac{1}{2}\xi_j(t)\sum_{m=1}^{n+1}\Omega_m\left(\left[t-t_m-\frac{1}{c}n_m z_j(t)\right]\Big/ T_m\right) \times\exp\{-i[\omega_m t - k_m z_j(t)] + i\varphi_m\}, \tag{11}$$

$$\frac{\partial}{\partial t}\xi_j(t) = -i\omega_{21}\xi(t)_j + i\frac{1}{2}b_j(t)\sum_{m=1}^{n+1}\Omega_m([t-t_m - n_m z_j(t)/c]/T_m)\exp\{i[\omega_m t - k_m z_j(t)] - i\varphi_m\}, \tag{12}$$

where $f_k(t\to-\infty)=f^{\text{in}}((\omega_k-\omega_{ph})/c)e^{-i\omega_k t}$ acts for a model.

Equations (10)–(12) are obtained from the Schrödinger equation by the multiplying $\langle A(t)|a_k$ or $\langle A(t)|P_{13}^j$ and integrated over all the atomic spatial coordinates $r_j\{j=1,2,3,\ldots,N\}$: $\int dr_1\ldots dr_N\langle A(t)|a_k$ and $\int dr_1\ldots dr_N\langle A(t)|P_{13}^j$.

The optical frequency fluctuations $\delta\omega_j(t)$ lead to the additional stochastic phase $\delta\varphi_j(t)=\int_0^t dt'\,\delta\omega_j(t)$ of the jth atom with average values $\langle\delta\varphi_j(t)\rangle_\gamma=0$. We consider that these fluctuations are slow enough with respect to the temporal duration of the light pulses that they should be considered first of all for the temporal intervals between the interactions with optical pulses. We use the following well-known formulas for the average values of the exponential functions including these fluctuations $\langle \exp\{-i\delta\varphi_j(t-t_0)\}\rangle_\gamma = \langle \exp\{-i\int_{t_0}^{t} dt'\,\delta\omega_j(t)\}\rangle_\gamma = \exp\{-\gamma(t-t_0)\}$, $\langle \exp\{-i\int_{t_2}^{t_2+\tau_2} dt'\,\delta\omega_j(t) - i\int_{t_1}^{t_1+\tau_1} dt'\,\delta\omega_j(t)\}\rangle_\gamma = \exp\{-\gamma(\tau_2+\tau_1)\}$.

In Sec. III, we will study the behavior of Eqs. (10)–(12) at the initial condition $b_j(t\to-\infty)=\xi_j(t\to-\infty)=0$. Note that Eqs. (10)–(12) describe the QM processes based on both EIT and photon echoes, where the QM process is affected by spatial, temporal, and spectral parameters of the photon wave packet and the classical laser pulses. Note that Eqs. (11) and (12) are transformed into the equations for the atomic coherence ρ_{13}^j and ρ_{12}^j if a weak classical probe field is considered instead of a single-photon field. This means that each of the two techniques can be used for QM with weak quantum fields.

III. QUANTUM MAPPING

Storing a photon state onto atomic states can be performed by complete resonant absorption for the atomic transition $|1\rangle \to |3\rangle$ in optically dense gases if the inhomogeneous (Doppler) broadening Δ_n is larger than the spectral width of the quantum light $\delta\omega_{\rm ph}:\Delta_n\rangle\delta\omega_{\rm ph}$. Due to the Doppler broadening, the complete absorption of the photon is accompanied by the fast dephasing of the excited atomic states, so that the subsequent coherent interactions between the atomic excitations and the light that is suppressed play a positive role for the indestructible storage of the quantum state of light.

Because a general analytical solution of Eqs. (10)–(12) is not yet known, we rewrite the equations in a simpler form, taking into account the absence of any overlapping of the WRITE and READ fields with the photon in the medium. The photon absorption and emission take place in the absence of overlapping with the laser fields: $\Omega_m(t)=0,\{m=1,\ldots,n,n+1\}$, which characterizes the considered photon echo of the QM technique. In this case the Eqs. (10) and (11) acquire the following form before the action of the WRITE pulse:

$$\frac{\partial}{\partial t} f_k = -i\omega_k f_k - ig\sum_{j=1}^{N}\beta_j \exp\{-ikz_j^0\}, \tag{13}$$

$$\frac{\partial}{\partial t}\beta_j = -i[\omega_{31}+\Delta_j+\delta\omega_j(t)]\beta_j - ig\mathcal{E}(t,z_j^0) \times\exp\{-i(\omega_{\rm ph}t - k_{\rm ph}z_j^0)\}, \tag{14}$$

where $\Delta_j=\omega_{31}(v_z^j/c)$ and $\beta_j(t,z_j^0;\Delta_j)=b_j(t,z_j^0;\Delta_j)\exp\{-i\Delta_j t\}$. We also use a new variable $\mathcal{E}(t,z)$ for the quantum field $\int_0^\infty dk f_k(t)e^{ikz}=\mathcal{E}(t,z)\exp\{-i\omega_{\rm ph}(t-z/c)\}$ with $\omega_{\rm ph}=\omega_{31}$ and $k=\omega_k/c$ in the chosen (z,t) model. Here we neglect the weak reflection of the photon $[f_{-|k|}(t)\cong 0]$, which results from the resonant interactions of the field with the atoms. We have also put $\exp\{i(k-\omega_{31}/c)v_z^j t\}\cong 1$ in the derivation of Eqs. (13) and (14), which limits the time scale of our theoretical approach by the inequality $t < t_{\rm echo}=\omega_{31}/(\delta\omega_{\rm ph}\alpha c)$ [31]. This condition means that atomic movement in the z direction cannot be any longer than the Beer's depth of the resonant absorption, so the excited atoms are localized in the volume where the quantum field was absorbed.

Multiplying Eq. (13) by e^{ikz} and integrating the product over $\int_{-\infty}^{\infty}dk$, we obtain the macroscopic equation for the function $\mathcal{E}(t,z)$ in the medium:

$$\left(\frac{\partial}{c\,\partial t}+\frac{\partial}{\partial z}\right)\mathcal{E}(t,z) = -i(2\pi g n_0/c)\exp\{i\omega_{\rm ph}(t-z/c)\} \times\langle\beta(t,z;\Delta)\rangle_\Delta, \tag{15}$$

$$\langle\beta(t,z;\Delta)\rangle_{\gamma,\Delta} = (Sn_0)^{-1}\sum_{j=1}^{N}\beta_j(t,z_j^0;\Delta_j;\delta\varphi_j(t))\,\delta(z-z_j^0), \tag{16}$$

where $n_0=N/(LS)$ is the atomic density, S is the cross section of the light beam, and Δ is frequency detuning from the center of the atomic transition $|1\rangle\leftrightarrow|3\rangle$. For the macroscopically large number of atoms we calculate the summation in Eq. (16) over the atoms using usual transfer to the continuum atomic distribution. Thus, we can replace the Eq. (16) as follows:

$$\langle\beta(t,z;\Delta)\rangle_{\gamma,\Delta}|_{N\to\infty}\to Q(z)\int_{-\infty}^{\infty} d\Delta\, G(\Delta/\Delta_n)\langle\beta(t,z;\Delta)\rangle_\gamma, \tag{16a}$$

where $G((\omega-\omega_{31})/\Delta_n)$ is spectral inhomogeneous broadening and the function $Q(z)$ determines the spatial size L of the gas tube with $Q(z)=1$ for $0<z<L$ and $Q(z)=0$ for $z<0$ and $z>L$. Note that the continuous distribution in Eq. (16a) is possible for the time interval t shorter than the lifetime of the excited optical atomic states, which is much longer than our timescale.

If the spectral width of the DATA pulse is much narrower than the inhomogeneous broadening, $\delta\omega_{\rm ph}\ll\Delta_n$ [12], we find that the simple solutions for Eqs. (14)–(16): the field decays exponentially in the medium $\mathcal{E}(t,z)=\mathcal{E}_0[t-(1/c)z]e^{-\alpha z}$ and $\beta_j=-ig\exp\{-i(\omega_{31}+\Delta_j)[t-(1/c)z_j^0]-\alpha z_j^0\}\tilde{\mathcal{E}}_0(\Delta_j,t_j)$, where $\tilde{\mathcal{E}}_0(\Delta_j,t_j)=\int_{-\infty}^{t_j}dt'\,\mathcal{E}_0(t')\exp\{i\Delta_j t'\}$ is a current spectrum of the field on the jth atom at $t_j=t-(1/c)z_j^0$, and $\mathcal{E}_0(t)=\exp\{i\omega_{ph}t\}\int_0^\infty dk f_k(t)$ is the temporal envelope of the wave packet in the entrance of the medium. We consider the optically dense medium $(\alpha L\gg 1)$ for all spectral components of light. The solution for the fields $[\mathcal{E}(t,z),f_k(t)]$ and atomic excitations β_j is given in Appendix A for arbitrary ratios of the field and atomic parameters. Immediately after the photon absorption these values are

$$b_j(t_j)|_{t\gg t_{\rm ph}} = \beta_j\exp\{i\Delta_j t\} = -i\beta^0(\Delta_j;z_j^0)\exp\{-i\delta\varphi_j(t_j) - i\omega_{31}t_j + i\Delta_j z_j^0/c\}, \tag{17}$$

$$\beta^0(\Delta_j;z_j^0) = (2\pi g/c)f^{\rm in}(\Delta_j/c)\exp\{-\alpha_{(+)}(\Delta_j)z_j^0\}, \tag{18}$$

$$\xi_j(t)=0,\quad f_k(t>\Delta_n^{-1},\delta\omega_{\rm ph}^{-1},\alpha^{-1}/c)\to 0, \tag{19}$$

where $\alpha_{(+)}(\Delta)=\alpha_0\langle(\gamma+i[\Delta'-\Delta])^{-1}\rangle_{\Delta'}$ [see also Eq. (16)], $\alpha_0=2\pi n_0 g^2/c$, and $\alpha_{(+)}(\Delta)=\mathrm{Re}\{\alpha_{(+)}(\Delta)\}+i\,\mathrm{Im}\{\alpha_{(+)}(\Delta)\}$, where $\mathrm{Im}\{\alpha_{(+)}(\Delta)\}$ reflects the dispersion of the field in the medium and determines the additional phase $\mathrm{Im}\{\alpha_{(+)}(\Delta_j)\}z_j$

for the jth atomic excitation with frequency detuning Δ_j.

