Influence of confining anisotropy on the unstable behavior of a Bose gas with attractive interaction

A. G. de Sousa,[1,2] A. B. F. da Silva,[1] G. C. Marques,[3] and V. S. Bagnato[2]
[1]*Instituto de Química de São Carlos, Universidade de São Paulo, Caixa Postal 369, São Carlos-SP 13560-970, Brazil*
[2]*Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, São Carlos-SP 13560-970, Brazil*
[3]*Instituto de Física de São Paulo, Universidade de São Paulo, Caixa Postal 66318, São Paulo-SP 05315-970, Brazil*
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We apply controlled perturbation theory to calculate the spectrum of the Gross-Pitaevskii equation for a system composed of attractive bosons confined in an anisotropic harmonic trap. The energy spectrum is calculated as a function of the coupling parameters for traps going from cigar to pancake shapes. The critical number of particles that ensures real values for the energy spectrum is obtained as a function of the potential anisotropic parameter, showing strong dependence of the critical number on the anisotropy of the trap. For a number of particles above the critical value the metastability of the system is characterized through the calculation of the condensate lifetime, using the imaginary part of the energy values. The obtained results are relevant for experiments where highly anisotropic traps are considered.

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I. INTRODUCTION

The vast interest in Bose-Einstein condensation (BEC) arises partly from the fact that this phenomenon touches several physical disciplines thus creating a link between them. In the quantum mechanics view BEC appears as a coherent matter wave arising from overlapping de Broglie waves and is analogous to conventional and "atom lasers," superfluidity, interferometry, holography, and lithography among others. In general, BEC is a new window to the quantum world.

A few years ago, BEC in weakly interacting confined atomic gases was achieved in laboratories [1–4]. These observations have now been confirmed by an impressive number of groups worldwide and have triggered an enormous amount of theoretical and experimental investigation on the subject. While the early works focused on the equilibrium thermodynamics of condensates close to the phase transition, recently the dynamical response of the condensate wave function to external perturbations was the subject of thorough investigation. Subsequently, general attention turned to the study of the superfluid characteristics of BECs, phenomena of quantum transport, and the interaction of BECs with light. Meanwhile, exotic states like multiple species condensates [5,6] and vortices [7,8] have been created. Feshbach resonances have been found [9–11] and various kinds of atom lasers have been constructed [12–16]. BEC interferometers have been realized [17], experiments on diffraction of BECs have been carried out [18], and nonlinear matter-wave amplification [19–21] has been observed. There is still room for many different studies involving a great variety of aspects. Since the first realization of BEC with attractive interaction [3] the metastable behavior of the system has suggested ways to study the condensate decay. Nowadays theoretical and experimental groups have investigated the macroscopic quantum tunneling of the Bose condensate and also have studied the methods to stabilize Bose condensate with attractive interactions [22–27]. The system with attractive interaction has still many features to be understood.

The aim of this paper is to consider an attractive Bose gas and to derive an approximate solution to the Gross-Pitaevskii equation with a negative effective interaction strength in a cylindrical symmetry, such that it would be accurate for arbitrary values of the coupling parameter. We also analyze the stable critical number of particles N_c and the lifetime as functions of the anisotropy of the confining potential. In terms of the use of controlled perturbation theory to solve the Gross-Pitaevskii equation, this is a similar approach to that of our previous work [28]. Here, however, we consider an arbitrary anisotropic harmonic trap that is a configuration widely used nowadays.

II. CONTROLLED PERTURBATION THEORY

Because controlled perturbation theory has been widely used, we used this opportunity to review this technique in some detail.

Atomic interactions for dilute trapped gases are well described by the Fermi contact potential because the ultralow energies make the interaction shape independent. The interatomic potential is therefore given by

$$\Phi(r) = A\delta(r), \quad A \equiv 4\pi\hbar^2 \frac{a_s}{m_0}, \tag{1}$$

where a_s is the s-wave scattering length and m_0 the atomic mass. For the external confining potential, we will consider a harmonic trap, of general shape,

$$U(r) = \frac{m_0}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{2}$$

The quantum description of the trapped atoms is well within the Gross-Pitaevskii (GP) equation, which for a system of N particles can be written as

$$\hat{H}(\varphi) = -\frac{\hbar^2\nabla^2}{2m_0} + U(r) + NA|\varphi|^2, \tag{3}$$

where A is the interaction parameter and $U(r)$ the confining potential.

