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Valence-bond states for quantum computation

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We propose a way of universal quantum computation by doing joint measurements on distributed singlets. We show how these joint measurements become local measurements when the singlets are interpreted as the virtual components of a large valence-bond state. This proves the equivalence of the cluster-state-based quantum computational model and the teleportation-based model, and we discuss several features and possible extensions. We show that all stabilizer states have a very simple interpretation in terms of valence-bond solids, which allows to understand their entanglement properties in a transparent way.

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The concept of teleportation [1] plays a crucial role in the understanding of entangled quantum systems. It does not only allow us to use entangled states as perfect quantum channels, but also to implement nonlocal unitary operations. Based on this idea it was shown that universal quantum computation can be achieved if one can prepare a separable initial state and implement joint two-qubit measurements [2–6]. In the same spirit, but somehow orthogonal to these schemes, Raussendorf and Briegel [7] showed that universal quantum computation is possible by implementing local measurements on the qubits of a highly entangled so-called cluster state [8]. These studies highlighted the central role of entanglement for quantum computation [9,10]. However, the structure of general multiparticle entanglement is, for the moment being, still very poorly understood, and it is somehow mysterious that the cluster states enable universal quantum computation. On this note, we show that the structure of entanglement in cluster states is particularly simple and can be well understood by looking at it as a so-called valence-bond solid [11] with only nearest-neighbor bonds. This enables us to show that the one-way computer [7] essentially works in an equivalent way as the other measurement-based proposals for quantum computation [12].

This paper is organized as follows: In the first part, we show how universal quantum computation can be achieved by doing joint measurements on a collection of maximally entangled states of two qubits. The scheme derived is very similar to the schemes presented in [3–6], but has the advantage to be deterministic. We proceed by showing that the joint measurements needed in the quantum computation scheme can be converted into local measurements, at the expense of initially preparing one big entangled initial state instead of many singlets, a so-called valence-bond state [11]. We next show how all stabilizer or graph states (including cluster states) have a very simple parametrization in terms of valence-bond states. We conclude by mentioning several features and extensions of valence-bond states.

Let us start with showing how universal quantum computation can be performed using joint measurements on a collection of singlets. Imagine a quantum computer with all logical qubits on a vertical line. It is well known that a universal set of quantum gates is given by arbitrary local unitary transformations and the phase gate

$$U_{ph} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$

between neighboring qubits.

A local unitary operation U on a qubit A [$|\Psi\rangle$ in Fig. 1(a)] can be implemented as follows: (1) take a singlet $|H\rangle = (|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$ of qubits B and C (any other maximally entangled would also be fine); (2) do a Bell measurement between qubits A and B in the basis

$$|\alpha\rangle = (U^{\dagger}\sigma_{\alpha} \otimes 1)|H\rangle, \quad \alpha = 0,1,2,3, \tag{1}$$

where σ_{α} denote the Pauli matrices (including $\sigma_0 = 1$); (3) the wave function corresponding to qubit C is now given by $\sigma_{\alpha}U|\phi\rangle$, which is the wanted transformation up to an extra multiplication with a Pauli operator conditioned on the measurement outcome. This extra left multiplication with Pauli operators, however, does not harm: A later one-qubit operation V can be chosen to be conditioned on the outcome (i.e., implementing $V\sigma_{\alpha}$ instead of V). Furthermore, right multiplication of the two-qubit phase gate U_{ph} with Pauli operators is equivalent to left multiplication of it with different ones. Therefore, the extra Pauli operators can be pushed through the quantum circuit without affecting the computation. Note also that, by linearity, exactly the same discussion holds true if qubit A was initially entangled with other qubits.

Similarly, the phase gate U_{ph} can be implemented by adding three extra pairs of maximally entangled states $|H\rangle$ as depicted in Fig. 1(b). Suppose two three-qubit measurements are done [see Fig. 1(b)] in the complete bases

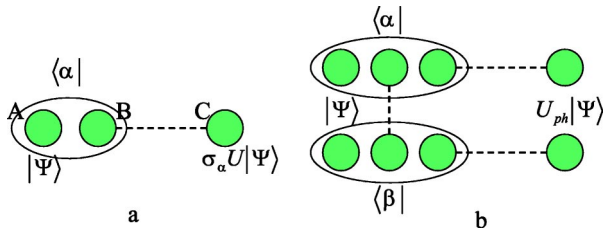

FIG. 1. (a) Implementation of a one-qubit gate by measuring in the two-qubit basis $|\alpha\rangle$ (1). The edges connected by the line denote the maximally entangled state $|H\rangle$. (b) Implementation of a two-qubit gate by three-qubit measurements in the basis $|\alpha\rangle$ and $|\beta\rangle$ (2).

