Quantum simulation of cosmic inflation in two-component Bose-Einstein condensates

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Generalizing the one-component case, we demonstrate that the propagation of sound waves in two-component Bose-Einstein condensates can also be described in terms of effective sonic geometries under appropriate conditions. In comparison with the one-component case, the two-component setup offers more flexibility and several advantages. In view of these advantages, we propose an experiment in which the evolution of the inflaton field, and thereby the generation of density fluctuations in the very early stages of our universe during inflation, can be simulated, realizing a *quantum simulation via analogue gravity models*.

I. INTRODUCTION

Within our present standard model of cosmology, basically all inhomogeneities—including the seeds for the formation of structures such as our galaxy—originate from quantum fluctuations of a single scalar field, the *inflaton* [1,2]. This (postulated) field drives inflation, which is a stage of very rapid expansion in the earliest evolutionary phase of our universe [3]. Tracing the inflaton fluctuations back in time and thereby undoing the redshift induced by the cosmic expansion, the anisotropies of the cosmic microwave background we observe today correspond to extremely short wavelengths during inflation. As a result, the fluctuations of the cosmic microwave background probe ultrahigh (e.g., Planckian) energy scales—which are experimentally inaccessible with the present-day and near-future available technology of, for example, particle accelerators. At such ultrahigh energies, quantum effects of gravity are expected to become important—but the underlying physical theory for the description of these effects is not known yet. Consequently, high-precision measurements of the cosmic microwave background might give us some insight into physics beyond well-established theories (in particular, beyond the standard model of particle physics).

In order to detect signatures of the new physics in the anisotropies of the cosmic microwave background, one has to investigate which kind of higher-order corrections and correlations could potentially be induced by deviations from the known laws of physics occurring at ultrahigh energies. One way to achieve this aim is to consider analogous systems, based on laboratory physics, which reproduce major features of the inflaton field, and which can therefore be used to simulate the generation of (quantum) fluctuations and further interesting effects—theoretically as well as experimentally. This line of approach has come to be known under the term *analogue gravity/cosmology,* see, e.g., Ref. [4]. The consideration of these analogues leads to a better understanding of the system to be simulated, in particular, regarding the possible impact of high-energy degrees of freedom. As cosmology is essentially a purely observational branch of science, because obviously we cannot do experiments on the real system, i.e., the universe, the analogues also allow an experimental verification of so far only theoretically predicted effects. Basically the same motivations underlie the idea of quantum black-hole analogues ("dumb holes" [5–7]), since the Hawking radiation of "real" gravity black holes has its origin in (trans-) Planckian modes: The quanta emitted necessarily come from regions very close to the black-hole horizon, and experience a large redshift when finally detected far away from the black hole [5,8].

A scalar field (such as the inflaton) within the curved space-time structure of an expanding universe can be simulated by propagating sound waves in single-component Bose-Einstein condensates (BEC's) [9]; previous considerations on such effective acoustic geometries in single-component BEC's can be found in Refs. [7,10–13]. The advantage of BEC's lies in the fact that the corresponding parameters (such as local density and speed of sound) can be controlled with atomic precision experimentally, and that the underlying physics is well understood on all energy scales in particular in the dilute case, for which the Gross-Pitaevskii equation provides a rather accurate description of the order-parameter dynamics. There are basically two possibilities for simulating an expanding universe within a BEC: changing the interparticle coupling or expanding the condensate (or a combination of both) [11–13]. However, both methods come with problems. First, a controlled expansion of the BEC cloud requires a specific time-dependent trap; and furthermore, the density of the cloud rapidly decreases during the expansion, leaving only a short time to do the experiment. Second, in order to change the interparticle coupling drastically via a Feshbach resonance, by rapidly sweeping a time-dependent external magnetic field, one has to go very close to the resonance; one then encounters the problem that the coupling constant effectively acquires an imaginary part due to molecule formation in three-body recombination processes [14,15], which spoils the desired effect. Third, a time-dependent interparticle coupling would also induce a variation of the density and the size of the BEC cloud—unless the trap is changed accordingly, which again is difficult (see first point). Finally, one has to be able to measure the generated fluctuations in order to simulate the inflaton field experimentally.

In the present paper, we investigate whether it is possible to overcome some of these problems in two-component Bose-Einstein condensates, which are readily experimentally

available; the various components can be realized by trapping different hyperfine ground states of the same atom [16–19]. We start by describing in Sec. II how the collective equations of motion for small fluctuations (i.e., sound waves) in the two-component gas are obtained. In Sec. III, we show that it is possible to map these equations of motion onto two effective sonic metrics (bimetricity) under certain conditions. The analogue of the Planck scale, where the concept of the effective geometry breaks down, is discussed in Sec. IV. Section V is devoted to the simulation of the de Sitter geometry—i.e., to constructing an analogue for the inflaton field during inflation. We discuss possible experimental realizations and measurement prescriptions to realize the desired inflation quantum simulation in Sec. VI. The advantages and drawbacks of the described method and further aspects are summarized in Sec. VII.