Equations (17)–(19) describe the evolution of the excited atomic system before the action of the WRITE laser pulse for time $t < T_2^{\rm col}$, γ^{-1}. The WRITE pulse propagates along the z direction and acts upon the medium with the time delay t_1 ($t_1 < \gamma^{-1}$) (see Figs. 1 and 2). We use the temporal profile $E_1(t) = E_{1;0}\text{sech}\{(t-t_1)/T_1\}$ and the carrier frequency ω_1 resonant to the transition $|2\rangle \leftrightarrow |3\rangle$. Assuming the quantum field is in the vacuum state $[f_k(t) \cong 0]$, Eqs. (10)–(12) can be rewritten as follows:

$$\frac{\partial}{\partial t}b_j(t) = -i[\omega_{31} + \delta\omega_j(t)]b_j(t) + i\frac{1}{2}\xi_j(t)\Omega_{1,0}\text{sech}\left[\left(t - t_1 - \frac{1}{c}n_1 z_j(t)\right)\bigg/ \mathrm{T}_1\right]\exp\{-i(\omega_1 t - k_1 z_j(t)) + i\varphi_1\}, \tag{20}$$

$$\frac{\partial}{\partial t}\xi_j(t) = -i\omega_{21}\xi(t)_j + i\frac{1}{2}b_j(t)\Omega_{1,0}\text{sech}\{(t - t_1 - n_1 z_j(t)/c]/T_1\}\exp\{i[\omega_1 t - k_1 z_j(t)] - i\varphi_1\}. \tag{21}$$

The most preferable conditions of the photon-echo QM is realized if the WRITE pulse completely transfers the atomic excitations from the level $|3\rangle$ onto the long-lived level $|2\rangle$ without their spectral selection. Such situation takes place if the WRITE pulse area is $\theta_1 = \pi$ and a sufficiently short pulse duration t_w less than $\delta\omega_{\rm ph}^{-1}$ (i.e., a large spectral width as compared to the spectrum of the stored field $\delta\omega_{ls,1} \approx t_w^{-1} \gg \delta\omega_{\rm ph}$). It is necessary obviously to exclude any transfer of the atomic excitation onto the level $|1\rangle$ by the WRITE pulse. For this, the energy splitting between the two lowest levels $|1\rangle$ and $|2\rangle$ must be larger than the WRITE spectral widths $\omega_{21} \gg \delta\omega_{ls,1,2,\ldots}$. Solution of Eqs. (20) and (21) in such conditions gives the following values for the atomic amplitudes after the action of the WRITE pulse with a pulse area of π (see Appendix B):

$$b_j(t) = 0, \quad \xi_j(t) = \beta^0(\Delta_j; z_j^0)\exp\{-i\omega_{21}t - i\delta\varphi_j(t_1) + i\phi_j\} \tag{22}$$

and $f_k(t) = 0$, where ϕ_j is the additional phase of the jth atom:

$$\phi_j = \omega_{21}(1 + \nu_z^j/c)(z_j^0/c) - (\nu_z^j/c)\omega_{32}t_1 - \varphi_1 \cong \omega_{21}z_j^0/c - \Delta_j t_1 - \varphi_1, \tag{23}$$

where we have also assumed that $(\nu_z^j/c)\omega_{32}t_1 = \Delta_j t_1(\omega_{32}/\omega_{31}) \cong \Delta_j t_1$ for $\omega_{21} \ll \omega_{31}$. The properties of atomic excitations $\xi_j(t)$ on the level $|2\rangle$ are determined by the spectrally varied amplitude $\beta^0(\Delta_j; z_j^0)$, which has the same exponential spatial behavior if the stored quantum field spectrum is narrower than the inhomogeneous broadening ($\delta\omega_f < \Delta_n$). Such type of quantum memory is completely different from the slow-light-based EIT QM technique, where the quantum information about the DATA pulse is recorded in the spatial profile of the atomic excitations $\xi_j(t, z_j)$ (polaritons). Due to this reason, the mechanism of stored field retrieval is also different from the slow-light-based EIT QM technique mechanism. For the reconstruction of the stored field in the slow-light-based EIT QM technique, it is realizable by applying only one control field to the atomic transitions, while short π pulse or multipulses are used in the proposed photon-echo QM scheme.

IV. TWO-PULSE MANIPULATIONS WITH A STORED PHOTON

Initially we considered the reconstruction of the stored DATA field by the first READ pulse (see Figs. 1 and 2), which acts upon the medium at $t = t_2$ and has the pulse area $\theta_2 \neq \pi$. The pulse propagates along the "$-z$" axis, opposite to the flight direction of the DATA photon. The READ pulse action will create a wave of polariton excitation in the atomic system propagating in the $-z$ direction. At the same time we assume that the atoms move in the gas tube without collisions and that the atomic dipoles get an opposite Doppler shifts to $-z$ direction with respect to the shifts which they had before the interaction with the WRITE pulse and the DATA photon. Therefore, the subsequent phase evolution of the atomic dipoles irradiating the field in the $-z$ direction will have the temporal behavior $d_j(t) \sim d_{31}\exp\{i\Delta_j(t - t_2) - i\Delta_j t_1\}$. Thus the atomic coherence will be recovered at $t = t_{e,1} = t_1 + t_2$ and resulting in photon echo irradiation in the opposite direction with respect to the initial DATA photon. The nontrivial physical aspect of such collective atomic irradiation takes place in the high optical dense medium, where the weak optical fields are usually strongly absorbed. In the present condition the echo signal is irradiated from the medium without reabsorption. Such emission is due to coherence retrieval of rephasing process, where each atom participates with the same efficiency both in the initial absorption and in the subsequent irradiation of the echo signal. Thus, the evolution stops leads at the ground level $|1\rangle$ with a complete phase recovery. This physical picture can be clarified additionally by the general physical explanation of the irradiation process of the echo signal as a completely reversible process. The action of all laser pulses plays the role of the Maxwell's demon, rotating the phases of the all atoms in the appropriate manner in order to revive both the atomic coherence and all the interactions between the atoms and field which are reversible in time. Since such coherence reversibility seems to be possible in general for multiparticle systems, it is worthwhile asking the following question: how may be repeatedly realize such reversibility without using a new photon?

To answer this question, let us consider the properties of the first echo irradiation [11], which appears after the first read pulse. The atomic state just after the first read pulse ($m = 2$), taking into account the initial conditions of Eqs. (18), (22), and (23) [see Eqs. (B3)–(B5)] is as follows. The amplitudes $b_j(t)$ and $\xi_j(t)$ for time $t > t_2 + T_2$ are

$$b_j(t)_{t>t_2+T_2} = b_j^0(t)\exp\left\{-i\left[\delta\varphi_j(t_1) + \delta\varphi_j\left(t - t_2 + \frac{1}{c}z_j\right)\right]\right\},$$

$$b_j^0(t) = i\,\sin\left(\frac{1}{2}\theta_2\right)\text{sech}(\pi\Delta_j T_2/2)\beta^0(\Delta_j, z_j^0)\exp\{i\mu_j(t)\}, \tag{24}$$

$$\xi_j(t)|_{t>t_2+T_2} = {}_2F_1((2\pi)^{-1}\theta_2, -(2\pi)^{-1}\theta_2; \gamma^1_{j,2}; 1)\beta^0(\Delta_j; z^0_j) \times \exp\{-i\omega_{21}t - i\delta\varphi_j(t_1) + i\varphi_j\}, \qquad (25)$$

$$\mu_j(t) = -\omega_{31}\left(t + \frac{1}{c}z^0_j\right) + 2\frac{1}{c}\omega_{21}z^0_j - \Delta_j\left(t_{e,1} - \frac{1}{c}z^0_j\right) + \varphi_{2,1}, \qquad (26)$$

where $\varphi_{2,1}=\varphi_2-\varphi_1$. Optical irradiation by the medium will be determined by the coherent properties of the atomic excitations $b_j(t)$. The new atomic phase $\mu_j(t)$ in Eq. (26) includes the term $-\omega_{31}(t+z^0_j/c)$, which reflects the appearance of the coherence polariton propagating in the backward z direction. Such polariton can irradiate the light coherently in the medium, if the term $2(1/c)\omega_{21}z^0_j$ in $\mu_j(t)$ does not suppress the phase matching in the irradiating volume of the medium. We have considered the echo irradiation, using for convenience new notation for the backward field modes $\vec{k}=-k\vec{e}_z$ and $\omega_{-k}=\omega_k$; thus, Eqs. (13) and (14) take the form $(t>t_2+T_2)$

$$\frac{\partial}{\partial t}f_{-k} = -i\omega_k f_{-k} - ig\sum_{j=1}^{N}\eta_j e^{ikz^0_j}, \qquad (27)$$

$$\frac{\partial}{\partial t}\eta_j = -i[\omega_{31} - \Delta_j + \delta\omega_j(t)]\eta_j - igF(t,z^0_j) \times \exp\{-i\omega_{31}(t + z^0_j/c)\}, \qquad (28)$$

where

$$\eta_j(t) = b_j(t)\exp\{i\Delta_j t\},$$

$$\eta^0_j(t) = b^0_j(t)\exp\left\{-i\left[\delta\varphi_j(t_1) + \delta\varphi_j\left(t - t_2 + \frac{1}{c}z^0_j\right)\right]\right\}\exp\{i\Delta_j t\}, \qquad (29)$$

$$\int_0^\infty dk f_{-k}(t)e^{-ikz} = F(t,z)\exp\{-i\omega_{31}(t + z/c)\}, \qquad (30)$$

where $\eta^0_j(t)$ corresponds to the atomic excitation after the first reading pulse in the new notations; $F(t,z)$ is an unknown slowly varying envelope of the irradiated field with initial condition $F(t=t_2+T_2,z)=0$. Taking into account Eqs. (28)–(30) we obtain the equation for the field envelope $F(t,z)$ similarly to the derivation of Eq. (15) at the photon mapping stage:

$$\left(\frac{\partial}{c\,\partial t} - \frac{\partial}{\partial z}\right)F(t,z) = A_1(t,z) + A_2(t,z) = -ig^{-1}\alpha_0\exp\{i\omega_{31}(t + z/c)\}\langle\eta^0(t,z;\Delta)\rangle_{\gamma\Delta} - \alpha_0\int_{-\infty}^{t}dt' F(t',z)\langle\exp\{i\Delta(t - t') - i\delta\varphi(t - t')\}\rangle_{\gamma,\Delta}. \qquad (31)$$

From Eq. (31) we extend the integration from the interval $\int_{t_2+T_2}^{t}dt\cdots$ to $\int_{-\infty}^{t}dt\cdots$ assuming that $F(t<t_2+T_2,z)=0$.