We shall consider a harmonic potential of cylindrical symmetry, with radial frequency

$$\omega_r \equiv \omega_x = \omega_y$$

and axial frequency ω_z such that the anisotropy parameter and oscillator length l_r, are defined as

$$\lambda \equiv \frac{\omega_z}{\omega_r}, \quad l_r \equiv \sqrt{\frac{\hbar}{m_0\omega_r}}. \tag{4}$$

The coupling parameter is defined as

$$g \equiv 4\pi\frac{a_s}{l_r}N. \tag{5}$$

The nonlinear eigenproblem for the Hamiltonian (3) cannot be solved exactly. The standard perturbation theory starting with the harmonic-oscillator approximation cannot be employed if an arbitrary amplitude of the coupling parameter defined in Eq. (5) is considered. It is possible to find accurate approximate expressions for the whole spectrum for arbitrary values of the coupling parameter by means of controlled perturbation theory [29–34], which will be outlined here.

Assume that we are looking for a function $E(g)$ of a coupling parameter g. For simplicity, we consider that the function and coupling parameter are real throughout the whole procedure. This, however, can be straightforwardly extended to the case of complex functions.

If one invokes standard perturbation theory, valid for small coupling parameters, one gets for the eigenvalues a sequence $\{p_k(g)\}$ of perturbative approximations ($k = 0,1,2,\ldots$), such that

$$E(g) \approx p_k(g) \tag{6}$$

However, the perturbative sequence $\{p_k(g)\}$ is usually divergent if an arbitrary value of g is considered. Moreover, the coupling parameter g is often not small, in which case the perturbative sequence $\{p_k(g)\}$ cannot, in principle, provide reasonable approximations.

The main idea of controlled perturbation theory consists in the introduction of control functions that optimize the convergence of the above calculational procedure. Instead of a divergent sequence $\{p_k(g)\}$, one would get a convergent sequence $\{E_k(g,u_k)\}$, whose convergence is governed by control functions $u_k = u_k(g)$. The control functions for making a perturbative sequence convergent were first published by Yukalov in 1976 [35]. The inclusion of control functions can be done in different ways. A straightforward way is to start the perturbation theory with an initial approximation containing a set of trial parameters u. Later they are then transformed into functions $u_k(g)$ such that the sequence $\{e_k(g)\}$ of the terms

$$e_k(g) \equiv E_k(g,u_k(g)) \tag{7}$$

becomes convergent. Perturbation theory, reorganized by introducing control functions [35], has been successfully applied to a variety of problems in quantum mechanics, statistical physics, and field theory [35–45]. Perturbation theory thus reorganized has different names, such as optimized perturbation theory, controlled perturbation theory, modified perturbation theory, renormalized perturbation theory, the oscillator-representation method, the δ expansion, and many others. The method of potential envelopes [46–48] is also closely related to this approach. More details about the method can be found in reviews and papers included among the references [49–52].

Only in a few simple cases such as zero- and one-dimensional anharmonic oscillators [53–55] can control functions be chosen from the direct observation of convergence. In contrast to this, the standard situation is when perturbative terms, of arbitrarily large orders, are not available. For realistic problems one is usually able to find just a couple of perturbative terms. Because of this, one usually defines control functions by invoking some heuristic reasons.

The foundation for the choice of control functions can be built in the frame of the self-similar approximation theory [56–61]. These functions are to be chosen so that they provide the optimal convergence, i.e., the convergence is as fast as possible. To derive the concrete equations defining the control functions, it is necessary to construct a dynamical system, called the approximated cascade [60,61], whose trajectory is bijective to the approximation sequence $\{e_k(g)\}$. The limit of the latter is one-to-one correspondence with an attractive point of the approximation cascade. Approaching the fixed point, the cascade velocity for the k order, $V_k(g)$, defined as

$$V_k(g) = E_{k+1}(g,u_k) - E_k(g,u_k) + (u_{k+1} - u_k)\frac{\partial}{\partial u_k}E_k(g,u_k), \tag{8}$$

tends to zero. Hence, the closer we are to the fixed point, the smaller is the modulus of the cascade velocity (8). In others words, to provide the fastest convergence for the control functions, one has to minimize the cascade velocity modulus

$$|V_k(g)| \leqslant |E_{k+1}(g,u_k) - E_k(g,u_k)| + \left|(u_{k+1} - u_k)\frac{\partial}{\partial u_k}E_k(g,u_k)\right|. \tag{9}$$