We note that the inflaton has been mentioned in the context of Bose-Einstein condensates previously [20]. However, the discussion there has been rather qualitative. In particular, the inflaton mode has not been related there to any effective space-time metric of cosmological character (for example, the de Sitter metric), in which its propagation ought to take place. The latter is necessary to appropriately describe the freezing process of the quantum fluctuations and the related concept of a horizon, both of which we shall investigate for de Sitter space-time in what follows.

II. EQUATIONS OF MOTION

In terms of the Madelung representation for the order parameter components $\psi_a(\mathbf{r}, t)$,

$$\psi_a(\mathbf{r}, t) = \sqrt{\varrho_a(\mathbf{r}, t)} \exp\{iS_a(\mathbf{r}, t)\}, \tag{1}$$

with the density $\varrho_a(\mathbf{r}, t)$ and the phase (eikonal) $S_a(\mathbf{r}, t)$, the Lagrangian density of a dilute two-component Bose-Einstein condensate reads (we put $\hbar = 1$) [16]

$$\mathcal{L} = -\sum_a \left(\varrho_a \partial_t S_a + \frac{\varrho_a}{2m_a} (\boldsymbol{\nabla} S_a)^2 + \frac{(\boldsymbol{\nabla} \sqrt{\varrho_a})^2}{2m_a} + V_a \varrho_a \right)$$
$$- \frac{1}{2} \sum_{a,b} g_{ab} \varrho_a \varrho_b. \tag{2}$$

Here, the two masses of the atoms are m_a ($a = 1, 2$), the one-particle trapping potentials (which are generally different) are given by $V_a(\mathbf{r}, t)$, and the (symmetric) two-particle interaction coupling matrix is denoted g_{ab}.

Linearizing around a given, stationary background solution $\varrho_a^0(\mathbf{r})$ and $S_a^0(\mathbf{r})$ with $\mathbf{v}_a^0 = \boldsymbol{\nabla} S_a^0 / m_a$ and neglecting the quantum pressure terms $\propto (\boldsymbol{\nabla} \sqrt{\varrho_a})^2$—which amounts to the local density (Thomas-Fermi) approximation in one-component Bose-Einstein condensates—leads to the second-order effective action

$$\mathcal{L}_{\text{eff}}^{(2)} = -\sum_a \left[\delta\varrho_a \partial_t \delta S_a + \frac{\varrho_a^0}{2m_a} (\boldsymbol{\nabla} \delta S_a)^2 + \delta\varrho_a \mathbf{v}_a^0 \cdot \boldsymbol{\nabla} \delta S_a \right]$$
$$- \sum_{a,b} \frac{1}{2} g_{ab} \delta\varrho_a \delta\varrho_b. \tag{3}$$

Varying the above action with respect to $\delta\varrho_a$ yields two Bernoulli-type equations for the fluctuations,

$$D_a \delta S_a + \sum_b g_{ab} \delta\varrho_b = 0, \tag{4}$$

where the co-moving derivative is defined to be

$$D_a \delta S_a = \partial_t \delta S_a + \mathbf{v}_a^0 \cdot \boldsymbol{\nabla} \delta S_a. \tag{5}$$

Using the above equation to eliminate $\delta\varrho_a$ from $\mathcal{L}_{\text{eff}}^{(2)}$, we obtain a phases-only effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}}^{S} = \sum_{ab} \frac{1}{2} (D_a \delta S_a) g_{ab}^{-1} (D_b \delta S_b) - \sum_a \frac{\varrho_a^0}{2m_a} (\boldsymbol{\nabla} \delta S_a)^2. \tag{6}$$

The general wave equations for the phase fluctuations δS_a then take the form

$$\sum_b D_a (g_{ab}^{-1} D_b \delta S_b) - \frac{\varrho_a^0}{m_a} \boldsymbol{\nabla}^2 \delta S_a = 0. \tag{7}$$

These wave equations, in the general case, do not yet have the pseudo-Lorentz invariance required to obtain effective space-time metrics of Lorentzian signature.

III. EFFECTIVE GEOMETRY

So far, we have involved no specific assumptions about the background densities and velocities as well as the constituent masses. We now come to discuss a simple case in which an effective metric description in terms of the Painlevé-Gullstrand-Lemaître type [21] is viable.

As demonstrated in Ref. [22], the introduction of effective geometries for multiple interacting fields is more involved than the single-field case—where rather general assumptions ensure the existence of an effective metric for the propagation of perturbations. In the case of multiple interacting fields, there are the following three main possibilities: (i) Owing to a lack of symmetry one cannot introduce a metric at all ("pre-geometry"); (ii) the perturbations effectively decouple and can be described by multiple metrics; or (iii) all the metrics coincide and there is one unique metric (reproducing the principle of equivalence). There is also the (fourth) possibility that the propagation of perturbations is not equivalent to scalar (i.e., spin-zero) fields in curved space-times, but to fields with higher, nonzero spin instead (e.g., Dirac [23] or vector [24] fields).

