Canonical and kinetic forms of the electromagnetic momentum in an *ad hoc* quantization scheme for a dispersive dielectric

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An *ad hoc* quantization scheme for the electromagnetic field in a weakly dispersive, transparent dielectric leads to the definition of canonical and kinetic forms for the momentum of the electromagnetic field in a dispersive medium. The canonical momentum is uniquely defined as the operator that generates spatial translations in a uniform medium, but the quantization scheme suggests two possible choices for the kinetic momentum operator, corresponding to the Abraham or the Minkowski momentum in classical electrodynamics. Another implication of this procedure is that a wave packet containing a single dressed photon travels at the group velocity through the medium. The physical significance of the canonical momentum has already been established by considerations of energy and momentum conservation in the atomic recoil due to spontaneous emission, the Cerenkov effect, the Doppler effect, and phase matching in nonlinear optical processes. In addition, the data of the Jones and Leslie radiation pressure experiment is consistent with the assignment of one $\hbar\mathbf{k}$ unit of canonical momentum to each dressed photon. By contrast, experiments in which the dielectric is rigidly accelerated by unbalanced electromagnetic forces require the use of the Abraham momentum.

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I. INTRODUCTION

In classical electrodynamics, a medium is traditionally described by its macroscopic linear susceptibility. The long history and great utility of this phenomenological method have inspired a substantial body of work aimed at devising a similar description for the quantized electromagnetic field in a dielectric medium [1–6]. This has proven to be a difficult and subtle task.

A useful *ad hoc* scheme for the quantization of the electromagnetic field in a dispersive dielectric has been independently suggested by Loudon [7] and Milonni [8]. In the present paper we will use Milonni's version of this scheme. This simple and plausible formulation leads in a natural way to the definition of several forms of electromagnetic momentum; a "canonical" momentum associated with spatial translations, and two "kinetic" momenta that result from quantizing the familiar Abraham or Minkowski momenta of classical electrodynamics. We shall see that all of these operators can be physically meaningful, but that they have different domains of applicability.

The existence of more than one form of momentum may seem surprising, but there is an analogous situation in semiclassical electrodynamics. In the nonrelativistic limit, the kinetic energy part of the Hamiltonian for this problem is

$$H = \frac{1}{2m}(\mathbf{p} - e\boldsymbol{\mathcal{A}})^2, \tag{1}$$

where m is the mass, e is the charge, $\boldsymbol{\mathcal{A}}$ is the classical vector potential (we shall use calligraphic symbols for all classical variables), and

$$\mathbf{p} = \frac{\hbar}{i}\boldsymbol{\nabla} \tag{2}$$

is the "canonical" momentum [9]. The Heisenberg equation of motion $i\hbar d\mathbf{r}/dt=[\mathbf{r},H]$ shows that the velocity operator $\mathbf{v}=d\mathbf{r}/dt$ is given by

$$m\mathbf{v} = \mathbf{p} - e\boldsymbol{\mathcal{A}}, \tag{3}$$

and this defines the "kinetic" momentum $m\mathbf{v}$.

The kinetic momentum in Eq. (3) evidently has the expected classical limit, i.e., the product of mass and velocity, but it does not serve as the generator of spatial translations. To see this, we note that spatial translations along different axes commute, so that the corresponding generators must also commute. An explicit calculation using Eq. (3) yields

$$[mv_i, mv_j] = i\hbar e\epsilon_{ijk}\mathcal{B}_k \neq 0, \tag{4}$$

where $\boldsymbol{\mathcal{B}}=\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}$ is the magnetic field, and the Einstein summation convention is used for repeated vector indices. This shows that $m\mathbf{v}$ cannot be the generator of spatial translations for $\boldsymbol{\mathcal{B}}\neq 0$. On the other hand, it is well known that the canonical momentum $\mathbf{p}$ in Eq. (2) is the operator that generates spatial translations, but solving Eq. (3) for $\mathbf{p}$ shows that it does not have the expected classical limit. Thus both the canonical and kinetic momenta are physically meaningful, but they play distinct roles in the theory.

In the following sections, we shall see that Milonni's quantization scheme leads to an analogous situation. In the electromagnetic case there is a unique "canonical" momentum operator $\mathbf{P}_{can}$ that generates spatial translations, but there are at least two possibilities for the kinetic momentum. This peculiar situation is related to the long standing controversy in classical electrodynamics regarding the "correct" definition of the electromagnetic momentum density in a medium [10,11]. The traditional contenders for this title are the Abraham,

$$\mathbf{g}_A(\mathbf{r},t) = \frac{\langle \boldsymbol{\mathcal{E}}(\mathbf{r},t) \times \boldsymbol{\mathcal{H}}(\mathbf{r},t)\rangle}{c^2}, \tag{5}$$

and the Minkowski,

$$\mathbf{g}_M(\mathbf{r},t) = \langle \boldsymbol{\mathcal{D}}(\mathbf{r},t) \times \boldsymbol{\mathcal{B}}(\mathbf{r},t)\rangle, \quad (6)$$