The first term A_1 on the right side of Eq. (31) is determined by the initial excitation of the medium [see Eq. (29)] and the second integral term $[A_2 \sim \int^t dt' F(t',z)\cdots]$ is the result of the echo field action on the medium that gives the polarization responsible for the irradiation absorption at usual conditions (see Appendix C). Using Eqs. (24), (29), and (18) we find A_1:

$$A_1(t,z) = (2\pi\alpha_0/c)\sin\left(\frac{1}{2}\theta_2\right)\exp\left\{-\gamma\left(t - t_2 + t_1 + \frac{1}{c}z\right)\right\}\exp\left\{i\left(2\frac{1}{c}\omega_{21}z + \varphi_{2,1}\right)\right\} \times\left\langle \operatorname{sech}\left(\frac{\pi}{2}\Delta T_2\right)f^{\,\mathrm{in}}\left(\frac{1}{c}\Delta\right)\exp\{-\alpha_{(+)}(\Delta)z\} \times\exp\left\{i\Delta\left(t - t_{e,1} + \frac{1}{c}z\right)\right\}\right\rangle_\Delta. \qquad (32)$$

Averaging Eq. (32) over the inhomogeneous broadening is proportional to the value $\langle\exp\{i\Delta[t-t_{e,1}+(1/c)z]\}\cdots\rangle_\Delta$; i.e., the polarization $A_1(t,z)$ will be recovered at the moment of time $t=t_{e,1}-(1/c)z$ that will be a source of the echo signal emission (see Fig. 2). The analytical solution of Eqs. (31) and (28) with initial condition of Eq. (29) is given in Appendix C. In the case of a spectrally narrow probe field $(\gamma, \delta\omega_{\mathrm{ph}} \ll T_m^{-1}, \Delta_n)$ we have the following equation for the field's envelope $F(t,z)$:

$$-\frac{\partial}{\partial z}F(\varsigma,z) = Q(z)\left\{\alpha\sin\left(\frac{1}{2}\theta_2\right)\exp\left(-\frac{1}{2}\alpha z - 2\gamma t_1 + i\varphi_{2,1} + 2i\frac{1}{c}\omega_{21}z\right)\mathcal{E}_0(t_{e,1} - \varsigma)e^{-\gamma(\varsigma - t_{t,1})} - \frac{1}{2}\alpha F(\varsigma,z)\right\}, \qquad (33)$$

where $\varsigma=t+(1/c)z$ is independent coordinate in the equation, which clearly shows the main temporal and spatial properties of the echo signal emission. The signal is generated in the medium by the atomic excitation whose amplitude increases exponentionally $[\sim\exp(-\frac{1}{2}\alpha z)]$ from $z=L$ to $z=0$. The term $-\frac{1}{2}\alpha F(\varsigma,z)$ in Eq. (32) gives the exponential absorption $\sim\exp(\frac{1}{2}\alpha z)$ for the field propagation without any initial coherence of the medium. Such behavior of these two exponents plays a crucial role for the complete emission of the excited energy from the atomic system. The solution of Eq. (33) is

$$F(t,z) = \frac{\sin\left(\frac{1}{2}\theta_2\right)}{[1 - 2i(1/c)\omega_{21}\alpha^{-1}]}\exp\left\{-2\gamma t_1 + i\varphi_{2,1} - Q(z)\left(\frac{1}{2}\alpha z - 2i\frac{1}{c}\omega_{21}z\right)\right\}\mathcal{E}_0\left(t_{e,1} - t - \frac{1}{c}z\right)e^{-\gamma(t - t_{e,1} + z/c)}. \qquad (34)$$

Thus, we see that the echo field amplitude increases exponentially $[\sim\exp(-\frac{1}{2}\alpha z)]$ from $z=L$ to $z=0$ and which revives the initial process of the photon absorption and temporal profile of the reconstructed field, which is temporally reversed with respect to the initial wave-packet envelope.

Complete reconstruction of the initial photon is limited by the value $\sin(\frac{1}{2}\theta_2)$, phase relaxation term $\exp\{-2\gamma t_1\}$, and the factor $[1-2i(1/c)\omega_{21}\alpha^{-1}]^{-1}$. The last term is determined by the influence of the phase matching conditions on the echo signal irradiation. A most interesting case occurs when the absorption coefficient α is sufficiently large, but the frequency splitting between the two lowest-energy levels ω_{21} is enough small. In such a case the phase matching will only be satisfied with any accuracy if $[1-2i(1/c)\omega_{21}\alpha^{-1}]^{-1}\approx 1$, which is possible for real gases with $\alpha \approx 1\ \text{cm}^{-1}$.

A general solution of Eqs. (28), (29), and (31) for the large spectral widths of the stored field $(\delta\omega_{\text{ph}}, T_m^{-1} \gg \gamma)$ is given at Appendix C using the Fourier representation $\widetilde{F}(\Delta,z)=(2\pi)^{-1}\int_{-\infty}^{\infty}dtF(t,z)e^{i\Delta(t+z/c)}$:

$$\widetilde{F}(\Delta,z) \cong \frac{\sin\left(\frac{1}{2}\theta_2\right)}{c\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(-\Delta)\right\}}\exp\{-2\gamma t_1 + i(\Delta t_{e,1} + \varphi_{2,1})\}\exp\left\{-Q(z)\left(\alpha_{(+)}(-\Delta) - 2i\frac{1}{c}\omega_{21}\right)z\right\}\text{sech}\left(\frac{\pi}{2}\Delta T_2\right)f^{\text{ in}}\left(-\frac{1}{c}\Delta\right), \qquad (35)$$

where $\alpha(-\Delta)=2\pi\alpha_0 G(-\Delta/\Delta_n)$ is the absorption coefficient at the frequency detuning $-\Delta$. Using Eq. (35) we find the solution for the field function $f^{(1)}_{-\omega/c}(t)(\omega=\omega_{31}+\Delta)$ after echo irradiation $(t>t_{e,1})$:

$$f^{(1)}_{-(\omega_{31}+\Delta)/c}(t>t_{e,1}) = c\widetilde{F}(-\Delta,z)_{z<0} \cong \frac{\sin\left(\frac{1}{2}\theta_2\right)\text{sech}(\pi T_2\Delta/2)}{\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(-\Delta)\right\}}\exp\{-2\gamma t_1 - i(\omega_{31} + \Delta)t + i(\Delta t_{e,1} + \varphi_{2,1})\}f^{\text{ in}}(-\Delta/c), \qquad (36)$$

with the probability to find a photon $P_{\text{ph}}(t>t_{e,1})$ in this echo signal:

$$P_{\text{ph}}(t>t_{e,1}) = \frac{1}{c}\int_{-\infty}^{\infty}d\Delta\left|f^{(1)}_{-(\omega_{31}+\Delta)/c}(t>t_{e,1})\right|^2\Big|^{\alpha(\Delta)\approx\alpha}_{T_2\delta\omega_{ph}\ll 1} = \frac{\sin^2\left(\frac{1}{2}\theta_2\right)\exp\{-4\gamma t_1\}}{\{1+4\omega_{21}^2(\alpha c)^{-2}\}}, \qquad (37)$$

where we have assumed a short enough reading laser pulse $(\text{sech}[(\pi/2)\delta\omega_{\text{ph}}T_2]\cong 1)$ with the assumption $\alpha(\Delta)\approx\alpha$ and the normalization of the initial state of light $(1/c)\int_{-\infty}^{\infty}d\Delta|f^{\text{ in}}(-\Delta/c)|^2=1$. As follows from Eqs. (36) and (37) the high-efficiency reconstruction of the initial photon state will take place for, and if, the phase matching condition fulfills all the reading of the spectral components $2\omega_{21}(\Delta) \ll c\alpha$ (where $-\delta\omega_{\text{ph}}/2 \leq \Delta \leq \delta\omega_{\text{ph}}/2$). The numerical analysis in Ref. [25] has shown that the condition of Eq. (34) can be fulfilled for real experimental parameters within the spectral interval $\delta\omega_{\text{ph}}\approx 10^8\ \text{s}^{-1}$. As seen in Eq. (36), we must stress that the dispersion effects, take place at the initial photon absorption and then in the echo signal irradiation completely compensate each other in the reconstructed field (for $z<0$). This effect is a direct consequence of the perfect reversibility of the echo process.