Phonons in arbitrary two-component condensates correspond to case (i) in general—even though one might diagonalize the dispersion relation, the full equations of motion do not allow the introduction of an effective geometry (in the most general situation). In order to arrive at an effective metric, certain requirements on the background solution are necessary. Let us assume that the parameters of the back-

ground solution satisfy the following conditions:

$$\mathbf{v}_1^0 = \mathbf{v}_2^0 \equiv \mathbf{v} \rightsquigarrow D_1 = D_2 \equiv D,$$

$$\frac{\varrho_1^0}{m_1} = \frac{\varrho_2^0}{m_2} \equiv \frac{\varrho_0}{m},$$

$$\text{eigenvectors}(g_{ab}) = \text{const}. \tag{8}$$

The diagonalization of the (real and symmetric) coupling matrix g_{ab} leads to eigenvalues $g_\pm$, given by

$$g_\pm = \frac{g_{11} + g_{22}}{2} \pm \sqrt{\left(\frac{g_{11} - g_{22}}{2}\right)^2 + g_{12}^2}. \tag{9}$$

Note that—in contrast to the eigenvectors of the matrix g_{ab}—its eigenvalues $g_\pm$ are not required to be constant. By virtue of the assumptions (8), the Lagrangian in Eq. (6) can be diagonalized,

$$\mathcal{L}_{\text{eff}}^S = \frac{1}{2} g_\pm^{-1} (D\phi_\pm)^2 - \frac{\varrho_0}{2m} (\boldsymbol{\nabla} \phi_\pm)^2 \equiv \frac{1}{2} \sqrt{-g_\pm} g_\pm^{\mu\nu} \partial_\mu \phi_\pm \partial_\nu \phi_\pm, \tag{10}$$

where $\phi_\pm$ denote the projections of the phase fluctuations δS_a onto the (constant) eigenvectors of g_{ab} and a summation convention over indices μ, ν and $\pm$ is implied. In this (highly symmetric) case, we therefore obtain two *independently propagating*, i.e., decoupled modes $\phi_\pm$, which feel effective space-time metrics of the conventional (covariant) Painlevé-Gullstrand-Lemaître form [25],

$$g_{\mu\nu}^\pm = \frac{\varrho_0}{c_\pm} \begin{pmatrix} c_\pm^2 - \mathbf{v}^2 & \mathbf{v} \\ \mathbf{v} & -\mathbf{1} \end{pmatrix}, \tag{11}$$

where the two sound velocities are $c_\pm = \sqrt{g_\pm \varrho_0 / m}$ and $\mathbf{1}$ represents the unit matrix. As long as these two sound velocities do not coincide $c_+ \neq c_-$, the system under consideration corresponds to the bi-metric case (ii) discussed at the beginning of this section.

The assumption in Eq. (8) that the eigenvectors of g_{ab} be constant can be satisfied, if g_{ab} itself is constant, or if it is sufficiently symmetric. We shall assume $g_{11} = g_{22}$ in the following, because it allows both for a bi-metric approach and an implementation of time-dependent $g_\pm = g_\pm(t)$. In this situation of $g_{11} = g_{22} \equiv g_{\text{diag}}$, the eigenvalues are simply given by $g_\pm = g_{\text{diag}} \pm g_{\text{off}}$, where $g_{12} = g_{21} \equiv g_{\text{off}}$, and the eigenvectors read

$$\phi_\pm = \frac{\delta S_1 \pm \delta S_2}{\sqrt{2}}. \tag{12}$$

If, in addition, $g_{\text{diag}} \approx g_{\text{off}}$, which can be fulfilled to a high degree of accuracy (on the level of 3%) between different hyperfine species in ^{87}Rb (Ref. [18]) as well as in ^{23}Na (Ref. [19]), we have one "hard" density mode ϕ_+, and one "soft" spin mode ϕ_-. This separation of energy scales occurs close to the point of spatial phase separation of the two components, due to the increased interspecies repulsion g_{off} [16,17].

IV. DISPERSION RELATION AND THE PLANCK SCALE

So far we discussed the effective geometry for low-energy excitations (sound waves). The full dispersion relation, without the Thomas-Fermi approximation (i.e., not neglecting the quantum pressure terms) can be obtained via the JWKB approximation, which amounts to the geometrical optics limit of quasiparticle propagation.

Combining the (linearized) equation of continuity,

$$i(\omega + \mathbf{v}_a^0 \cdot \mathbf{k}) \delta\varrho_a = \frac{\varrho_a^0}{m_a} \mathbf{k}^2 \delta S_a, \tag{13}$$

and the Bernoulli-type equation (4) augmented with the quantum pressure term on the right-hand side,

$$i(\omega + \mathbf{v}_a^0 \cdot \mathbf{k}) \delta S_a + \sum_b g_{ab} \delta\varrho_b = -\frac{\mathbf{k}^2}{4m_a\varrho_a^0} \delta\varrho_a, \tag{14}$$

gives us two Bogoliubov dispersion relations of the usual type. Using the assumptions (8), we obtain

$$\left(\frac{m}{\varrho_0}(\omega_\pm + \mathbf{v} \cdot \mathbf{k})^2 + \frac{\mathbf{k}^4}{4m\varrho_0}\right) = g_\pm \mathbf{k}^2. \tag{15}$$