$$\{|\alpha\rangle\} = \{|\beta\rangle\} = \{(\sigma_x)^i \otimes (\sigma_x)^j \otimes 1(|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle)\}, \tag{2}$$

with $i,j \in \{0,1\}$ and $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. This implements the gate $(H \otimes H)U_{ph}$ with H the Hadamard gate $H = |+\rangle\langle 0| + |-\rangle\langle 1|$, up to a harmless extra multiplication with Pauli operators. Together with the possibility of implementing local unitaries, this proves that universal quantum computation can be done by doing only generalized Bell measurements on two and three qubits.

Let us summarize the ingredients needed for being able to implement quantum computing along the lines sketched: (1) it must be possible to create ancillary singlets; (2) two- and three-qubit measurements of the form (1) or (2) can be implemented between halves of these extra singlets and the logical qubits. It is then trivial to translate a quantum circuit into a measurement scheme on distributed singlets to perform universal quantum computation.

In practical implementations, it is very hard to perform joint measurements. Let us therefore investigate whether the joint measurements can be converted into local ones at the expense of creating an initial highly entangled state suitable for universal quantum computation. This would correspond to transforming the teleportation-based quantum computation [3–6] into a cluster-state quantum computation [7]. The point of Bell or GHZ measurements is exactly the fact that the outcome of the measurement does not contain any information about the state, and hence it does not matter which outcome one obtains. Therefore, even if for one or the other reason only a two-dimensional (2D) subspace spanned by two states in (1) or (2) would be physically accessible, the whole procedure would work equally well. Indeed, in the case of GHZ measurements, measurements in, e.g., the $|000\rangle \pm |111\rangle$ basis would allow to implement a phase gate up to a local Pauli operator; in the case of Bell measurements, measurements in the basis $|00\rangle \pm \exp(-i2\xi)|11\rangle$ would allow to implement all local unitaries of the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(i\xi) & -\exp(-i\xi) \\ \exp(i\xi) & \exp(-i\xi) \end{pmatrix} \tag{3}$$

which form a complete set of quantum gates if supplemented by the phase gate.

Consider now a quantum circuit translated into the measurement-based computational model with distributed singlets and joint measurements as explained before. The trick now consists of interpreting the qubits in this scheme as virtual qubits, in such a way that joint measurements on virtual qubits correspond to single-qubit measurements on a physical qubit. More specifically, consider a configuration of singlets of virtual qubits on which joint measurements are implemented. Everywhere one has to implement a two- or three-qubit measurement, one projects the qubits under consideration on a one-qubit subspace with the projector $P = |\tilde{0}\rangle\langle 00| + |\tilde{1}\rangle\langle 11|$ or $P = |\tilde{0}\rangle\langle 000| + |\tilde{1}\rangle\langle 111|$. The tildes denote the physical Hilbert space. Starting from a configuration of singlets and doing the appropriate projections on all places such as to reduce the Hilbert space, one gets a state on which local measurements allow for implementing a specific quantum computing circuit.

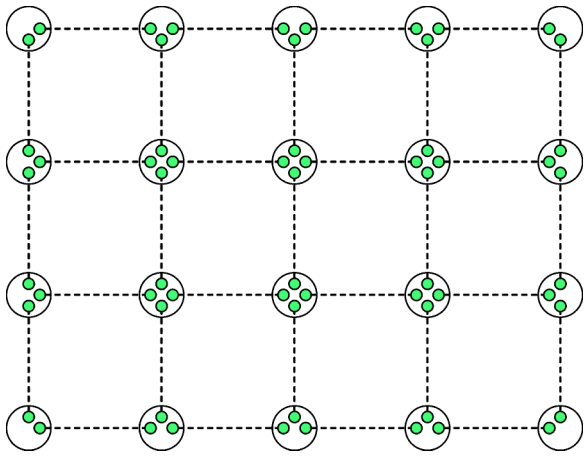

FIG. 2. Representation of a valence-bond solid. The solid circles connected with a dotted line denote virtual singlets; a bigger open circle denotes a projection P of all virtual qubits inside it with Hilbert space $H_2^{\otimes n} (n \in \{2,3,4\})$ to a physical single qubit $\tilde{H}_2$. In the present paper, P is always of the form $P = |\tilde{0}\rangle\langle 00\ldots 0| + |\tilde{1}\rangle\langle 11\ldots 1|$.