Using the solution of $F(t,z)$ we can analyze the behavior of the atomic excitation $\eta_j(t)$. The solution for the atomic parameters just after the echo pulse irradiation $(t>t_{e,1})$

$$\eta_j(t) = \eta_j^0(t) - \eta_j^1(t) \cong \left(\exp\left\{-i\left[\delta\varphi_j(t_1) + \delta\varphi_j\left(t - t_2 + \frac{1}{c}z_j^0\right)\right]\right\} - \frac{\exp\left[-i\delta\varphi_j\left(t + \frac{1}{c}z - t_{e,1}\right)\right]}{\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(\Delta)\right\}}\exp(-2\gamma t_1)\right) \times b_j^0(t)\exp\{i\Delta_j t\}, \qquad (38)$$

where $-\eta_j^1(t)$ is the additional term in the atomic excitation due to the interaction with the echo signal. After averaging over the inhomogeneous broadening and phase fluctuations we find the probability $P_{33}(t>t_{e,1})$ to find the atomic system on the excited state (see Appendix C)

$$P_{33}(t>t_{e,1}) = \sum_{j=1}^{N}\eta_j(t) = \sum_{j=1}^{N}\{|\eta_j^0(t)|^2 + |\eta_j^1(t)|^2 - (\eta_j^0(t))^*\eta_j^1(t) - (\eta_j^0(t))(\eta_j^1(t))^*\} = Sn_0\int_0^L dz\int_{-\infty}^{\infty}d\Delta G(\Delta/\Delta_n)\left\{1 - \frac{\exp(-4\gamma t_1)}{\{1+4\omega_{21}^2(\alpha(\Delta)c)^{-2}\}}\right\} \times |b^0(t,z,\Delta)|^2\Big|^{\alpha(\Delta)\cong\alpha(0)=\alpha}_{\alpha L>1} = \sin^2(\theta_2/2)\left(1 - \frac{\exp(-4\gamma t_1)}{1+4\omega_{21}^2(\alpha c)^{-2}}\right). \qquad (39)$$

Thus for a very slow phase relaxation $4\gamma t_1 \ll 1$ and for a good phase matching condition $4\omega_{21}^2(\alpha c)^{-2}\ll 1$, we have found that $P_{33}\approx 0$; i.e., all atomic excitations will be transferred from level $|3\rangle$ onto the ground level $|1\rangle$ with the probability for the echo field emission being $P_{\text{ph}}\approx\sin^2(\theta_2/2)$. Such behavior of atomic systems and their fields can be explained only if we cannot separate atoms one from another in their interactions with the echo signal, where the one group of atoms stay initially on the level $|1\rangle$ while the another group of atoms is excited on the level $|3\rangle$ before the echo emission. Thus, we have an example of coherent atomic behavior where the interaction with the echo signal cancels the field absorption due to the perfect phase matching between the field and atomic phases.

Below we have considered only the most preferable conditions $4\gamma t_1 \ll 1$ and $4\omega_{21}^2(\alpha c)^{-2} \ll 1$ for the read process of the stored state. After echo signal emission, the wave function of the field plus the atoms will be [see Eqs. (8), (9), (22), (23), and (36)]

$$|\psi(t)\rangle|_{t>t_{e,1}} = |\psi_m^{(2)}(t)\rangle + |\psi_f^{(1)}(t)\rangle,$$

$$|\psi_f^{(m)}(t)\rangle = \int dk f_{-k}^{(m)}(t) a_{-k}^{+}|0\rangle|A(t)\rangle. \quad (40)$$

Thus some atomic excitations in Eq. (40) still exist only on the level $|2\rangle$. We can initiate a similar emission of a new photon echo signal by applying the second READ pulse at time $t=t_3>t_{e,1}$. Let us assume that this laser pulse also propagates in the backward z direction. The evolution of the total quantum system in the laser pulse can be described by Eqs. (18) and (19) with the new initial conditions of Eqs. (25), (36), and (40) [i.e., $b_j(t)=0$, $\xi_j(t)\neq 0$]. Following the solutions of Appendixes B and C and ignoring the frequency fluctuations ($\gamma t \ll 1$), and assuming the phase matching fulfilment we may find the wave function after the second echo irradiation,

$$|\psi(t)\rangle = |\psi_m^{(2)}(t)\rangle + \sum_{m=1}^{2} |\psi_f^{(m)}(t>t_{e,m})\rangle, \quad (41)$$

where

$$b_j(t) \cong 0, \xi_j(t)|_{t>t_3+T_3} = {}_2F_1(\theta_3/(2\pi), -\theta_3/(2\pi); \gamma_3^{(1)}(\Delta); 1)\, {}_2F_1(\theta_2/(2\pi), -\theta_2/(2\pi); \gamma_2^{(1)}(\Delta_j); 1)\beta(\Delta_j; z_j^0)\exp\{-i\omega_{21}t + i\phi_j\}, \quad (42)$$

$$f_{-(\omega_{31}+\Delta)/c}^{(2)}(t)_{t>t_{e,2}} = \frac{\sin(\theta_3/2)}{\cosh(\pi\Delta T_3/2)}\, {}_2F_1(\theta_2/(2\pi), -\theta_2/(2\pi); \gamma_2^1(\Delta); 1) f^{\,\mathrm{in}}(-\Delta/c)\exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{31}+\Delta t_{e,3})\}. \quad (43)$$

Let us suppose that the second READ pulse has the pulse area $\theta_3=\pi$ and its spectral width is larger than the photon spectrum width ($\delta\omega_3 \sim T_3^{-1} \gg \delta\omega_{\mathrm{ph}}$); thus, we obtain

$$\frac{\sin(\theta_3/2)}{\cosh(\pi\Delta\chi T_3/2)}_{\Delta T_3 \ll 1} \cong \sin(\theta_3/2)|_{\theta_3=\pi} = 1,$$

$$ {}_2F_1((2\pi)^{-1}\theta_3, -(2\pi)^{-1}\theta_3; \gamma_3^{(1)}(\Delta); 1)|_{\Delta T_3 \ll 1} \cong \cos(\theta_3/2)|_{\theta_3=\pi} = 0. \quad (44)$$

With such parameters of the READ pulse we obtain

$$f_{-(\omega_{31}+\Delta)/c}^{(2)}(t)|_{t>t_{e,2}} = {}_2F_1(\theta_2/(2\pi), -\theta_2/(2\pi); \gamma_2^{(1)}(\Delta); 1) \times f^{\,\mathrm{in}}(-\Delta/c)\exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{31}+\Delta t_{e,3})\}, \quad (45)$$

$$\xi_j(t)|_{t>t_3+T_3} = 0, \quad (|\psi_m^{(2)}(t)\rangle = 0). \quad (46)$$

Thus the final wave function, Eq. (41), evolves from Eq. (40) to the following quantum superposition of the two photon wave packets [see Eqs. (36) and (45)]:

$$|\psi(t)\rangle = \sum_{m=1}^{2} |\psi_f^{(m)}(t>t_{e,m})\rangle, \quad (47)$$

where, using Eqs. (36) and (45), we find

$$\langle\psi(t)|\psi(t)\rangle \cong \sum_{m=1}^{2} \langle\psi_f^{(m)}(t>t_{e,m})|\psi_f^{(m)}(t>t_{e,m})\rangle = 1. \quad (48)$$

Equations (47) and (48) show that a photon is reradiated by the medium with a probability close to one in the two wave packets separated in time by the value $t_{e,2}-t_{e,1}=t_3-t_2 \gg \delta t_{\mathrm{ph}}$, which is much larger than the wave packet durations. If the spectrally broadened first read pulse has the pulse area of $\theta_2 \neq \pi$, we obtain the amplitudes of the two echo signals using formulas (44):

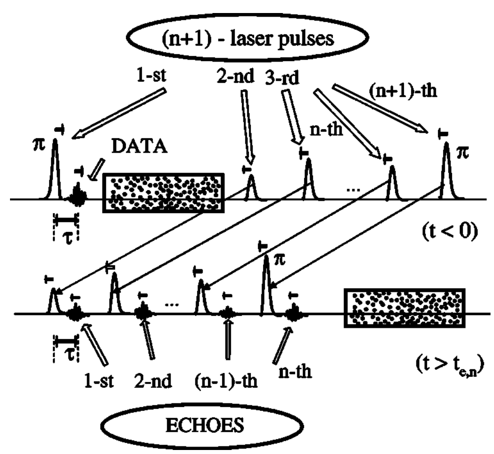

FIG. 3. A general scheme of n-pulse quantum memory with irradiation of a photon in a sequence of n echo fields. The amplitude and the phase of each echo signal can be controlled by controlling the READ pulses.

$$f^{(1)}_{-\omega/c}(t>t_{e,1}) \cong \sin(\theta_2/2) f^{\text{in}}(-\Delta/c)\exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{21}+\Delta t_{e,1})\}, \quad (49)$$

$$f^{(2)}_{-\omega/c}(t>t_{e,2}) \cong \cos(\theta_2/2) f^{\text{in}}(-\Delta/c)\exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{31}+\Delta t_{e,3})\}, \quad (50)$$

where the terms $\sin(\theta_2/2)$ and $\cos(\theta_2/2)$ determine the amplitude probability of the echo signal, and the terms $\exp\{i(\varphi_{21}+\Delta t_{e,1})\}$ and $\exp\{i(\varphi_{31}+\Delta t_{e,2})\}$ determine the phase and the time $t_{e,1}$ of these signal emissions. Thus Eqs. (49) and (50) show how we can control the phases and the amplitudes of the two wave packets by varying the pulse area θ_2 of the first readout laser pulse and the laser field phase φ_1, φ_2, and φ_3.

V. GENERALIZATION AND DISCUSSION

We can now generalize the obtained results for the case of $n+1$ READ pulses. In a similar way to the previous section we obtain the solution for the wave function $|\psi(t)\rangle$ after the $(n+1)$th pulse excitation with the irradiation of n echo signals for $t>t_{e,n}$ (see Fig. 3) [$t_{e,m}=t_{m+1}+t_1$ is the time of the echo signal emission, where $m=1$ in Eqs. (35) and (38)] (see also Eqs. (42) and (43)]:

$$|\psi(t)\rangle = |\psi_m^{(2)}(t)\rangle + \sum_{m=1}^{n} |\psi_f^{(m)}(t)\rangle,$$

where

$$b_j(t)=0,$$

$$\xi_j(t)|_{t>t_3+T_3} = \prod_{m=2}^{n+1} {}_2F_1(\theta_m/(2\pi), -\theta_m/(2\pi); \gamma^{(1)}_{j,m}; 1)\beta(\Delta_j; z_j^0) \times \exp\{-i\omega_{21}t + i\phi_j\}, \quad (51)$$

$$f^{(n)}_{-(\omega_{31}+\Delta)/c}(t)|_{t>t_{e,n}} = \frac{\sin(\theta_{n+1}/2)}{\cosh(\pi\Delta T_{n+1}/2)} \times \prod_{m=2}^{n} {}_2F_1(\theta_m/(2\pi), -\theta_m/(2\pi); \gamma^{(1)}_m(\Delta); 1) f^{\text{in}}(-\Delta/c) \times \exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{n+1,1}+\Delta t_{e,n+1})\}. \quad (52)$$