The deviation from the linear (rest frame) dispersion $\omega^2 \propto \mathbf{k}^2$ occurs at the two healing lengths,

$$\xi_\pm^2 = \frac{1}{4m\varrho_0 g_\pm}. \tag{16}$$

Accordingly, the analogues of the Planck length, the two coherence lengths $\xi_\pm$, scale in the same way as the inverse spin and density mode velocities,

$$\xi_\pm^2 c_\pm^2 = \frac{1}{4m^2}. \tag{17}$$

V. DE SITTER SPACE-TIME AND ANALOGUE INFLATON

The extreme dependence of the sound velocity of the spin mode on the coupling matrix g_{ab} can be profitably used to simulate a rapidly expanding universe via small temporal changes $g_{ab} = g_{ab}(t)$. A "spin horizon" for the spin mode should be easier to realize experimentally than the sound horizon of a one-component BEC, in view of the possibility to manipulate the "spin" velocity such that it closely approaches zero.

The line elements for a background at rest, $\mathbf{v} = \mathbf{0}$, read

$$ds_\pm^2 = \varrho_0\left(c_\pm dt^2 - \frac{1}{c_\pm} d\mathbf{r}^2\right). \tag{18}$$

From now on we focus on one particular mode, the spin mode, drop the subscripts $\pm$ in most of the following formulas, and furthermore set $m = 1$ for convenience. We suppose the background density ϱ_0 to remain essentially constant during rapid variations of g_- (which is possible for $V_a = V_b$, cf. Sec. VII).

Assuming a time dependence of the propagation velocities/coupling constants,

$$c = \frac{c_0}{H^2 t^2} \Leftrightarrow g = \frac{g_0}{H^4 t^4}, \qquad (19)$$

with H being the condensed-matter analogue of the Hubble parameter in cosmology, we obtain the de Sitter metric

$$ds^2 = \varrho_0 c_0 \left(d\tau^2 - \left[\frac{e^{H\tau}}{c_0} \right]^2 d\mathbf{r}^2 \right), \qquad (20)$$

with a transformed de Sitter time coordinate $\tau = H^{-1} \ln(Ht)$, representing proper time (not equal to the laboratory time), and the prefactor $\varrho_0 c_0 = \text{const.}$

The corresponding Klein-Fock-Gordon equation

$$\Box \phi = \frac{1}{\sqrt{-\mathsf{g}_-}} \partial_\mu (\sqrt{-\mathsf{g}_-} \mathsf{g}_-^{\mu\nu} \partial_\nu \phi) = 0, \qquad (21)$$

with g_- denoting the determinant of the metric $\mathsf{g}_{\mu\nu}^-$ and $\mathsf{g}_-^{\mu\nu}$ the inverse of the metric, assumes in this case the simple form (in three spatial dimensions)

$$\left(\frac{\partial^2}{\partial \tau^2} + 3H \frac{\partial}{\partial \tau} - e^{-2H\tau} [c_0 \boldsymbol{\nabla}]^2 \right) \phi = 0. \qquad (22)$$

After a spatial mode expansion into plane waves (taking into account isotropy and homogeneity), each mode behaves as a damped harmonic oscillator with a time-dependent potential

$$\left(\frac{\partial^2}{\partial \tau^2} + 3H \frac{\partial}{\partial \tau} + e^{-2H\tau} [c_0 \mathbf{k}]^2 \right) \phi_{\mathbf{k}} = 0. \qquad (23)$$

The evolution induced by the above equation of motion can roughly be split up into three regimes.

(i) Oscillation: for very early times, we have $e^{-2H\tau} [c_0 \mathbf{k}]^2 \gg H^2$; hence the damping term can be neglected and the modes oscillate almost freely.

(ii) Horizon crossing: at some point in time, the monotonously decreasing term $e^{-2H\tau}$, corresponding to the expansion of the universe, becomes small enough and the damping term starts to play a role. Since the length scale c_0/H corresponds to the size of the particle horizon in an expanding (de Sitter) universe, this is the point where the modes cross the horizon and hence do not oscillate freely anymore.

(iii) Freezing: for late times, we have $e^{-2H\tau} [c_0 \mathbf{k}]^2 \ll H^2$, and hence the potential term can be neglected. This corresponds to a strongly over-damped oscillator and thus the modes do effectively not evolve anymore.

The same three essential stages undergoes the inflaton field during the epoch of inflation in the very early universe, according to our (present) standard model of cosmology [2]. Since the Hubble parameter during inflation is expected to be significantly smaller than the Planck scale—otherwise the (semiclassical) notion of a (quantum) field within a curved space-time would not apply—the modes leave the Planck regime (are "born") nearly freely oscillating. Later on, with increasing τ, they cross the horizon and freeze. At the end of inflation, the frozen quantum fluctuations of the inflaton field are transferred into (classical) density fluctuations (decoher-ence of the quantum state due to interaction with other degrees of freedom), which are in turn supposed to be the seeds for structure formation in our universe represented, e.g., by our galaxy.