Consider now a two-dimensional $N \times M$ grid of vertices with singlets that connect all nearest-neighbor vertices (see Fig. 2). Projecting the four virtual qubits (two or three at the boundaries) to one physical qubit using the projectors $P = |\tilde{0}\rangle\langle 0000| + |\tilde{1}\rangle\langle 1111|$, one obtains a big state of $N \cdot M$ qubits. The open circles correspond to the physical qubits, and we call the connections between neighboring qubits (being singlets of virtual qubits) bonds. A measurement of a (physical) qubit in the $|\tilde{0}\rangle, |\tilde{1}\rangle$ basis destroys all bonds emanating from it, i.e., the virtual singlets emanating from it disappear and the projectors of the neighboring particles change into $|\tilde{0}\rangle\langle 000| + (-1)^m |\tilde{1}\rangle\langle 111| = \sigma_z^m (|\tilde{0}\rangle\langle 000| + |\tilde{1}\rangle\langle 111|)$, depending on the measurement outcome m (note that one virtual qubit disappears due to the broken bond). Any initial state that would implement a specific quantum circuit can, in this way, be generated from these $N \cdot M$ qubits by doing appropriate local measurements in this basis. This proves that it is possible to do universal quantum computation starting from the specific entangled state depicted in Fig. 2 on which local measurements are implemented. It turns out that this state is exactly the cluster state [8], and therefore the derived model for doing quantum computation is exactly equivalent to the one-way computer introduced by Raussendorf and Briegel [7].

States obtained by projecting halves of singlets onto lower-dimensional Hilbert spaces are known in condensed-matter physics as valence-bond solids or valence-bond states (VBS) [11,13,14]. They are very interesting as they are always ground states of local Hamiltonians, and their entanglement properties can easily be characterized. Cluster states, on the other hand, are a subset of the so-called stabilizer states [15], which are defined by specifying a complete set of commuting observables O_i, where each O_i is a tensor product of the Pauli matrices $\sigma_0, \sigma_x, \sigma_y, \sigma_z$. The stabilizer states are the

FIG. 3. Implementing a global unitary transformation on qubits 1 and 2 by doing local projections P_1 and P_2 on them and a maximally entangled state $|H\rangle$.

common eigenstates of these operators. Let us show that any stabilizer state can be interpreted as a valence-bond state. Stabilizer states can efficiently be prepared from a completely separable state by applying appropriate two-qubit unitary operations to it (see, e.g., [16]). The reason that stabilizer states are very simple and manageable to work with is due to the fact all these two-qubit unitary operations can be chosen to commute with each other [17]. The trick is now to implement these commuting two-qubit unitary transformations by a teleportation-like principle that consists of adding virtual singlets, and then doing appropriate projections [2,18]. More specifically, consider the two qubits 1 and 2 in Fig. 3; an extra singlet $|H\rangle_{\bar{1}\,\bar{2}}$ is added, and then any unitary transformation between 1 and 2 can be performed by projecting the two-qubit spaces labeled by 1, $\bar{1}$ (2, $\bar{2}$) onto the qubits 1 (2) with appropriate projectors $P_1(P_2)$. Iterating this scheme, one sees that every stabilizer state can be interpreted as a VBS, possibly with bonds extending over all sites. It would be interesting in this respect to find a normal form for stabilizer states that minimizes this number of bonds [20]. In the case of the cluster states, however, only unitaries between the nearest neighbors have to be implemented, and hence a simple VBS as depicted in Fig. 2 is obtained.

As an example, let us explicitly construct the valence-bond states corresponding to arbitrary cluster and graph states [19,20], which form, up to local unitaries, the class of all stabilizer states. To each graph state, one can associate a graph parametrized by its adjacency matrix Γ. The number of virtual qubits at each site in the VBS is of course equal to the number of bonds on the given site, and is equal to the number of vertices emanating from a given physical qubit. The bonds are maximally entangled states $|H\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$, and the projectors on each site are all of the form $P = |\tilde{0}\rangle\langle 00\ldots 0| + |\tilde{1}\rangle\langle 11\ldots 1|$. This simple construction describes all possible graph and cluster states. As an example, the graph state corresponding to the five-qubit error-correcting code [22] is depicted in Fig. 4.