We set the last pulse area in the READ pulse sequence to be $\theta_{n+1}=\pi$ and its spectral width to be larger than the photon spectrum width: $\delta\omega_{ph} \ll \delta\omega_{n+1} \sim T^{-1}_{n+1}$. So ${}_2F_1((2\pi)^{-1}\theta_{n+1}, -(2\pi)^{-1}\theta_{n+1}; \gamma^{(1)}_{n+1}(\Delta\to 0); 1)|_{\Delta T_{n=1}\ll 1} \cong \cos(\theta_{n+1}/2)=0$. Thus we obtain $\xi_j(t>t_{e,n})=0$, so

$$|\psi(t>t_{e,n})\rangle|_{\theta_{n+1}=\pi, \Delta T_{n+1}\ll 1} = \sum_{m=1}^{n} |\psi_f^{(m)}(t>t_{e,m})\rangle, \quad (53)$$

with the normalization

$$\langle\psi(t)|\psi(t)\rangle \cong \sum_{m=1}^{n} \langle\psi_f^{(m)}(t>t_{e,m})|\psi_f^{(m)}(t>t_{e,m})\rangle = 1. \quad (54)$$

Thus, after reirradiation of the initial single-photon wave packet the state (6) is decomposed into the quantum superposition, Eq. (53), of the n different states of the nonoverlapping wave packets $|\psi_f^{(m)}(t>t_{e,m})\rangle$ $(m=1,\ldots,n)$. Obviously, any subsequent action upon the medium by other laser pulse with the frequency ω_{32} will not generate the photon echo signal because all the atoms have already been transferred onto the ground levels $|1\rangle$. We can control the parameters of the each term in the quantum superposition, which is easy to demonstrate in the case of all short pulses, where the (n')th term $(n'\leqslant n)$ in Eq. (52) will be [see also Eq. (44)]

$$f^{(n')}_{-(\omega_{31}+\Delta)/c}(t)|_{t>t_{e,n}} = \sin(\theta_{n'+1}/2)\prod_{m=2}^{n'} \cos(\theta_m/2) f^{\text{in}}(-\Delta/c) \times \exp\{-i(\omega_{31}+\Delta)t + i(\varphi_{n'+1,1}+\Delta t_{e,n'})\}.$$

From this we may deduce that the wave packet phase can be controlled only by the phases of the first and $(n'+1)$th laser pulse whereas the amplitude of the (n')th packet is determined by the laser pulses from $m=1$ to $m=n'$. It is also possible to calculate the probability of finding a photon in the (n')th echo pulse, which is equal to $P_{n'} = \cos^2(\theta_2/2)\cdots\cos^2(\theta_{n'}/2)\sin^2(\theta_{n'+1}/2)$. For equal probabilities, $P_1=P_2=\cdots=P_n=1/n$, the following parameters of the laser pulse areas must be satisfied: $\sin^2(\theta_1/2)=1$, $\sin^2(\theta_2/2)=1/n$, $\sin^2(\theta_3/2)=1/(n-1)$, $\ldots, \sin^2(\theta_{n'}/2)=(n+2-n')^{-1}$ (where $2\leqslant n'\leqslant n+1$).

Here, it should be noted that the obtained results are based on the phase matching, while in a three-level system with Σ transitions the phase mismatching condition for photon-echo QM processes [35] can be satisfied by tuning the laser pulse wave vectors orientation. So in such atomic systems the proposed manipulations with a photon can be successfully realized without any limitations imposed by the phase mismatching.

It is also possible to generalize the result of Eq. (53) for a three-dimensional spatial case and for the spectrally selective excitation of the resonant atomic line. The simplest scheme possible is shown in Fig. 4. The main difference of this scheme is that the laser does not propagate exactly backward along the z axis. The laser field wave vectors, however, must be sufficiently close to the $-z$ axis, if the phase matching condition is to hold well enough, in order that the irradiation of the echo signals are still able to be effective. It is important that these echo fields can be irradiated in different directions with respect to each other, in order that the formula for the final state of the echo fields after the $(n+1)$th laser pulse action will be able to take on a more general form:

$$|\psi(t)\rangle|_{t>t_{e,n}} \cong \sum_{m=1}^{n} |\psi_f^{(m)}(k_{e,m}; t>t_{e,m})\rangle, \quad (55)$$

where we have introduced the wave vectors for each mth wave packet $k_{e,m}=k_m-k_1+k_{\text{ph}}$.

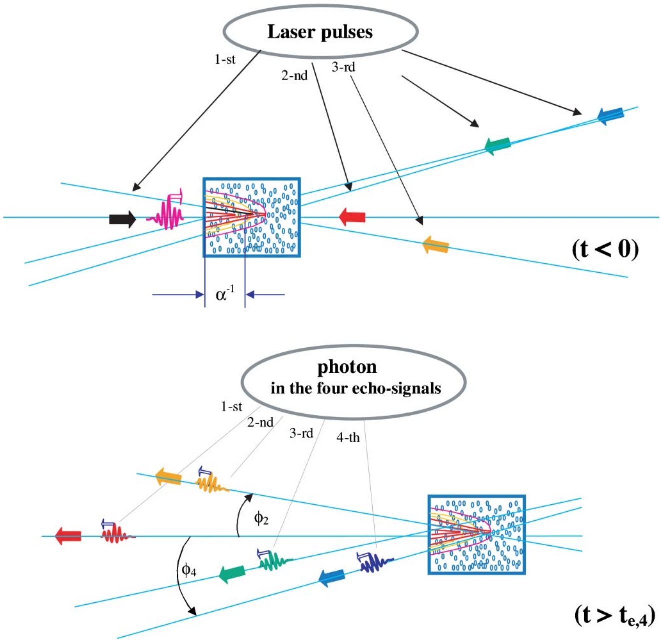

FIG. 4. (Color online) A three-dimentional spatial scheme of reconstruction of a stored photon as a sequence of the four echo pulses irradiated in different spatial directions. The angles of the irradiations ϕ_2 and ϕ_4 must be sufficiently small that the phase matching condition at the stage of the echo signal emission can be reliably held.

Formula (55) expands the initial wave function of Eq. (2) into a superposition of the wave packets with different wave vectors $k_{e,m}$ and time moments of their irradiations $t_{e,m}$.

Another scheme of generalization is obtained by applying readout laser pulses $(m=2,3,\ldots,n)$ with different carrier frequencies ω_m within the spectral interval $\delta\omega_m$, which is sufficiently small compared to the DATA signal spectrum $\delta\omega_{\text{ph}}$. At the same time the last, $(n+1)$th READ pulse must satisfy π pulse area in order to have a complete reconstruction of the storied state. In such a case each term in the formula (55) acquires its own carrier frequency $\omega_{e,m}$ and the spectral width $\delta\omega_{e,m}$, so that we can rewrite Eq. (55) as following form (see Figs. 4 and 5):

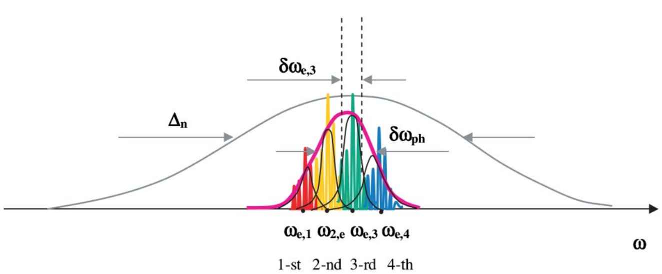

FIG. 5. (Color online) Spectral parameters of the READ pulses vs spectrum of the irradiated echo signals, carrier frequencies, and the spectral widths. The total spectrum of all of the echo signals corresponds to the spectrum of the initial photon where the complete reconstruction of the stored field takes place.

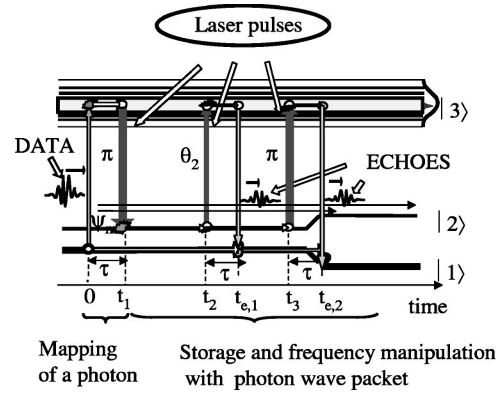

FIG. 6. Schematic of selective control of the echo signal. The additional shift of energies for the two lowest levels $|1\rangle$ and $|2\rangle$, caused by additional magnetic and electrical fields, is shown. The energy shifts change only the carrier frequency of the second echo signal. The frequency shift $\delta\omega_{31}$ must be smaller than ω_{21} for the Λ scheme of the proposed QM technique.

$$|\psi(t)\rangle|_{t>t_{e,n}} \cong \sum_{m=1}^{n} |\psi_f^{(m)}(k_{e,m},\omega_{e,m},\delta\omega_{e,m};t>t_{e,m})\rangle \quad (56)$$

in Eq. (56), $\delta\omega_{e,m}$ and $\omega_{e,m}$, which can be changed by varying the pulse area and the carrier frequency of the READ pulse. Accordingly, we can manipulate the echo amplitude by choosing an arbitrary value for Eq. (56).

Finally, we should note that the proposed quantum manipulations can be greatly enriched by using additional pulsed electrical and magnetic fields and by shifting the energy levels in the three-level system. An idealized scheme is shown in Fig. 6. Using the pulses of magnetic or electrical fields we can selectively change both the phase and carrier frequency of each wave packet in the final quantum superposition. In the case of a Λ-type atomic transition, the frequency can be changed within the spectral interval determined by the value ω_{21}. Detailed analysis of this process is beyond the scope of this paper.

VI. CONCLUSION

We have shown that the proposed photon-echo QM technique can be used for quantum manipulations of a stored quantum field. Here we must mention that the proposed photon-echo QM technique can be applied to an arbitrarily weak quantum field from the general properties of Eqs. (10)–(12). The technique has been developed on the basis of the strengths of the quantum approach, for qualitative QM processes using a quantum state of a field interacting with a macroscopic medium. In a semiclassical approach, only the reconstruction of an optical field envelope has been usually studied. Therefore, it is especially important in cases where it acts as an amplifier at the reconstruction process of the stored light [12]. In effect, the authors in Ref. [12] consider these imperfect cases of the QM technique because the initial ground quantum state of the medium could not be reconstructed completely after the recovery of the optical pulse profile, where it was necessary to check the quality of the arbitrary quantum state reconstruction in the technique.