Two variants of the fixed-point conditions can be considered to minimize either the minimal-difference condition

$$E_{k+1}(g,u_k) - E_k(g,u_k) = 0 \tag{10}$$

or the minimal-sensitivity condition

$$(u_{k+1} - u_k)\frac{\partial}{\partial u_k}E_k(g,u_k) = 0. \tag{11}$$

The latter, since in general $u_{k+1} \neq u_k$, reduces to the variational condition

$$\frac{\partial}{\partial u_k}E_k(g,u_k)=0. \tag{12}$$

Both conditions (10) and (12) are widely used within the various applications. When it happens that Eq. (10) or (12) has no solutions, these equations can be generalized to the condition

$$\min_u |E_{k+1}(g,u)-E_k(g,u)|, \tag{13}$$

or to the condition

$$\min_n \left|\frac{\partial}{\partial u}E_k(g,u)\right|. \tag{14}$$

The accuracy of the controlled approximants (7), as compared to the known value $E(g)$, is characterized by the percentage errors

$$\varepsilon_k(g)\equiv\frac{e_k(g)-E(g)}{|E(g)|}\times 100\% . \tag{15}$$

Let us emphasize the difference between controlled perturbation theory and a variational procedure based on the minimization of the internal-energy functional. First, the latter has sense solely for the ground state while the former is valid for the whole spectrum of the eigenproblem. Second, the latter implies the case of a zero-temperature condensate, while the former is independent of temperature. Third, the minimization of the internal energy yields a control value for the energy itself, but the described method provides control approximants for the spectrum.

III. CYLINDRICAL SYMMETRY MODEL

We csonsider here a cylindrical trap, which serves as an illustration of the applicability of the method and also because such geometries are often employed in experiments.

To solve the eigenproblem $\hat{H}\Psi_n=E_n\Psi_n$, with H given by Eq. (3), we invoke the controlled perturbation theory described in Sec. II, starting with the initial Hamiltonian

$$\hat{H}_0=-\frac{1}{2}\nabla^2+\frac{1}{2}(u^2r^2+v^2z^2) \tag{16}$$

containing two control parameters u and v [60,61]. The eigenvalues for the operator in Eq. (16) are easily determined as

$$E^{(0)}_{nmk}=(2n+|m|+1)u+\left(k+\frac{1}{2}\right)v, \tag{17}$$

with the quantum numbers

$$n=0,1,2\ldots,\quad m=0,\pm1,\pm2,\ldots,\quad k=0,1,2\ldots.$$

The related eigenfunctions are

$$\Psi^{(0)}_{nmk}(r,\varphi,z)=\left[\frac{2n!u^{|m|+1}}{(n+|m|)!}\right]^{1/2}r^{|m|}\exp\left(-\frac{1}{2}ur^2\right)L_n^{|m|}\times(ur^2)\frac{e^{im\varphi}}{\sqrt{2\pi}}\frac{(v/\pi)^{1/4}}{\sqrt{2^kk!}}\exp\left(-\frac{1}{2}vz^2\right)\mathrm{H}_k(\sqrt{v}z)$$

where $L_n^m(\cdot)$ are the Laguerre polynomials and $H_k(\cdot)$ are Hermite polynomials.

In first order, we have

$$E^{(1)}_{nmk}(g,u,v)=(\Psi^{(0)}_{nmk},\hat{H}\Psi^{(0)}_{nmk}). \tag{18}$$

To write down this integral explicitly, it is convenient to use the notation

$$I_{nmk}\equiv\frac{1}{u\sqrt{v}}\int|\Psi^{(0)}_{nmk}(\vec{r})|^4d\vec{r},$$

in which $\vec{r}=(r,\varphi,z)$ is the dimensionless space variable in cylindrical coordinates. We get

$$I_{nmk}=\frac{2}{\pi^2}\left[\frac{n!}{(n+|m|)!\,2^kk!}\right]^2\int_0^\infty x^{2|m|}e^{-2x}[L_n^{|m|}(x)]^4dx\times\int_0^\infty e^{-2t^2}H_k^4(t)dt.$$

It is also convenient to introduce the notation

$$p\equiv 2n+|m|+1,\quad q\equiv 2k+1, \tag{19}$$

in which the effective interaction strength is represented by

$$s\equiv 2p\sqrt{q}I_{nmk}\lambda g. \tag{20}$$

In this way, the energy levels (18) can be written as

$$E^{(1)}(g,u,v)=\frac{p}{2}\left(u+\frac{1}{u}\right)+\frac{q}{4}\left(v+\frac{\lambda^2}{v}\right)-\frac{1}{2}\frac{su\sqrt{\lambda}}{vp\sqrt{q}}, \tag{21}$$

where, for simplicity, the quantum indices n, m, and k in the left-hand side are dropped.

