Bell's inequalities for particles of arbitrary spin in fixed analyzers

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We propose a new set of observables for experiments on entangled particles of arbitrarily large spin that produce significant Clauser-Horne-Shimony-Holt inequality violations for fixed analyzer settings over a wider range of spins than was previously possible. These observables are better suited for experiments where analyzer orientations must be chosen before the spin of the entangled particles is known, such as experiments using polarization entangled downconverted photons.

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Bell's inequalities are a class of mathematical statements, which were derived assuming local realism, to point out contradictions between local realistic and quantum-mechanical predictions [1]. For instance, the Clauser, Horne, Shimony, and Holt (CHSH) inequality [2] is a statement about the joint probabilities of spin measurements on entangled spin-$1/2$ particles that must hold for a locally realistic theory. Quantum mechanics predicts violations of the CHSH inequality for certain analyzer orientations. Over the past few decades, violations of the CHSH inequality have been observed in several experiments [3–6]. Entangled particles have been used to achieve quantum teleportation [7], quantum cryptography [8], and quantum dense coding [9]. The actual violation of the CHSH inequality in Ekert's quantum cryptography protocol verifies the security of the quantum information channel.

While entanglement experiments have traditionally focused on spin-$1/2$ particles because of their simple two-dimensional Hilbert spaces, higher-spin entangled particles have recently generated much interest because of their ability to carry more quantum information [10]. Generalizations of the CHSH inequality for higher-spin particles provide tests for the entanglement and a generalization of the Ekert protocol for these particles. Peres first demonstrated that quantum mechanics predicts a finite violation of a CHSH-like inequality with particles of arbitrarily large spin [11,12]. The generalized CHSH statements are of the form

$$S(n,\theta,f) \leqslant 2. \tag{1}$$

Here, S is an observable that depends upon the spin of the entangled particles (n), the orientations of the analyzers (θ), and the combination of joint probabilities of spin measurements (f).

Howell, Linares, and Bouwmeester (HLB) first experimentally demonstrated a CHSH inequality violation for a spin-1 particle using polarization entangled photons generated by type-II downconversion [13]. HLB noted that polarization measurements on $2n$ pairs of polarization entangled photons—that is, $2n$ photons in each of two spatial modes—are formally equivalent to spin measurements on one pair of entangled spin-n particles, provided the $2n$-photon pairs were generated simultaneously and are therefore quantum mechanically indistinguishable [17,18]. The equivalence arises because there are exactly $2n+1$ eigenstates for both spin measurements on spin-n particles and polarization measurements on $2n$ indistinguishable photons; for instance, a spin measurement on a spin-1 particle can yield either $|1\rangle$, $|0\rangle$, or $|-1\rangle$ while a polarization measurement on two photons can yield either $|2H\rangle$, $|HV\rangle$, or $|2V\rangle$. With appropriate manipulation of birefringent elements, the second-order contribution of the downconversion field produced four photon states of the form

$$\begin{aligned}|\Psi\rangle &= |2H,2V\rangle - |HV,HV\rangle + |2V,2H\rangle \\ &\equiv |1,-1\rangle - |0,0\rangle + |-1,1\rangle. \end{aligned} \tag{2}$$

Here, $|A,B\rangle$ represents the eigenstate where the two entangled particles in spatial modes 1 and 2 are in states A and B, respectively. HLB measured the joint probabilities of polarization measurements with photon detectors and observed a violation of the CHSH inequality.

HLB were limited to experiments with two photon pairs because, for the most part, the relatively high quantum efficiency Geiger-mode detectors can only detect the presence of photons in the channel, not the actual number of photons. As a result, differentiating between the possible polarization states of multiple photons is difficult and requires many detectors. However, a new generation of high quantum efficiency detectors is being developed that will be able to determine the actual number of photons in the channel. With these detectors, experiments with multiple photon pairs will be much easier and CHSH inequalities for higher-spin particles can be tested. Further, these higher-order correlations can be used to increase quantum information capacity.