It is worthwhile comparing the parameters in Eqs. (18), (22), and (23) for atomic excitations with Eqs. (7)–(9), in the context of the quantum information storage, with the atomic parameters of dark states excited in the EIT technique after the storage of a similar quantum light in the medium [10,20]. In accordance with Eq. (23), the stored photon wave-packet state includes a specific phase $\Delta_j t_1$. Thus, the atomic excitations on level $|2\rangle$ are dephased with respect to each other due to the inhomogeneous broadening of the transition $|1\rangle$-$|3\rangle$. Another major point of interest in Eqs. (18) and (22) is the spatial localization of the stored light with exponential decay with the absorption coefficient $\alpha(\Delta_j)$. The spatial localization in Eq. (18) is independent of the spatial profile of the storied light, while in the slow-light-based EIT QM technique the light is stored within a small volume of the medium excitation whose spatial shape is determined by the shape of the field envelope. For comparison we also note that the slow-light-based EIT QM technique cannot secure nonlocal-in-time reconstruction of the stored states since any reconstruction in it is inevitably connected with the complete emission of the stored information from the medium.

It is important to further develop the proposed technique for various states of light. The photon-echo QM technique can be put into practice with sufficiently weak classical optical fields, where the ground level depopulation is noncrucial. We have shown that the photon-echo QM technique is based on the complete reversibility of the quantum evolution. The complete reversibility can be proven for intensive probe fields, in particular using the area theorem for the photon-echo signals [36]. This complete reversibility can be useful for quantum information processing with intensive quantum probe (DATA) fields, for example, in squeezed states [37].

We have demonstrated that the proposed photon-echo QM method with multipulse reconstruction can be effectively applied to single photon fields. The properties of complicated multiphoton states representation for manipulating and generating entangled multiphoton quantum states of light and for testing fundamental problems of the mupliparticle decoherence are determined by the interactions and absorption of the photons by the macroscopic media. We note that these fundamental physical problems are studied now using also the concept of Loschmidt echo [38] and experiments with Hahn spin echo in the nuclei magnetic resonance. The proposed photon-echo QM experiments are very close to this area, which is devoted to a more precise theoretical investigation of the echo phenomena in mutiparticle systems. Further investigation of the reversibility problems represents a special interest.

ACKNOWLEDGMENT

The authors acknowledge that this work was supported by the Brain Pool Program (Grant No. 032-1-5) of KOSEF, S. Korea.

APPENDIX A

The wave function for the quantum mapping of a photon field can be derived using Eqs. (13)–(16). By substituting Eq. (14) for $\beta(t,z_j^0;\Delta_j)$,

$$\beta_j(t) = -ig\exp\{-i[\omega_{31}t - z_j^0/c]\}\int_{-\infty}^{t} dt'\, E(t',z_j^0) \times\exp\{-i[\Delta_j(t-t') + \delta\varphi_j(t-t')]\}, \tag{A1}$$

into Eqs. (16) and (15), we obtain

$$\begin{aligned}\left(\frac{\partial}{c\,\partial t} + \frac{\partial}{\partial z}\right)\mathcal{E}(t,z) &= -\alpha_0 Q(z)\int_{-\infty}^{t} dt' E(t',z)\int_{-\infty}^{\infty} d\Delta G(\Delta/\Delta_n) \\ &\quad\times\langle\exp\{-i\Delta(t-t')\}\exp\{\delta\varphi(t-t')\}\rangle_\gamma \\ &= -\alpha_0 Q(z)\int_{-\infty}^{t} dt' E(t',z)\int_{-\infty}^{\infty} d\Delta G(\Delta/\Delta_n) \\ &\quad\times\exp\{-(i\Delta+\gamma)(t-t')\},\end{aligned} \tag{A2}$$

where $\alpha_0 = 2\pi n_0 g^2/c$. Using the Fourier transform for the function $\mathcal{E}(t,z) = \int_{-\infty}^{\infty} du\tilde{\mathcal{E}}(u,z)\exp\{-iut\}$ we obtain the equation for the spectral component $\tilde{\mathcal{E}}(u,z)$:

$$\left(-i\frac{u}{c} + \frac{\partial}{\partial z}\right)\tilde{\mathcal{E}}(u,z) = -Q(z)\alpha_{(+)}(u)\tilde{\mathcal{E}}(u,z), \tag{A3}$$

where [see also Eq. (16a)]

$$\alpha_{(\pm)}(u) = \alpha_0\langle(\gamma \pm i[\Delta - u])^{-1}\rangle_\Delta = \alpha_0\int_{-\infty}^{\infty} d\Delta \frac{G(\Delta/\Delta_n)}{[\gamma \pm i(\Delta - u)]}. \tag{A4}$$

Equation (A3) has the following solution:

$$\tilde{\mathcal{E}}(u,z) = \tilde{\mathcal{E}}_0(u,0)\exp\left\{i\frac{u}{c}z - \alpha_{(+)}(u)\int_{-\infty}^{z} Q(z)dz\right\}, \quad \tilde{\mathcal{E}}_0(u,0) = \frac{1}{c}f^{\,\mathrm{in}}(u/c). \tag{A5}$$

Using Eqs. (A1) and (A5) and considering that the temporal duration of the pulse $\delta t_{\mathrm{ph}} \ll \gamma^{-1}$, we reach the following, for the stage of complete absorption of a photon $(t > \delta t_{\mathrm{ph}})$ in an optically dense medium **(**$\mathrm{Re}\{\alpha_+(\delta\omega_{\mathrm{ph}})\}L \approx \alpha(0)L \gg 1$, $\mathcal{E}(t \gg \delta t_{\mathrm{ph}}, z) \to 0$**)**:

$$\begin{aligned}\beta_j(t \gg \delta t_{\mathrm{ph}}) &\cong -ig\exp\{-i\delta\varphi_j(t)\}\int_{-\infty}^{\infty} dt' E(t',z_j^0) \\ &\quad\times\exp\{-i\Delta_j(t-t')\} \\ &= -i\beta^0(\Delta_j;z_j^0)\exp\{-i\delta\varphi_j(t) \\ &\quad - i(\omega_{31}+\Delta_j)(t - z_j^0/c)\},\end{aligned} \tag{A6}$$

where

$$\beta^0(\Delta_j;z_j^0) = (2\pi g/c)f^{\,\mathrm{in}}(\Delta_j/c)\exp\{-\alpha_{(+)}(\Delta_j)z_j^0\} \tag{A7}$$

and

$$b_j = \beta_j\exp\{i\Delta_j t\}, \quad f_k(t \gg \delta t_{\mathrm{ph}}) \to 0, \quad \xi_j(t) = 0. \tag{A8}$$

APPENDIX B

Equations (10)–(12) are transformed into the following equations for the atomic parameters at the action of each mth laser pulse:

$$\frac{\partial}{\partial t}b_j(t) = -i[\omega_{31}+\delta\omega_j(t)]b_j(t) + i\frac{1}{2}\xi_j(t)\Omega_{m,0}\,\text{sech}\left[\left(t-t_m - \frac{1}{c}n_m z_j(t)\right)\Big/ T_m\right]\exp\{-i[\omega_m t - k_m z_j(t)] + i\varphi_m\}, \tag{B1}$$

$$\frac{\partial}{\partial t}\xi_j(t) = -i\omega_{21}\xi(t)_j + i\frac{1}{2}b_j(t)\Omega_{m,0}\,\text{sech}\left[\left(t-t_m - \frac{1}{c}n_m z_j(t)\right)\Big/ T_m\right]\exp\{i[\omega_m t - k_m z_j(t)] - i\varphi_m\}, \tag{B2}$$

We assume that the spatial sizes of the laser (WRITE or READ) pulses in the xy plane are large enough (see Figs. 1 and 2) as compared to the size of the DATA quantum field. Thus, we can consider uniform Rabi frequency applied to the DATA quantum field: $\Omega_m(t,r)\cong\Omega_m([t-t_m-(1/c)n_m r]/T_m)$. It is enough to have the solutions for the parameters $b_j(t)$ and $\xi_j(t)$ after the action of the arbitrary mth laser pulse $[t>t_m+(1/c)n_m z_j(t)+T_m]$. Assuming that the pulse duration of the mth laser pulse is short enough so the phase fluctuations $\delta\varphi_j(t)=\int_0^t dt'\,\delta\omega_j(t')\cong\delta\varphi_j(t_m)$, the solution of Eqs. (B1) and (B2) is $b_j(t)=\tilde{b}_j(t)e^{-i\delta\varphi_j(t)}\cong\tilde{b}_j(t)e^{-i\delta\varphi_j(t_m)}$, where

$$\chi_j^m(t) = \tilde{b}_j(t)\exp\left\{i\omega_{31}t - \frac{1}{2}i[\phi_j^m + \delta\varphi_j(t_m)]\right\},$$

$$\varsigma_j^m(t) = \xi_j(t)\exp\left\{i\omega_{21}t + \frac{1}{2}i[\phi_j^m + \delta\varphi_j(t_m)]\right\},$$

$$\phi_j^m = k_m r_j(0) + \varphi_m - \Delta_j^m t_j^m, \quad \Delta_j^m = \omega_m - \omega_{32} - \omega_m v_{n_m}^j/c,$$

$$v_{n_m}^j = \vec{n}_m\vec{v}_j, \quad m=1,2,\ldots \quad . \tag{B3}$$

In the interaction picture, Eq. (B3) has the following equations:

$$\frac{\partial}{\partial t}\chi_j^m(t) = i\frac{1}{2}\varsigma_j^m(t)\Omega_{m,0}\,\text{sech}[(t-t_j^m)/T_m^j]\exp\{-i\Delta_j^m(t-t_j^m)\}, \tag{B4}$$

$$\frac{\partial}{\partial t}\varsigma_j^m(t) = i\frac{1}{2}\chi_j^m(t)\Omega_{m,0}\,\text{sech}[(t-t_j^m)/T_m^j]\exp\{i\Delta_j^m(t-t_j^m)\}, \tag{B5}$$

where

$$T_m^j = T_m(1-v_{n_m}^j/c)^{-1}\cong T_m, t_j^m = \left(t_m + \frac{1}{c}n_m r_j(0)\right)(1-v_{n_m}^j/c)^{-1} \cong \left(t_m + \frac{1}{c}n_m r_j(0)\right)\left(1+v_{n_m}^j/c\right). \tag{B6}$$