The fixed-point conditions are therefore obtained from the conditions

$$\frac{\partial}{\partial u}E^{(1)}(g,u,v)=0,\quad\frac{\partial}{\partial v}E^{(1)}(g,u,v)=0. \tag{22}$$

These yield to the control-function equations

$$p\left(1-\frac{1}{u^2}\right)-\frac{s}{p\lambda}\sqrt{\frac{v}{q}}=0,\quad q\left(1-\frac{\lambda^2}{v^2}\right)-\frac{s}{p\lambda\sqrt{\lambda q}}=0. \tag{23}$$

Substituting the control functions $u=u(s)$ and $v=v(s)$, defined by Eqs. (23), into Eq. (21), we obtain the controlled approximant

$$E(s) \equiv E^{(1)}[g(s),u(s),v(s)]. \tag{24}$$

It is instructive to analyze the weak-coupling and strong-coupling limits in detail. In the weak-coupling limit, s very small, and Eq. (23) gives the radial control function

$$u(s) \approx -1 - \frac{s}{2\sqrt{q\lambda}p^2} + \frac{s^2}{8q^2\lambda^2p^3} - \frac{3s^2}{8q\lambda p^4} + \frac{s^3}{4q^2\lambda^2\sqrt{q\lambda}p^5} - \frac{5s^3}{16q\lambda\sqrt{q\lambda}p^6} - \frac{3s^3}{64q^3\lambda^3\sqrt{q\lambda}p^4} \tag{25}$$

and the corresponding axial control function

$$v(s) \approx \lambda - \frac{s}{2pq\sqrt{q\lambda}} + \frac{s^2}{4q^3\lambda^2p^2} - \frac{s^2}{4q^2\lambda p^3} - \frac{3s^3}{16q^2\lambda p^5\sqrt{q\lambda}} + \frac{5s^3}{16q^3\lambda^2p^4\sqrt{q\lambda}} - \frac{7s^3}{64q^4\lambda^3p^3\sqrt{q\lambda}}. \tag{26}$$

In the strong-coupling limit, with s very large, the radial control function is

$$u(s) \approx ps^{-2/5} - \frac{p(\lambda^2q^2+3p^2)}{5}s^{-6/5} + \frac{p(-\lambda^4q^4-\lambda^2q^2p^2+3p^4)}{5}s^{-2}, \tag{27}$$

and for the axial control function we get

$$v(s) \approx \lambda^2qs^{-2/5} + \left(\frac{4q\lambda^2(-\lambda^2q^2+3p^2)}{5} - 2p^2q\lambda^2\right)s^{-6/5} + \left[p^4q\lambda^4\frac{2[(-6\lambda^4q^4+\lambda^2q^2p^2+6p^4)/25p^2+(-\lambda^2q^2+3p^2)^2/25p^2]}{p^2} + \frac{4(-\lambda^2q^2+3p^2)^2}{25p^4} + q\lambda^2\left(p^4+\frac{4p^2(-\lambda^2q^2+3p^2)}{5}\right) - \frac{8p^2q\lambda^2(-\lambda^2q^2+3p^2)}{5}\right]s^{-2}. \tag{28}$$

Finally, for the weak-coupling limit, the energy (24) becomes

$$E(s) \approx \left(\frac{p(\lambda+1/\lambda)}{2} + \frac{q(-1-\lambda^2)}{4}\right) + \left(\frac{p[1/2pq\sqrt{q\lambda}-1/2\lambda^2pq\sqrt{q\lambda}]}{2} - \frac{1}{4}\frac{e^{\left\{\frac{1}{2}\ln(4)-\frac{1}{2}Ic\ \mathrm{sgn}[(-4+2s/\sqrt{q\lambda}p^2)I]\pi\right\}}}{p\sqrt{q}} + \frac{q(1/2\sqrt{q\lambda}p^2-\lambda^2/2\sqrt{q\lambda}p^2)}{4}\right)s. \tag{29}$$

For the strong-coupling limit, we find

$$E(s) \approx \left(\frac{3}{4} - \frac{\sqrt{qv^2}}{2v\sqrt{q}}\right)s^{2/5} + \left(\frac{p^2}{2} + \frac{q^2v^2}{4}\right)s^{-2/5}. \tag{30}$$

The derived expressions (29) and (30) are valid for any combination of quantum numbers.