This VBS interpretation of cluster states makes their nice and appealing properties very explicit. The fact that, e.g., a singlet can be created between two arbitrary qubits by doing appropriate local measurement on the other ones can readily be understood by the concept of entanglement swapping [21]. The entropy of a block of spins can readily be seen to scale roughly as the number of qubits with emanating bonds from it (i.e., proportional to the area of the surface of the block). The fact that the sensitivity to noise of a cluster state does not scale with the number of (physical) qubits [23], is due to the fact that it is effectively made up by *local* singlet pairs. This insight also enables to construct distillation protocols for cluster states by translating bipartite distillation protocols to the valence-bond picture [24]. Note, however,

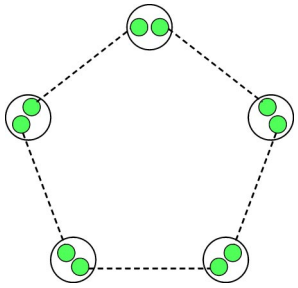

FIG. 4. The valence-bond picture of the state $|\tilde{0}\rangle$ in the five-qubit error-correcting code; the state $|\tilde{1}\rangle$ is obtained by applying local σ_z operators to all physical qubits.

that complications can arise due to the fact that local noise and operations can create correlations between the virtual singlets.

The description of valence-bond states in terms of stabilizer states is also interesting from the point of view of condensed-matter theory. It is, e.g., well known that operations of the Clifford group acting on a stabilizer state can efficiently be simulated classically. This implies that evolutions generated by the Clifford group on VBS states can be simulated efficiently, and correlation functions of products of Pauli operators can be calculated.

The present study also opens the question whether there exist ground states of (gapped) Hamiltonians involving only two-body short-range interactions on a lattice that would enable to implement the presented measurement scheme (this is not the case for cluster states). Such 2D valence bond solids indeed exist for higher spins (e.g., spin $3/2$), and it is trivial to devise a toy model for which this holds. Consider, e.g., a hexagonal lattice with spin-$7/2$ particles at each vertex. To each particle corresponds an eight-dimensional Hilbert space, which we can interpret as a system of three virtual qubits. We associate each outgoing edge to one of these qubits, and associate the Hamiltonian $\vec{S}\vec{S}+31$ to two of these qubits connected by an edge. The ground state on such a hexagonal lattice with this two-body local Hamiltonian will be unique, and the teleportation scheme can be implemented perfectly on it. Note that the cluster state is very similar to that construction, but there the three qubits are interpreted as virtual qubits and a smart projection was used to reduce the dimension of the effective Hilbert space.

More interestingly, the trick used to implement two-qubit unitary gates by introducing a virtual singlet followed by a projection—this is the way cluster states can be generated from completely separable ones—can also be extended to the case where the unitaries do not commute with each other. Indeed, the cluster state can be made in the lab if an Ising interaction can be implemented on neighboring qubits [25]. However, in some experimental set ups, it is not always possible to implement such commuting gates, as is the case, e.g., for quantum dots [26]; here one is essentially restricted to implement two-qubit gates generated by the Heisenberg interaction, which certainly do not commute when acting on neighboring spins. However, if one can apply these unitary gates sequentially (i.e., one has control over the sites on which one implements the gate), then it is also possible to

construct valence-bond solids that could be suitable for quantum computation.

The present results also show that the valence-bond solid picture is very useful for understanding multipartite entanglement. Indeed, VBS are particularly interesting from the point of view of quantum information theory, as the simple and elegant tools developed for bipartite quantum systems can be applied to it (see, e.g., [14]). Moreover, one can readily see that the VBS form a dense subset of all possible quantum states if the singlets are replaced by higher-dimensional maximally entangled states $|I\rangle=\sum_{i=1}^{D}|i\rangle|i\rangle$ and if the projectors can be chosen arbitrarily (e.g., in the case of three qubits, every state can be made by considering two singlets and projecting two qubits of them onto a qubit space [27]). It would be very interesting to develop a general theory of multiparticle entanglement based on this VBS picture, where one could construct entanglement measures that quantify the valence-bond resources needed to describe the state.

In conclusion, we have identified the entanglement properties of the cluster states that are responsible for the possibility of universal quantum computation. The main insight was given by the fact that the structure of entanglement in these states is essentially bipartite and can be understood in terms of valence bonds. This allowed to prove the equivalence of the one-way computer with teleportation-based computation schemes, and to clarify the special features of the cluster states.

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