The exact solution of Eqs. (B4) and (B5) is well known [32–34]. The general solution for $\chi_j^m(t\gg t_m+T_m)$ and $\zeta_j^m(t\gg t_m+T_m)$ has the following form after interaction with the mth laser pulse action:

$$\zeta_j^m = \zeta_j^m(-\infty)\,{}_2F_1((2\pi)^{-1}\theta_m, -(2\pi)^{-1}\theta_m;\gamma_{j,m}^{(1)};1) + i\chi_j^m(-\infty)\frac{\sin\left(\frac{1}{2}\theta_m\right)}{\cosh\left(\frac{\pi}{2}\Delta_j^m T_m\right)}, \tag{B7}$$

$$\chi_j^m = \chi_j^m(-\infty)\,{}_2F_1((2\pi)^{-1}\theta_m, -(2\pi)^{-1}\theta_m;\gamma_{j,m}^{(2)};1) + i\zeta_j^m(-\infty)\frac{\sin\left(\frac{1}{2}\theta_m\right)}{\cosh\left(\frac{\pi}{2}\Delta_j^m T_m\right)}, \tag{B8}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function; $\theta_m=\pi\Omega_{m,0}T_m$ is the pulse area,

$$\gamma_{j,m}^{(1)} = 1/2 - i\Delta_m^j T_m^j/2, \quad \gamma_{j,m}^{(2)} = (\gamma_{j,m}^{(1)})^*,$$

$$\gamma_m^{(1)}(\Delta) = 1/2 - i\Delta T_m/2,$$

$$(\gamma_m^{(1)}(\Delta))^* = \gamma_m^{(2)}(\Delta), \tag{B9}$$

$$\chi_j^m(t\ll t_m)\to\chi_j^m(-\infty), \quad \varsigma_m^j(t\ll t_m)\to\varsigma_j^m(-\infty). \tag{B10}$$

We introduce the initial conditions $\chi_j^m(-\infty)$ and $\varsigma_j^m(-\infty)$ for Eqs. (B4) and (B5) using the approximation of sufficiently short laser pulses as compared to the time delays between the pulses $t_{m,m-1}/T_m\gg1$, $t_{m+1,m}/T_{m\gg1}$. In particular, for the WRITE pulse ($m=1$) it corresponds to the approximation $t_1/T_1\gg1$ with the following initial conditions for Eqs. (B4) and (B5):

$$\chi_j^1(-\infty) = -i\beta^0(\Delta_j;z_j^0)\exp\left\{-\frac{1}{2}i[\phi_j^1+\delta\varphi_j(t_1)]\right\}, \quad \varsigma_j^1(-\infty)\to 0, \tag{B11}$$

based on Eqs. (B3), (17), and (18). We assume $(t-t_1)/T_1\gg1$. After the transition to laboratory coordinate system we reach the solution

$$b_j(t) = -i\beta^0(\Delta_j;z_j^0)\,{}_2F_1((2\pi)^{-1}\theta_1, -(2\pi)^{-1}\theta_1;\gamma_{j,1}^{(2)};1)\times\exp\{-i[\omega_{31}t - (\omega_{31}+\Delta_j)z_j^0/c] - i\delta\varphi_j(t)\}, \tag{B12}$$

$$\xi_j(t)=\beta^0(\Delta_j;z_j^0)\frac{\sin(\theta_1/2)}{\cosh(\pi\Delta_1^jT_1/2)}\exp\left\{-i\left[\omega_{21}t-\frac{1}{c}(\omega_{31}+\Delta_j)z_j^0\right]-i(\phi_j^1+\delta\varphi_j(t_2)]\right\}$$
$$=(2\pi g/c)f^{\,\text{in}}_{k=(\omega_{31}+\Delta_j)/c}\frac{\sin(\theta_1/2)}{\cosh(\pi\Delta_1^jT_1/2)}\exp\{-\alpha_+(\Delta_j)z_j^0-i[\omega_{21}t+\delta\varphi_j(t_2)]+i\mu_1(t_1;v_j;r_j)\},\qquad\text{(B13)}$$

where

$$\mu_1(t_1;v_j;r_j)=\omega_1t_1-\varphi_1-\omega_{32}(1+v^j_{n_1}/c)t_1+\delta\mu_1(r_j),\qquad\text{(B14)}$$

$$\delta\mu_1(r_j)=\omega_{31}(1+v_z^j/c)z_j/c-\omega_{32}(1+v^j_{n_1}/c)n_1r_j(0)/c.\qquad\text{(B15)}$$

We note that ${}_2F_1\big((2\pi)^{-1}\theta_m,-(2\pi)^{-1}\theta_m;\gamma^{(2)}_{j,1};1\big)\cong\cos\{\theta_m/2\}$ for very short laser pulses $\left(\frac{1}{2}\pi\Delta_m^jT_m/2<\frac{1}{2}\pi\delta\omega_{ph}T_m\ll1,\ \cosh\{\frac{1}{2}\pi\Delta_m^jT_m\}\rightarrow1,\omega_{ph}=\omega_{31}=\omega_1\right)$, and $b_j(t>t_1)=0$ for $\theta_m=\pi$, which takes place for the WRITE pulse action ($m=1$).

In the case of the parallel wave vectors of the photon wave packet with the WRITE pulse, $k_{\text{ph}}//k_1$ ($n_1=e_z$), we have $\delta\mu_1(r_j)=\omega_{21}(1+v_z^j/c)z_j^0/c$. For $\omega_{21}L/c\ll c/v_{\max}$ it is $\delta\mu_1(r_j)\cong\omega_{21}z_j^0/c$.

APPENDIX C

We are interested in the solution of Eqs. (27), (28), and (31) for the medium localized in the space $0<z<\infty$ [where $Q(z)=1$ and $Q(z)=0$ if $z<0$] with the initial conditions of Eq. (29). First of all we find the solution of Eq. (31) for $F(t,z)$ $[F(t,z)|_{\alpha Z\gg1}\rightarrow0]$:

$$\left(\frac{\partial}{c\,\partial t}-\frac{\partial}{\partial z}\right)F(t,z)=Q(z)\{A_1(t,z)+A_2(t,z)\}.\qquad\text{(C1)}$$

We write the formulas for A_1 and A_2 taking into account [see Eqs. (29), (24), and (18)]

$$\eta_j^0(t)=\exp\left\{-i\left[\delta\varphi_j(t_1)+\delta\varphi_j\left(t-t_2+\frac{1}{c}z_j^0\right)\right]\right\}b_j^0(t)\exp\{i\Delta_jt\},$$

$$b_j^0(t)\exp\{i\Delta_jt\}=i(2\pi g/c)\sin\left(\frac{1}{2}\theta_2\right)\text{sech}(\pi\Delta_jT_2/2)f^{\,\text{in}}(\Delta_j/c)\times\exp\{-\alpha_{(+)}(\Delta_j)z_j^0\}\exp\left\{i\left[-(\omega_{31}-\Delta_j)\times\left(t+\frac{1}{c}z_j^0\right)+2\frac{1}{c}\omega_{21}z_j^0-\Delta_jt_{e,1}+\varphi_{21}\right]\right\}.\qquad\text{(C2)}$$

Averaging of Eq. (31) over the phase fluctuations and inhomogeneous broadening with initial condition of Eq. (C2) gives

$$A_1(t,z)=A_{01}\exp\left\{i\left(2\frac{1}{c}\omega_{21}z\right)\right\}\exp\left\{-\gamma\left(t-t_2+t_1+\frac{1}{c}z\right)\right\}\times\left\langle X(\Delta,z)\exp\left\{i\Delta\left(t-t_{e,1}+\frac{1}{c}z\right)\right\}\right\rangle_\Delta,\qquad\text{(C3)}$$

$$A_2(t,z)=-\alpha_0\int_{-\infty}^t dt'F(t',z)\langle\exp\{i\Delta(t-t')\}\rangle_\Delta\exp\{-\gamma(t-t')\},\qquad\text{(C4)}$$

where $A_{01}=(2\pi\alpha_0/c)\sin(\frac{1}{2}\theta_2)\exp\{i\varphi_{2,1}\}$, $X(\Delta,z)=\text{sech}[(\pi/2\Delta T_2)]f^{\,\text{in}}[(1/c\Delta)]\exp\{-\alpha_{(+)}(\Delta)z\}$, $\alpha_0=2\pi n_0g^2/c$.

For the analysis, let us consider two different cases.