For experiments with these detectors, however, current generalizations of the CHSH inequality are limited. While observables like Peres' can be easily measured with the new detectors, the range of analyzer orientations θ that produce violations of Eq. (1) depends heavily on the spin of the particles n, or in our case, the number of photons generated. When photon pairs are generated through downconversion, however, there is a probability that any number of photons will downconvert simultaneously. In these experiments, the analyzer orientation must be chosen before the number of photon pairs is known. While the Peres observables were not specifically designed for downconversion experiments, the relative ease with which these experiments will soon be implemented makes a new set of observables desirable. As a note, the discussion herein does not include the recently reported noise-resistant effects of multiport schemes [15,16].

In this paper, we present a search for a set of observables that will be possible to measure with the new multiple photon detectors and will also be able to achieve a significant CHSH violation for experiments where the orientation of the analyzers must be chosen before the spin of the entangled particles is known. That is, we are searching f space for a combination of joint probabilities that will reduce the n dependence of $S(n,\theta,f)$ in Eq. (1) for a fixed set of orientations θ. The generalized observables found produce inequality violations over a significantly wider range of spins than was previously possible with other generalizations, making them better suited for entanglement experiments with downconverted photons. This is a somewhat novel idea, because the observable is defined after the data are collected. Thus, there is a local redefinition of the observable dependent on the dimension of detected spin for a fixed analyzer setting. The local redefinition is allowed because it does not affect the outcome of the measurement, it simply changes the mathematics used to determine the violation of the inequality. It should be also noted that the motivation for this work is to determine analyzer settings to increase the CHSH inequality violation for maximally entangled states (such as those which can be produced by downconversion). Further, as will be shown, the analyzer settings will be dependent on the average spin dimensionality of the field. Thus, in the case of the downconverted field, the experimenter must determine the average number of coherent pairs created to best determine the appropriate analyzer settings.

Local realism asserts that the correlations of spacelike separated measurements on entangled particles do not indicate that nature is nonlocal, but rather that the wave-function description of the particles is incomplete. More precisely, there is some additional information, called a local hidden variable, that accounts for the correlations. Mathematically, this means that the joint probabilities P of the measurements can be factored into local probabilities that depend on the hidden variable λ,

$$P(A,B|\alpha,\beta,\lambda) = P(A|\alpha,\lambda)P(B|\beta,\lambda). \tag{3}$$

Here, A and B are possible outcomes of measurements on each of the two spatial modes where analyzers are oriented at angles α and β, respectively.

CHSH considered any joint observable $E(\alpha,\beta)$ that can be factored into local observables $\overline{A}(\alpha)$ and $\overline{B}(\beta)$ under the assumption of local realism,

$$E(\alpha,\beta) = \int \overline{A}(\alpha,\lambda)\overline{B}(\beta,\lambda)f(\lambda)d\lambda. \tag{4}$$

Provided that the two local observables satisfy $|\overline{A}(\alpha,\lambda)| \leqslant 1$ and $|\overline{B}(\beta,\lambda)| \leqslant 1$ for all analyzer orientations, the joint observables E must obey the CHSH inequality in a locally realistic world,

$$S = |E(\alpha,\beta) - E(\alpha,\beta') + E(\alpha',\beta) + E(\alpha',\beta')| \leqslant 2. \tag{5}$$

Quantum mechanics, however, predicts that for certain observables and analyzer orientations $\theta=\{\alpha,\beta,\alpha',\beta'\}$, this inequality does not hold. Here we study three different sets of analyzer orientations of the form $\theta=\{0,x,2x,3x\}$. These are the settings used by Peres in his generalization and commonly used in experiments where the spin of the entangled particles being analyzed is known. The three sets analyzed have $x=\pi/16$, $\pi/24$, and $\pi/32$.