VI. MEASUREMENT OF FLUCTUATIONS

In order to discuss the quantum state of the fluctuations $\hat{\phi}$, it is convenient to introduce yet another time coordinate, i.e., the conformal time $\eta = -e^{-H\tau}/H = -1/(H^2 t)$, in terms of which the de Sitter metric (20) can be cast into the conformally flat form

$$ds^2 = \frac{\varrho_0 c_0}{H^2 \eta^2} \left(d\eta^2 - \frac{1}{c_0^2} d\mathbf{r}^2 \right). \qquad (24)$$

Expanding the phase operator $\hat{\phi}$ into plane-wave solutions of the Klein-Fock-Gordon equation (21) in conformal time,

$$\left(\frac{\partial^2}{\partial \eta^2} - \frac{2}{\eta} \frac{\partial}{\partial \eta} + [c_0 \mathbf{k}]^2 \right) \phi_{\mathbf{k}}(\eta) = 0, \qquad (25)$$

we have, for an arbitrarily chosen quantization volume V, the analytical expression

$$\hat{\phi}(\mathbf{r}, \eta) = H \sqrt{\frac{g_0}{2Vc_0^3}} \sum_{\mathbf{k}} \frac{i - c_0 k \eta}{\sqrt{k^3}} e^{i\mathbf{k}\cdot\mathbf{r} - ic_0 k \eta} \hat{a}_{\mathbf{k}} + \text{H.c.} \qquad (26)$$

We introduced creation and annihilation operators $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$, where the $\hat{a}_{\mathbf{k}}$ annihilate the "adiabatic" vacuum state $|0\rangle_{\text{ad}}$ (see, e.g., Ref. [26]),

$$\hat{a}_{\mathbf{k}} |0\rangle_{\text{ad}} = 0. \qquad (27)$$

For early times (in the oscillating regime $\eta \downarrow -\infty$), the adiabatic quantum vacuum state coincides to zeroth order with the instantaneous ground state and is therefore a natural candidate for the vacuum state. As indicated above, after horizon crossing, the fluctuations are frozen at late times $t \uparrow \infty \rightsquigarrow \eta \uparrow 0$. In this regime, we obtain from Eq. (26) the expectation value

$$\langle \hat{\phi}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}} \rangle = \frac{H^2 g_0}{2Vc_0^3 k^3} \equiv |\phi_{\mathbf{k}}^0|^2. \qquad (28)$$

We display the evolution of a $\phi_{\mathbf{k}}$ component of Eq. (26) and the approach to the above "frozen" value in Fig. 1.
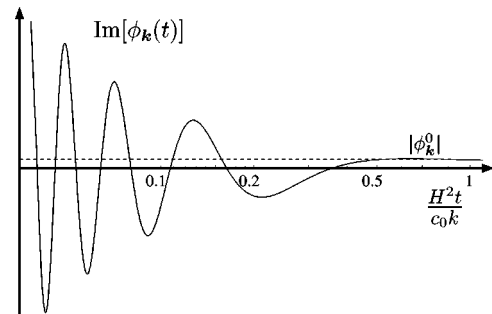

FIG. 1. Freezing process for the modes $\phi_{\mathbf{k}}$ according to expression (26), with the absolute value of the frozen $\phi_{\mathbf{k}}^0$ given by Eq. (28), displayed on a logarithmic laboratory time scale t.

In complete analogy to the cosmic inflaton field, the expression (28) generates a scale-invariant spectrum: In view of the d^3k integration, the corresponding spatial correlation function is (within the region of validity of the above approximations) invariant under rescaling $\langle \hat{\phi}_-(\mathbf{r}) \hat{\phi}_-(\mathbf{r}') \rangle \simeq \langle \hat{\phi}_-(\lambda \mathbf{r}) \hat{\phi}_-(\lambda \mathbf{r}') \rangle$, see, e.g., Ref. [27].

A. Phase-phase correlations

Let us investigate the impact of the frozen quantum fluctuations on the (spatially) Fourier-transformed two-point phase-phase correlation function defined as

$$C_\phi(\mathbf{k},t) = \int d^3r \exp\{i\mathbf{k} \cdot \mathbf{r}\} \langle \hat{\phi}_-(\mathbf{0},t) \hat{\phi}_-(\mathbf{r},t) \rangle. \quad (29)$$

Inserting Eq. (28) we obtain (after freezing)

$$C_\phi(\mathbf{k}) = \frac{H^2 g_0}{2 c_0^3 k^3}. \quad (30)$$

It appears that one could generate an arbitrarily strong effect by increasing the Hubble parameter H, but this is a fallacy. The validity of the approximations made requires that the conditions $c_0 k \gg H \gg c_{\text{final}} k$ on the frequency scales hold (with c_{final} denoting the final value of the time-dependent speed of sound), as well as that the long-wavelength limit $k\xi_-(t) \ll 1$ is fulfilled. In order to estimate the maximally obtainable effect, we insert $H \sim c_0 k$ as well as $k \sim k_{\text{final}}^{\text{Planck}}$ with $k_{\text{final}}^{\text{Planck}}$ denoting the inverse of the final healing length, i.e., the final Planck scale. The maximum effect of the dimensionless quantity $(\Delta k)^3 C_\phi(\mathbf{k})$, i.e., the correlation within some wave-number interval $(\Delta k)^3$, then reads

$$(\Delta k)^3 C_\phi^{\text{max}}(\mathbf{k}) \sim \sqrt{\frac{g_0}{g_{\text{final}}}} \frac{(\Delta k)^3}{\varrho_0} \sim \sqrt{\frac{g_0}{g_{\text{final}}}} \sqrt{\varrho_0 g_{\text{final}}^3} \left(\frac{\Delta k}{k_{\text{final}}^{\text{Planck}}} \right)^3. \quad (31)$$