(1) Case I: a spectrally narrow DATA field ($\gamma,\delta\omega_{\text{ph}}\ll T_m^{-1},\Delta_n$). Under the approximations we obtain

$$\langle X(\Delta,z)\exp\{\Delta[t-t_{e,1}+(1/c)z]\}\rangle_\Delta\cong\langle f^{\,\text{in}}[(1/c)\Delta]\exp\{i\Delta[t-t_{e,1}+(1/c)z]\}\rangle_\Delta\exp\{-\alpha_{(+)}(0)z\}$$

and the following formulas for A_1 and A_2:

$$A_1(t,z)\cong\alpha\sin\left(\frac{1}{2}\theta_2\right)\exp\left\{-\frac{1}{2}\alpha z+i\varphi_{2,1}+2i\frac{1}{c}\omega_{21}z\right\}\times\mathcal{E}_0\left(t_{e,1}-t-\frac{1}{c}z\right),\qquad\text{(C5)}$$

$$A_2(t,z)\cong-\alpha_0F(t,z)\int_{-\infty}^t\text{d}t'\left\langle\exp\{i\Delta(t-t')\}\right\rangle_\Delta\exp\{-\gamma(t-t')\}=-\frac{1}{2}\alpha F(t,z),\qquad\text{(C6)}$$

where we use

$$\mathcal{E}_0(t)=\exp\{i\omega_{\text{ph}}t\}\int_0^\infty dkf_k(t)=\frac{1}{c}\int_{-\infty}^\infty f^{\,\text{in}}\left(\frac{1}{c}\Delta\right)e^{-i\Delta t}d\Delta.$$

If $\text{G}(-\Delta/\Delta_n)=\text{G}(\Delta/\Delta_n)$, then we obtain

$$\alpha_{(\pm)}(0)=\alpha_0\int_{-\infty}^\infty d\Delta'\ \frac{G(\Delta'/\Delta_n)}{\gamma\pm i\Delta'}\bigg|_{\gamma\ll\Delta_n}\cong\pi\alpha_0G(0)\equiv\frac{1}{2}\alpha.\qquad\text{(C7)}$$

Using the new independent variables ζ and z [where $\varsigma=t+(1/c)z$], we find simpler form for Eq. (C1):

$$-\frac{\partial}{\partial z}F(\varsigma,z)=Q(z)\left\{\alpha\sin\left(\frac{1}{2}\theta_2\right)\exp\left[-\frac{1}{2}\alpha z-2\gamma t_1+i\varphi_{2,1}+2i\frac{1}{c}\omega_{21}z\right]\mathcal{E}_0(t_{e,1}-\varsigma)e^{-\gamma(\varsigma-t_{t,1})}-\frac{1}{2}\alpha F(\varsigma,z)\right\}. \quad \text{(C8)}$$

After integrating Eq. (C8) over z (from L to z) and transfering to the laboratory coordinates t and z, we get the following solution for $F(t,z)$:

$$F(t,z)=\frac{\sin\left(\frac{1}{2}\theta_2\right)}{\left(1-2i\frac{1}{c}\omega_{21}\alpha^{-1}\right)}\exp\left\{-2\gamma t_1+i\varphi_{2,1}-Q(z)\left[\frac{1}{2}\alpha z-2i\frac{1}{c}\omega_{21}z\right]\right\}E_0\left(t_{e,1}-t-\frac{1}{c}z\right)e^{-\gamma(t-t_{e,1}+z/c)}. \quad \text{(C9)}$$

The solution is given for the arbitrary z coordinate of the field from $-\infty$ to L.

(2) Case II: a large spectral width of the stored field and the laser pulses ($\delta\omega_{\mathrm{ph}},T_m^{-1}\gg\gamma$). Using Fourier transformation $F(t,z)=\int_{-\infty}^{\infty}d\Delta\widetilde{F}(\Delta,z)e^{-i\Delta(t+z/c)}$ in Eq. (C1), we obtain the following equation for $\widetilde{F}(\Delta,z)$:

$$\frac{d}{dz}\widetilde{F}(\Delta,z)=-Q(z)\{\widetilde{A}_1(\Delta,z)+\widetilde{A}_2(\Delta,z)\}, \quad \text{(C10)}$$

where

$$\widetilde{A}_1(\Delta,z)=A_{01}\exp\left\{i\left(2\frac{1}{c}\omega_{21}z\right)\right\}\exp\{-2\gamma t_1\}\exp\{i\Delta t_{e,1}\}\times\left\langle\frac{\gamma}{\pi\{(\Delta+\Delta')^2+\gamma^2)}X(\Delta',z)\right\rangle_{\Delta'}, \quad \text{(C11)}$$

$$\widetilde{A}_2(\Delta,z)=-\alpha_{(-)}(-\Delta)\widetilde{F}(\Delta,z). \quad \text{(C12)}$$

Taking into account boundary condition of the irradiated field $[\widetilde{F}(\Delta,z\to\infty)|_{\alpha(\Delta)z\gg1}\to0]$, we get the following from Eq. (C10):

$$\begin{aligned}\widetilde{F}(\Delta,z)&=\int_{z>0}^{L}dz'\widetilde{A}_1(\Delta,z')\exp\{\alpha_{(-)}(-\Delta)(z-z')\}\\&=A_{01}\exp\{-2\gamma t_1\}\exp\{i\Delta t_{e,1}\}\times\int_{z>0}^{L}dz'\exp\{\alpha_{(-)}(-\Delta)(z-z')\}\\&\quad\times\exp\left\{i\left(2\frac{1}{c}\omega_{21}z'\right)\right\}\times\left\langle\gamma\frac{\mathrm{sech}\left(\frac{\pi}{2}\Delta' T_2\right)f^{\mathrm{in}}\left(\frac{1}{c}\Delta'\right)}{\pi\{(\Delta+\Delta')^2+\gamma^2\}}\times\exp\{-\alpha_{(+)}(\Delta')z'\}\right\rangle_{\Delta'}.\end{aligned} \quad \text{(C13)}$$

In accordance with the previous approximations [see Sec. III (quantum mapping): $\gamma\ll T_m^{-1},\delta\omega_{\mathrm{ph}},\Delta_n$ $m=1,2,3,\ldots$, for the laser pulse], we find

$$\left\langle\gamma\frac{\mathrm{sech}\left(\frac{\pi}{2}\Delta' T_2\right)f^{\mathrm{in}}\left(\frac{1}{c}\Delta'\right)}{\pi\{(\Delta+\Delta')^2+\gamma^2)}\exp\{-\alpha_{(+)}(\Delta')z'\}\right\rangle_{\Delta'}\cong\mathrm{sech}\left(\frac{\pi}{2}\Delta T_2\right)f^{\mathrm{in}}\left(-\frac{1}{c}\Delta\right)\exp\{-\alpha_{(+)}(-\Delta)z'\}G(-\Delta/\Delta_n) \quad \text{(C14)}$$

and

$$\alpha_{(-)}(-\Delta)+\alpha_{(+)}(-\Delta)=2\alpha_0\int_{-\infty}^{\infty}d\Delta'\left.\frac{\gamma G(\Delta'/\Delta_n)}{\gamma^2+(\Delta'+\Delta)^2}\right|_{\gamma\ll\Delta_n}\cong2\pi\alpha_0G(-\Delta/\Delta_n)\equiv\alpha(-\Delta), \quad \text{(C15)}$$

where $\alpha(-\Delta)$ is the absorption coefficient on the transition $|1\rangle$-$|3\rangle$ at the detuning $-\Delta$ from the central frequency of transition $|1\rangle$-$|3\rangle$. Using Eqs. (C13) and (C14) for Eq. (C12) and integrating Eq. (C13) over z' (from z to L), we get

$$\widetilde{F}(\Delta,z)\cong\frac{\sin\left(\frac{1}{2}\theta_2\right)}{c\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(-\Delta)\right\}}\exp\{-2\gamma t_1+i(\Delta t_{e,1}+\varphi_{2,1})\}\exp\left\{-Q(z)\left[\alpha_{(+)}(-\Delta)-2i\frac{1}{c}\omega_{21}\right]z\right\}\mathrm{sech}\left(\frac{\pi}{2}\Delta T_2\right)f^{\mathrm{in}}\left(-\frac{1}{c}\Delta\right). \quad \text{(C16)}$$

Using Eq. (C16) and Fourier transformation, we find the solution for $F(t,z)$, which coincides with Eq. (C8), if we assume $\alpha(-\Delta)\cong\alpha,\alpha_+(-\Delta)\cong1/2\alpha$, and $\mathrm{sech}[(\pi/2)\Delta T_2]\cong1$. The additional phase $i\Delta t_{e,1}$ in Eq. (C16) reflects the time of the echo signal irradiation.

For the atomic excitations at the echo irradiation by using the solution for $F(t,z)$, we determine the atomic excitations $\eta_j(t)$ just after the echo pulse irradiation. The formal solution of Eq. (28) for $\eta_j(t)$ is

$$\eta_j(t) = \eta_j^0(t) - \eta_j^1(t), \quad \eta_j^1(t) = ig\, \exp\{-i[(\omega_{31} - \Delta_j)t + \omega_{31} z_j^0/c]\}\int_{-\infty}^{t} dt F(t',z_j^0)\exp\{-i\Delta_j t' - i\delta\varphi_j(t-t')\}, \tag{C17}$$

where $\eta_j^0(t)$ [see Eq. (C2)]. The function $F(t',z_j^0)$ has a sharp maximum at $t'=t_{e,1}-(1/c)z_j^0$, so taking into account slowly varying fluctuations $i\delta\varphi_j(t-t')$ we obtain

$$\eta_j^1(t) \cong ig\, \exp\left\{-i\left[(\omega_{31}-\Delta_j)\left(t+\frac{1}{c}z_j^0\right) + i\delta\varphi_j\left(t+\frac{1}{c}z - t_{e,1}\right)\right]\right\}\int_{-\infty}^{t} dt F(t',z_j^0)\exp\left\{-i\Delta_j\left(t'+\frac{1}{c}z_j^0\right)\right\}, \tag{C18}$$

with the same accuracy as it shown in Eqs. (C9) and (C16). The behavior of atoms after echo emission ($t \gg t_{e,1}+\delta t_{e,1}$) we find the following by the stretching the upper limit of the integration in Eq. (C10) to the infinity:

$$\eta_j^1(t)|_{t>t_{e,1}} \cong 2\pi i g\, \exp\left\{-i\left[(\omega_{31}-\Delta_j)\left(t+\frac{1}{c}z_j^0\right) + i\delta\varphi_j\left(t+\frac{1}{c}z-t_{e,1}\right)\right]\right\}\tilde{F}(-\Delta_j,z_j^0) = \frac{\exp\left\{-i\delta\varphi_j\left(t+\frac{1}{c}z-t_{e,1}\right)\right\}}{\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(\Delta)\right\}} \times \exp\{-2\gamma t_1\} b_j^0(t)\exp\{i\Delta_j t\}. \tag{C19}$$

Thus, the probability amplitude of the jth atom on the optically excited level $|3\rangle$ after the first echo emission will be

$$\eta_j(t) = \eta_j^0(t) - \eta_j^1(t) \cong \left(\exp\left\{-i\left[\delta\varphi_j(t_1) + \delta\varphi_j\left(t-t_2+\frac{1}{c}z_j^0\right)\right]\right\} - \frac{\exp\left[-i\delta\varphi_j\left(t+\frac{1}{c}z-t_{e,1}\right)\right]}{\left\{1-2i\frac{1}{c}\omega_{21}\alpha^{-1}(\Delta)\right\}}\exp(-2\gamma t_1)\right) b_j^0(t)\exp\{i\Delta_j t\}. \tag{C20}$$

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