The observables that can be measured with the new generation of photon detectors and that maximize CHSH violations are linear combinations of joint probabilities with coefficients of +1 and −1 [14]. For a spin-n measurement, E takes the form

$$E(\alpha,\beta) = \sum_{k,l=-n}^{+n} i(k)j(l)P(k,l|\alpha,\beta), \tag{6}$$

where the local observables are

$$\overline{A}(\alpha,\lambda) = \sum_{k=-n}^{+n} i(k)P(k|\alpha,\lambda), \tag{7}$$

$$\overline{B}(\alpha,\lambda) = \sum_{l=-n}^{+n} j(l)P(l|\alpha,\lambda).$$

Here, $i(k)$ and $j(l)$ are either +1 or −1 for each possible spin measurement k and l and determine the coefficient of each of the local outcomes for $\overline{A}$ and $\overline{B}$. Observables of this form meet the CHSH assumption since both $|\overline{A}(\alpha)| \leqslant \sum_{k=-n}^{+n} P(k|\alpha,\lambda)=1$ and $|\overline{B}(\beta)| \leqslant \sum_{l=-n}^{+n} P(l|\beta,\lambda)=1$. The pair of functions $f=\{i,j\}$ is an element of the f space over which we maximize S in Eq. (1).

The CHSH derivation holds more generally for observables where there are two functions $i(k)$ and $j(l)$ for $\overline{A}$ and $\overline{B}$, respectively, and where $|i(k)| \leqslant 1$ and $|j(l)| \leqslant 1$. However, functions which take on values strictly between −1 and +1 do not maximize S. Consider a pair of functions $i(k)$ and $j(l)$ that define an observable where $-1 < j(l') < 1$ for some measurement outcome l'. We can construct an observable that achieves a greater value of S by setting $j(l')$ to either −1 or +1. Let $F_l(\alpha,\beta)$ be another observable,

$$F_l(\alpha,\beta) = i(-n)P(-n,l|\alpha,\beta) + i(-n+1) \times P(-n+1,l|\alpha,\beta) \ldots i(+n)P(+n,l|\alpha,\beta). \tag{8}$$

The quantity $x_l=F_l(\alpha,\beta)-F_l(\alpha,\beta')+F_l(\alpha',\beta)+F_l(\alpha',\beta')$ is the sum of all the terms in Eq. (5) that are multiplied by a factor of $j(l)$. That is, $S=|\sum_{l=-n}^{+n} j(l)x_l|$. We can assume without loss of generality that the sum $\sum_{l=-n}^{+n} j(l)x_l$ is positive as we can change the sign of each value $j(l)$, thereby changing the sign of the sum. Since $x_{l'}$ is a real number, we have that either $x_{l'}<0$ or $x_{l'} \geqslant 0$, in which case we can achieve a larger value for S by changing the value of $j(l')$ to −1 or +1, respectively. Proceeding inductively through the $2n+1$ possible outcomes l', we see that at least one of the functions j that maximize S takes on values of ±1. A similar argument illustrates that functions i which maximize S have $i(k)=\pm 1$, so we need only consider functions of this type in our search of f space for the new observables.

We compute the quantum-mechanical predictions of $E(\alpha,\beta)$ from Eq. (6) by calculating each of the individual joint probabilities from the wave function for a system of $2n$ entangled photon pairs (which is formally equivalent to a system of two entangled spin-n particles). Using the notation developed by Kok and Braunstein [19], we define an operator which creates a singlet state from vacuum,

$$L_+ = a_h^\dagger b_v^\dagger - a_v^\dagger b_h^\dagger. \tag{9}$$

Here, $a_h^\dagger$, $a_v^\dagger$ $(b_h^\dagger, b_v^\dagger)$ are the creation operators for horizontal and vertical photons in spatial mode 1 (mode 2), respectively. Applying this operator $2n$ times to the vacuum state $|0\rangle$ yields the wave function $|\Psi_n\rangle$ (up to renormalization by a factor N_n) for a system of $2n$ entangled photon pairs and the corresponding system of entangled spin-n particles,

$$|\Psi_n\rangle = N_n L_+^{2n}|0\rangle = \frac{1}{\sqrt{n+1}}\sum_{m=0}^{2n}(-1)^m|mH(n-m)V,(n-m)HmV\rangle \equiv \frac{1}{\sqrt{n+1}}\sum_{k=-n}^{n}(-1)^{k+n}|k,-k\rangle. \tag{10}$$