Since the concept of the effective geometry only applies for $k \ll k_{\text{final}}^{\text{Planck}}$, the maximum impact of the frozen quantum fluctuations on the two-point phase-phase correlation function is limited by the product of the ratio $\sqrt{g_0/g_{\text{final}}} \gg 1$ and the final diluteness parameter $\sqrt{\varrho_0 g_{\text{final}}^3}$—which is bound to be a quite small quantity; for the spin mode of relevance here it is of order $10^{-2} \cdots 10^{-4}$.

B. Density-density correlations

In contrast to phase-phase correlations, which can only be measured by time-of-flight measurements [29], density-density correlations can be obtained directly, i.e., *in situ,* for example via (state-selective) absorption (such *in situ* measurements are generally more difficult to be carried out, though [29]).

The spatially Fourier-transformed two-point density-density correlation function is defined, in a manner analogous to its phase-phase counterpart in Eq. (29), to be

$$C_\varrho(\mathbf{k},t) = \int d^3r \exp\{i\mathbf{k} \cdot \mathbf{r}\} \langle \delta\hat{\varrho}_-(\mathbf{0},t) \delta\hat{\varrho}_-(\mathbf{r},t) \rangle. \quad (32)$$

Note that the expectation values in Eqs. (29) and (32) are realized via statistical averages in any measurement since the quantum state is not an eigenstate.

To calculate the density fluctuations, we have to derive the time dependence of the phase fluctuations beyond the zeroth-order (frozen) part $\phi_{\mathbf{k}}^0$. The approximate solution of Eq. (23) respectively Eq. (25) for frozen inflaton modes, transforming back to laboratory time, reads

$$\phi_{\mathbf{k}}(t) = \phi_{\mathbf{k}}^0 \left(1 + \frac{c_0^2 k^2}{2H^4 t^2} \right) + O\left(\frac{1}{t^3} \right). \quad (33)$$

Since $g \propto 1/t^4$, the density fluctuations increase linearly in time, cf. Eq. (4):

$$\delta\varrho_{\mathbf{k}} = -\frac{\dot{\phi}_{\mathbf{k}}}{g} = \phi_{\mathbf{k}}^0 \frac{c_0^2 k^2}{g_0} t + O(t^0). \quad (34)$$

The overdot denotes a partial derivative with respect to laboratory time. The sub-leading-term $O(t^0)$ ensures the validity of the canonical commutation relation for hydrodynamical density and phase operators, $[\delta\hat{\varrho}(t,\mathbf{r}), \hat{\phi}(t,\mathbf{r}')] = i\delta(\mathbf{r}-\mathbf{r}')$.

Using the above relations (28) and (34), and inserting into Eq. (32) yields

$$C_\varrho(\mathbf{k},t) = \frac{H^2 c_0 k}{2 g_0} t^2. \quad (35)$$

In addition to the assumptions already discussed in the case of phase-phase correlations ($c_0 k \gg H \gg c_{\text{final}} k$ and $k\xi_-(t) \ll 1$), the total time duration of the sweep is limited by the maximally possible variation of the coupling $t_{\text{final}}^2 = c_0/(c_{\text{final}} H^2)$, according to Eq. (19). Again we estimate the maximally obtainable effect by setting $H \sim c_0 k, k \sim k_{\text{final}}^{\text{Planck}}$, and $t_{\text{final}}^2 = c_0/(c_{\text{final}} H^2)$. Inserting into Eq. (35), the maximally obtainable relative change induced in the two-point density-density correlation function in the wave-number interval Δk reads

$$\frac{(\Delta k)^3 C_\varrho^{\text{max}}(\mathbf{k},t_{\text{final}})}{\varrho_0^2} \sim \frac{H}{c_{\text{final}} c_0} \frac{(\Delta k)^3}{\varrho_0} \sim \sqrt{\varrho_0 g_{\text{final}}^3} \left(\frac{\Delta k}{k_{\text{final}}^{\text{Planck}}} \right)^3, \quad (36)$$

where the last relation also uses, from $k_{\text{final}}^{\text{Planck}} \sim \sqrt{2\varrho_0 g_{\text{final}}}$, that $H \approx c_{\text{final}} \sqrt{\varrho_0 g_0} = c_{\text{final}} c_0$.

The density-density correlations are increasing with time. However, the maximal relative change is still by a factor of $\sqrt{g_0/g_{\text{final}}} \gg 1$ smaller than in the case of the phase-phase correlations, Eq. (31). For density-density as well as phase-phase quantum correlations, the final outcome is rather small owing to the intrinsic smallness of the final diluteness parameter $\sqrt{\varrho_0 g_{\text{final}}^3}$.