For atoms with negative scattering length, as in the case of ^{7}Li or a few states of ^{85}Rb, the coupling parameter Eq. (5) becomes negative. If $s<0$, the control function equations (23) have real solutions only in the interval $s_c<s<0$, where $s_c=-4$ obtained from Eqs. (22), (23), and (30).

Associated with this critical value for $s=s_c$, where $E(s)$ becomes complex, we have a critical value for the coupling parameter g_c, such that

$$g_c \equiv -\frac{s_c}{2p\sqrt{q}\lambda I_{nmk}}. \tag{31}$$

Since the complex part of the eigenvalue $E(s)$ is related to an unstable system, for numbers of particles producing $g>g_c$ the system will collapse as observed in [3].

For the ground-state level, with $n=m=k=0, p=q=1$, and $I_{000}=0.063\,494$, one finds

$$g_c = -\frac{31.50}{\lambda}. \tag{32}$$

Varying the anisotropy parameter λ, the critical coupling parameter g_c varies following (32).

The fact that there is a critical value for the coupling parameter Eq. (5) can be reformulated as the existence of a critical number of particles for stability of the condensate:

$$N_c = \frac{l_r g_c}{4\pi a_s}. \tag{33}$$

Thus, for the parameters of the experiments [3,62] with ^{7}Li, for $g_c=-31.5$, we get $N_c \sim 10^3$.

Substituting Eq. (31) in Eq. (32), we have the critical number of particles as a function of the anisotropy parameter:

$$N_c = \left|\frac{l_r - 31.5}{4\pi a_s\lambda}\right|. \tag{34}$$

Varying the anisotropy parameter λ, we can analyze the behavior of the critical number of particles N_c numerically. The result is presented in Fig. 1.

For $g>g_c$, the energy becomes complex, which implies the instability of the system. The lifetime of such a metastable system can be estimated [63,64] as

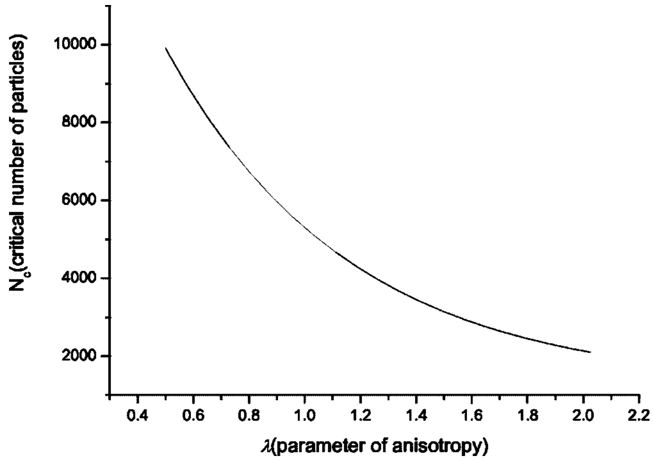

FIG. 1. The values of the critical number of particles N_c as functions of the different trap shapes for controlled perturbation theory.

$$\tau(g) \equiv \frac{1}{w_0|\mathrm{Im}\ e(g)|}, \tag{35}$$

where $e(g)=E(s(g))$. In the limit $g\to-\infty$, we have [28] for the real and imaginary parts of the energy

$$\begin{aligned}\mathrm{Re}\ e(g) \approx \{&0.251\,592\,384\,9(0.154\,508\,497\,2)\\ &+[(0.077\,254\,248\,60\times10^{-1})\lambda^2]\lambda^{2/5}\}(-g)^{2/5}\\ &+0.307\,061\,150\,3\lambda^{-2/5}g^{-2/5},\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\ e(g) \approx [&0.251\,592\,384\,9(0.475\,528\,258\,2)\\ &+(0.237\,764\,129\,1\lambda^2)\lambda^{2/5}](-g)^{2/5}\\ &-0.945\,037\,06\,\lambda^{-2/5}g^{-2/5}.\end{aligned}$$