The amplitude squared of the coefficients of each of the eigenkets $|k,-k\rangle$ gives the joint probabilities $P(k,-k|0,0)$ of spin measurements on the two entangled particles.

To calculate the general joint probabilities $P(A,B|\alpha,\beta)$ of measurements along different analyzer orientations α and β, we simply apply a standard two-dimensional rotation transformations to the vector bases $\{a_h^\dagger, a_v^\dagger\}$ and $\{b_h^\dagger, b_v^\dagger\}$. Using the quantum-mechanical predictions of the joint probabilities, we numerically probed the pair of functions $f=\{i(k),j(l)\}$ to find observables that maximize S for several sets of analyzer orientations θ (with $x=\pi/16$, $\pi/24$, and $\pi/32$ as remarked earlier) and for particles with spin n between 1/2 and 9/2 by trying every function possible.

Our search of f space revealed a set of observables that are less dependent on the spin of the particles being analyzed than previous CHSH generalizations. The functions $i(k)$ found to produce the largest CHSH violations are listed in Table I. In each case, the function $j(l)$, which achieved the largest CHSH violation, had the property that $i(k)=j(-k)$. In Fig. 1, we plot the quantum-mechanical predictions of S as a function of particle spin n using both the new observables and the Peres observables for comparison purposes. Here, the analyzer orientation with $x=\pi/32$ was used. While the Peres observables were not specifically designed for experiments where the spin of the particle is indeterminant at the time of measurement like those using type-II downconversion, they provide a useful illustration of the importance of the optimization routine performed. The wide range of spins for which the new observables produce CHSH violations will enable high-spin Bell's inequalities to be violated using multiple pairs of polarization entangled photons in the near future.

We searched the complete f space for spins up to $n=9/2$. As noted earlier, each of the functions had a symmetry $i(k)=j(-k)$. By restricting our search of f space to the much smaller space of only those functions with this symmetry, we were able to find observables that produce significant CHSH violations for experiments with even higher spin particles (spins $n\leqslant 7$). We hypothesize that these are in fact the observables yielding the largest CHSH violation because of the inductive evidence provided by the smaller spin cases. Moreover, a search of this restricted space in the larger spin cases did yield observables that achieve significant CHSH violations. In Fig. 2, quantum-mechanical predictions for S are

TABLE I. The values for $i(k)$ for each spin n and each set of analyzer orientations are listed. In each case, the function $j(k)$ had $i(k)=j(-k)$. The strings of 0's and 1's listed give the sign of $i(k)$ in ascending order of k read from left to right. The string of digits $d_1d_2d_3\cdots d_{2n}$ corresponds to the function $i(-n+j)=(-1)^{d_j}$ for j between 0 and $2n$. For instance, for $n=1$ and $x=\pi/16$, $d_1d_2d_3=101$, implying that the best-suited observable has $i(-1)=-1$, $i(0)=+1$, and $i(+1)=-1$.

n	$x=\pi/16$	$x=\pi/24$	$x=\pi/32$
1/2	10	10	10
1	101	101	101
3/2	1001	1010	1010
2	10001	10010	10101
5/2	100001	100110	101101
3	1000001	1000110	1011001
7/2	10000001	10001110	10011001
4	100000001	100011110	100111001
9/2	1000000001	1000011110	1001110001

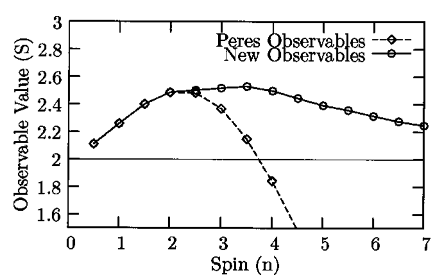

FIG. 1. The quantum mechanical prediction of S is plotted as a function of the entangled particle spin n for both the Peres and the new observables when the analyzers are oriented so $x=\pi/32$.