C. Amplification in the unstable regime

It became evident in the preceding considerations that the impact of the frozen vacuum fluctuations is in principle observable, but rather weak. Therefore it is desirable to find

some mechanism for the amplification of the fluctuations before measuring them. As one possibility, we propose changing the sign of g_- after the inflation phase, i.e., switching to the unstable regime, for a short period of time. If this change in g_- occurs much faster than the (frozen) dynamics of the field ϕ_-, we may apply the sudden approximation, assuming a step-function-like change of g_- from a small and positive g_-^{in} to a small negative value g_-^{out}.

Within the sudden approximation, the relation of the behavior of the field ϕ_- just before $(\phi_-^{in}, \dot{\phi}_-^{in})$ and after $(\phi_-^{out}, \dot{\phi}_-^{out})$ the rapid change of g_- from $g_-^{in} > 0$ to $g_-^{out} < 0$ can be obtained from the equations of motion

$$\phi_-^{in}(\mathbf{r}, t) = \phi_-^{out}(\mathbf{r}, t),$$

$$\frac{1}{g_-^{in}} \dot{\phi}_-^{in}(\mathbf{r}, t) = \frac{1}{g_-^{out}} \dot{\phi}_-^{out}(\mathbf{r}, t). \tag{37}$$

The latter equality is equivalent to $\delta \varrho_-^{in}(\mathbf{r}, t) = \delta \varrho_-^{out}(\mathbf{r}, t)$. Assuming a constant $g_-^{out} < 0$ after the transition to the unstable region, the subsequent time evolution is simply given by the superposition of the two independent solutions of imaginary spin mode frequency,

$$\phi_{\mathbf{k}}^{out}(t) = A_{\mathbf{k}} \exp\left\{ + \sqrt{\frac{\varrho_0 |g_-^{out}| \mathbf{k}^2}{m}} t \right\}$$
$$+ B_{\mathbf{k}} \exp\left\{ - \sqrt{\frac{\varrho_0 |g_-^{out}| \mathbf{k}^2}{m}} t \right\}. \tag{38}$$

Fluctuations with large wave numbers (but still small compared to the Planck scale) grow faster. Note that this behavior is opposite to that of the gravitational (Jeans) instability in the early universe, where small wave numbers experience a stronger amplification—larger structures collapse faster due to the gravitational attraction than inhomogeneities on smaller length scales [27,28].

If we had $\dot{\phi}_{\mathbf{k}}^{out}/\phi_{\mathbf{k}}^{out} = -\sqrt{\varrho_0 |g_-^{out}| \mathbf{k}^2/m}$, the factor $A_{\mathbf{k}}$ and hence the growing component of the solution would exactly vanish. However, during the phase of freezing, the time derivative $\dot{\phi}_{\mathbf{k}}$ decreases and is finally suppressed by a factor of $\sqrt{c_{final}/c_0}$. Hence the exponentially increasing part $A_{\mathbf{k}}$ does contribute in a roughly equal amount, $A_{\mathbf{k}} \approx B_{\mathbf{k}}$. As a result, we obtain a drastic amplification of the phase (and also density) fluctuations until the nonlinear regime (that is, phase separation [17]) is reached, where the fluctuations are not small compared to the order parameter itself anymore. Consequently, one would expect the fluctuations to be measurable—with state-selective absorption imaging, for example—provided that all other perturbations are small enough (see the next section).

VII. CONCLUSIONS

In summary, it is possible to simulate the behavior of the inflaton field—more accurately, its quantum fluctuations—within two-component Bose-Einstein condensates. If we decrease g_- with time according to Eq. (19), the phase fluctuations ϕ_- (i.e., the spin mode) experience an effective de

Sitter metric, and hence undergo the three stages of free oscillation, horizon crossing, and freezing explained in Sec. V. The frozen initial quantum fluctuations can be measured after an amplification phase, in which the coupling matrix is tuned to $g_- < 0$.

The difference of the spectra (in k space) of the frozen initial quantum fluctuations (with an approximately scale invariant spectrum $\propto 1/k^3$) and the usual quantum fluctuations of a condensate at rest ($\propto 1/k$) provides one possibility to distinguish the signal from other effects. As another option for a consistency check, one could keep the coupling $g_- > 0$ constant for a given time duration after the de Sitter phase (which has a decreasing $g_- \propto t^{-4} > 0$) and before switching to the unstable regime $g_- < 0$. This way, one would allow the modes to freely oscillate again before amplification, and thereby suppress those modes which are in the second or fourth quarter of their period (i.e., have a decreasing amplitude) at the moment of switching to the unstable regime. This intermediate phase of small constant $g_- > 0$ would be roughly analogous to the epoch after inflation, and the suppression mechanism is similar to the suppression of modes in the cosmic microwave background, leading to the peaks and valleys in its power spectrum [30].