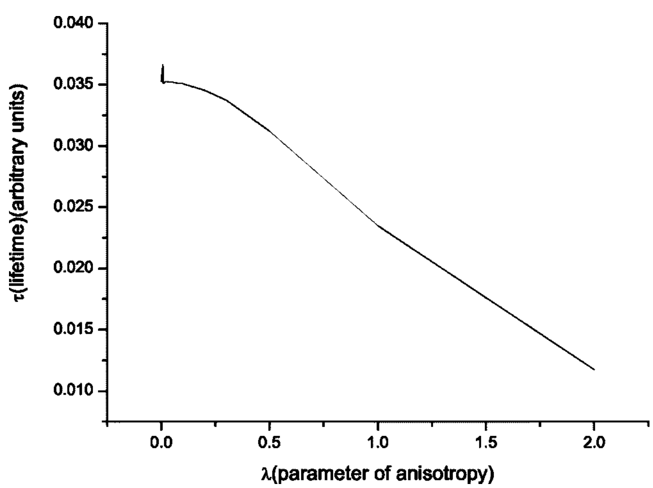

FIG. 2. The values of the lifetime τ as functions of the different trap shapes for controlled perturbation theory.

Varying the anisotropy parameter λ, we can analyze the behavior of the lifetime τ numerically (see Fig. 2). The geometric mean frequency is $\omega_0=2\pi\times145$ Hz and we are in the intermediate regime $g\approx4$.

As we observe, the critical number of particles N_c is also very sensitive to the trap shape, depending on the aspect ratio $\lambda\equiv\omega_z/\omega_r$. For the cigar-shape ($\lambda\ll1$) and spherical-shape ($\lambda=1$) traps, N_c is larger than for a disk shape ($\lambda\gg1$).

To estimate the critical number of particles, consider a spherical-shaped trap, as was previously used [3,62] for condensed ^{7}Li. With the radial frequency $\omega_r=2\pi\times150.6$ Hz and axial frequency $\omega_z=2\pi\times131.5$ Hz, $\lambda\approx0.9$. The scattering length of ^{7}Li is $a_s=-1.5\times10^{-7}$ cm. Since the oscillator length in this case is $l_r=3.2\times10^{-4}$ cm, the critical number of particles $N_c\sim5\times10^3$. However, if one takes a disk-shaped trap with aspect ratio $\lambda=10$, the critical number of particles can be as small as $N_c\sim530$, and in the cigar shape with aspect ratio $\lambda=0.1$, the critical particle number is $N_c\sim53\times10^3$. The critical number of particles is larger than studied in Ref. [28], because compared to that work we exchange the sign of the effective interaction strength, which is negative, and thus the critical coupling parameter undergoes change in its numeric value. So we verify a favorable influence to condense a large number of atoms, when the anisotropy is toward the cigar shape. We conclude that $\omega_{\mathrm{r}}>\omega_{\mathrm{z}}$ results in larger particle numbers inside the trap in the critical limit.

If the number of particles with a negative scattering length exceeds the critical particle number given by Eq. (31), such trapped atoms form a metastable state, and then only N_c particles can form a stable Bose condensate, excess particles being expelled out of the condensate during a time on the order of the time provided in Eq. (34). This lifetime is also sensitive to the trap shape. For the cigar-shape ($\lambda\ll1$) and spherical-shape ($\lambda=1$) traps, the lifetime is larger than for a disk shape ($\lambda\gg1$). When the number of particles N exceeds a critical one N_c, the Gross-Pitaevskii equation (1) with negative scattering length, which governs the condensate wave function, develops a singularity, a phenomenon known as "self-focusing" in plasma physics [65], which indicates that the system will rapidly collapse.

Recently Bose-Einstein condensation has been achieved with ^{85}Rb [66] by means of Feshbach resonance, which allowed wide tuning of the scattering length from negative to positive values. The ability to control the scattering length is used to control and measure the stability condition with the corresponding critical number of atoms.

There was a proposal [67] for stabilizing the Bose condensate with attractive interactions by driving a quadrupole collective excitation. Our consideration related in [28] suggests that it could, probably, be also possible to stabilize such condensates by transferring the atoms with the help of a resonance pumping field to excited states.

IV. CONCLUSION

Using the method of controlled perturbation theory [68] it was possible to find accurate approximate expressions for the whole spectrum of the trapped Bose gas for arbitrary values of the coupling parameter. Also, we studied the properties of Bose condensed gases in magnetic traps of high anisotropy. The behavior of the critical atom number and critical coupling parameter as a function of the anisotropy parameter shows interesting dynamics which varies with potential shape [66]. When the ground-state energy becomes complex, which implies instability in the system, we determined the lifetime at different trap anisotropic conditions.

Hence, through controlled perturbation theory we can study the spectral properties of the trapped Bose gases and finally investigate the behavior of Bose condensed gases in magnetic traps when the system becomes unstable.

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