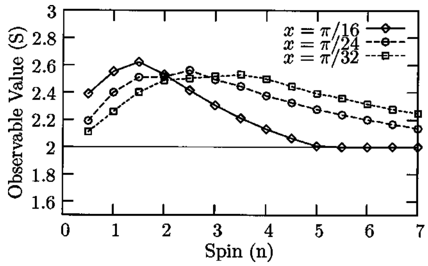

FIG. 2. Quantum mechanical predictions of S plotted as a function of particle spin n for each of the three analyzer orientations using the new observables. The observables are studied over the whole of f space for spins $n \leq 9/2$ and are optimized only over the restricted f space for $9/2 < n \leq 7$.

plotted with respect to n for all three analyzer orientations studied for n between $1/2$ and 7. We believe the violations achieved by these new observables for spins $1/2 \leq n \leq 9/2$ represent the upper bound for violations of the CHSH inequality in experiments involving polarization entangled photons or maximally entangled spin particles.

The results of Fig. 2 show that the experimenter must determine the expectation value of the spin dimensionality. For example, if downconversion is the means of realizing the spin states, the experimenter must know the expectation value of the number of coherent pairs before determining the analyzer settings most appropriate for the experiment. As shown in the figure, a larger number of average pairs would then require smaller analyzer settings.

With the improved generalization of the CHSH inequality, we can decrease the spin dependence of the range of analyzer orientations that produce violations of CHSH-like inequalities. These observables are ideal for experiments where the spin of the particles is unknown before the analyzers are oriented such as experiments on multiple pairs of polarization entangled photons generated by downconversion. In these experiments, the nonlocal properties of entangled particles can be observed for a wider range of spins than was previously possible with other generalizations. This will facilitate the generalization of many quantum applications, like Ekert's cryptographic protocol, in the near future with the advent of a new generation of photon detectors.

[1] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[3] J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).
[4] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[5] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).
[6] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).
[7] D. S. Naik, C. G. Peterson, A. G. White, A. J. Berglund, and P. G. Kwiat, Phys. Rev. Lett. **84**, 4733 (2000); T. Jennewein, C. Simon, G. Weihs, H. Weinfurter, and A. Zeilinger, *ibid.* **84**, 4729 (2000); W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, *ibid.* **84**, 4737 (2000).
[8] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[9] K. Mattle, H. Weinfurter, P. G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. **76**, 4656 (1996).
[10] G. A. Durkin, C. Simon, and D. Bouwmeester, Phys. Rev. Lett. **88**, 187902 (2002).
[11] A. Peres, Phys. Rev. A **46**, 4413 (1992).
[12] N. Gisin and A. Peres, Phys. Lett. A **162**, 15 (1992).
[13] J. C. Howell, A. Lamas-Linares, and D. Bouwmeester, Phys. Rev. Lett. **88**, 030401 (2002).
[14] W. J. Munro and M. D. Reid, Phys. Rev. A **50**, 3661 (1994).
[15] D. Kaszlikowski *et al.*, Phys. Rev. Lett. **85**, 4418 (2000).
[16] M. Zukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997).
[17] A. Lamas-Linares, J. Howell, and D. Bouwmeester, Nature (London) **412**, 887 (2001).
[18] P. D. Drummond, Phys. Rev. Lett. **50**, 1407 (1983).
[19] P. Kok and S. L. Braunstein, Phys. Rev. A **61**, 042304 (2000).