A. Advantages

In comparison with the one-component case, the realization of an effective geometry simulating the expansion of the universe in a two-component Bose-Einstein condensate has several advantages: First, in order to generate large relative variations of the spin coupling g_-, it is not necessary to go near the Feshbach resonance, and therefore the problems associated with molecule formation (inducing an imaginary part in the coupling constant) mentioned in the introduction are not relevant. Second, it is possible to change g_- while sustaining a constant background density: Assuming $V_a = V_b$ and $g_+ = \text{const}$, we obtain $\mathbf{v}_a^0 = 0$ and $\varrho_-^0 = 0$, $\dot{\varrho}_+^0 = 0$ as a valid solution even for time-dependent $g_-(t)$. Third, it is possible to amplify the fluctuations in the unstable regime g_- without necessarily destroying the condensate by phase separation.

B. Experimental parameters

In order to discuss the experimental feasibility of the proposed quantum simulation of the inflaton, it is necessary to provide an estimate for the experimental parameters involved. If we assume that we may achieve essentially equal initial healing lengths for the spin and the density mode $\xi_-^{in} \approx \xi_+^{in} = O(100 \text{ nm})$, we may simulate an expansion of the universe corresponding to more than one e-folding (i.e., $H\Delta\tau > 1$), arriving at the final healing lengths $\xi_-^{out} = O(1 \ \mu\text{m})$ and $\xi_+^{out} = \xi_+^{in} = O(100 \text{ nm})$. Although one e-folding is tiny compared to the expansion during inflation (which involves many orders of magnitude), one can still reproduce generic features (such as freezing). Since the concept of the effective geometry only applies for wavelengths large compared to the healing length (the analogue of the Planck scale), the characteristic size of the interesting structures (and hence the size of the condensate cloud) should be several micrometers—

which can be resolved with optical methods (e.g., state-selective resonant absorption imaging).

Furthermore, the measurability of the quantum fluctuations requires the absence of any other fluctuations (noise) which may swamp the signal. Assuming an initial speed of sound of about 1 mm/sec, a mode with a wavelength of a few μm corresponds to an initial frequency somewhere below 1 kHz. Hence one should avoid temperatures above 100 nK, for which the thermal fluctuations exceed the (to be measured) quantum fluctuations (assuming no thermalization during the experiment). For the parameters specified above, the Hubble parameter determining the timescale for changing g_- would also be below 1 kHz. This should represent no problem as it is possible to change the coupling constants (i.e., the external magnetic field) on the time scale of some microseconds.

Although quite challenging, the proposed experiment should thus in principle be feasible with present-day technology.

C. Planckian problem

Apart from simulating the inflaton field experimentally (quantum simulation via analogue), one of the long-term motivations of the present work was to be able to ultimately investigate the impact of the behavior at the effective Planck scale on the frozen fluctuations (as explained in the introduction). For dilute Bose-Einstein condensates, the behavior for small wavelengths (i.e., below the healing length) is in some sense trivial—the excitations are free particles of mass m (the constituent atoms), with $\omega = \mathbf{k}^2/(2m)$. More importantly, though, the healing length increases with time according to Eqs. (16) and (19)—i.e., it grows during the expansion—which is (almost) certainly not a realistic feature of the real Planck length. In order to circumvent this obstacle, one could introduce another cutoff scale which is sensitive to frequencies instead of wave numbers. For example, via exposing the condensate to monochromatic radiation with a detuned radio frequency $\omega = \omega_0 - \Delta\omega$ with ω_0 being the frequency of a suitable hyperfine transition and $\Delta\omega$ the detuning, one may frequency-selectively couple phonons with the frequency $\Delta\omega$ to that transition. In that case, $\Delta\omega$ would be a reasonable analogue of the Planck scale, by fixing a frequency scale at which free phonon propagation is cut off.

D. Outlook

After succeeding to measure the main effect—the fact that quantum fluctuations of the inflaton field in de Sitter space-time get frozen at a specific value—we can start to manipulate the behavior at the effective Planck scale and study its impact on the frozen fluctuations. The exciting prospect of undertaking such manipulations of Planck scale physics is that one might be able to experimentally investigate analogue signatures of new (trans-Planckian) physics in the anisotropies of the cosmic microwave background; such possible signatures of trans-Planckian physics are discussed, e.g., in Ref. [31].

We, finally, mention that a further interesting aspect of the scale separation of independently propagating spin and density modes is the possible implementation of a variant of the supersonical "warp drives" proposed in Ref. [32]. The reference frame, against which the "superluminal" motion of the ϕ_- mode is measured, is not provided by the laboratory frame, like in Ref. [32], but by the ϕ_+ mode, which has effectively instantaneous signal transfer on the time scales of the ϕ_- mode. In addition, the "hard" ϕ_+ mode is essentially flat on the curvature radius scale of the ϕ_- mode, i.e., appears to be essentially Minkowski space-time as seen from the ϕ_- mode. The Minkowski background of the ϕ_+ mode represents a necessary prerequisite for "superluminal" travel in warped space-times to be operational and definable, because it enables the comparison of two metrics, one of them flat and the other curved, on the same manifold [33]. In contrast to the single Euler-fluid case of Ref. [32], in the presently discussed two-component Bose-Einstein-condensed gas both of these metrics obey Lorentz invariance.

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