forms of the momentum density, where $\langle \cdots \rangle$ indicates an average over the period of the carrier wave. At present there seems to be a fairly strong consensus that the Abraham form is to be preferred for the electromagnetic momentum density [11–16], but new proposals continue to appear. In the work of Obukhov and Hehl [17], for example, the energy momentum tensor is automatically symmetric, and it leads to the momentum density $\mathbf{g}_{OH}(\mathbf{r},t)=\varepsilon_0\langle \boldsymbol{\mathcal{E}}(\mathbf{r},t)\times\boldsymbol{\mathcal{B}}(\mathbf{r},t)\rangle$. In the present paper we only consider nonmagnetic materials ($\mu=\mu_0$), for which $\mathbf{g}_{OH}(\mathbf{r},t)=\mathbf{g}_A(\mathbf{r},t)$, but it would be interesting to see an application of the Obukhov-Hehl approach to dispersive dielectrics. As pointed out by Loudon [6], the various forms of the momentum are potentially useful in different contexts. It should also be noted that Brevik [18] has argued that there is no unique solution to the problem of identifying the "true" electromagnetic energy-momentum tensor, since there is no unique prescription for the separation of the total energy-momentum tensor into a field part and a matter part. DeGroot and Suttorp [19] have pointed out that the problem of deriving the forms of the energy, the linear momentum, and the angular momentum for polarized media cannot be solved as long as macroscopic arguments are utilized; microscopic arguments starting from statistical mechanics are necessary.

In Sec. II, we present Milonni's procedure for the quantization of electromagnetic fields in a weakly dispersive, transparent dielectric medium. In Sec. III, we show that identifying the total electromagnetic momentum with the uniquely defined generator of spatial translations is equivalent to assuming that a photon with wave vector $\mathbf{k}$ has momentum $\hbar\mathbf{k}$, just as in the vacuum. The importance of the generator of spatial translations in this connection was previously noted by Brevik and Lautrop [20], but their work was limited to nondispersive materials. In Sec. IV, we show that quantization of the familiar Abraham and Minkowski versions of the total electromagnetic momentum leads to alternative suggestions for the form of the single-photon momentum. In Sec. V we discuss experimental tests of the predictions of this quantization method.

II. QUANTIZATION IN A DISPERSIVE DIELECTRIC

Milonni's method of quantization of the electromagnetic field in a weakly dispersive, transparent dielectric has the twin virtues of simplicity and agreement with the much more elaborate formalisms developed in some of the other references cited in the Introduction. This approach is directly based on the approximations used in the classical theory, so we begin by considering a classical field described by the vector potential,

$$\boldsymbol{\mathcal{A}}(\mathbf{r},t) = \boldsymbol{\mathcal{A}}^{(+)}(\mathbf{r},t) + \text{c.c.}, \quad (7)$$

where the analytic signal $\boldsymbol{\mathcal{A}}^{(+)}(\mathbf{r},t)$ is given by

$$\boldsymbol{\mathcal{A}}^{(+)}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\sum_s \boldsymbol{\mathcal{A}}_s(\mathbf{k})\mathbf{e}_s(\mathbf{k})e^{\mathbf{i}(\mathbf{k}\cdot\mathbf{r}-\omega(k)\mathbf{t})}. \quad (8)$$

For the quasimonochromatic fields of interest, the power spectrum, $|\boldsymbol{\mathcal{A}}_s(\mathbf{k})|^2$ is concentrated at a particular frequency ω_0 with spectral width $\Delta\omega \ll \omega_0$. The medium is assumed to be weakly dispersive with respect to this wave packet, i.e.,

$$\Delta n = \Delta\omega \left| \left(\frac{\partial n(\omega)}{\partial \omega}\right)_{\omega=\omega_0} \right| \ll |n(\omega_0)|. \quad (9)$$

For classical fields satisfying Eqs. (7)–(9) the effective energy is [12]

$$\mathcal{U}_{em} = \frac{d[\omega_0\epsilon(\omega_0)]}{d\omega_0}\frac{1}{2}\int d^3r\langle \boldsymbol{\mathcal{E}}^2(\mathbf{r},t)\rangle + \frac{1}{2\mu_0}\int d^3r\langle \boldsymbol{\mathcal{B}}^2(\mathbf{r},t)\rangle, \quad (10)$$

where $\langle \cdots \rangle$ denotes an average over the carrier period $2\pi/\omega_0$. By using Eq. (8) one can carry out the volume integrals to get

$$\mathcal{U}_{em} = \int \frac{d^3k}{(2\pi)^3}\sum_s \left\{\omega^2(k)\frac{d[\omega_0\epsilon(\omega_0)]}{d\omega_0} + \frac{k^2}{\mu_0}\right\}|\boldsymbol{\mathcal{A}}_s(\mathbf{k})|^2, \quad (11)$$

and the narrow width of the power spectrum allows this to be rewritten in the more suggestive form

$$\mathcal{U}_{em} = \int \frac{d^3k}{(2\pi)^3}\sum_s \left\{\omega^2(k)\frac{d[\omega(k)\epsilon(\omega(k))]}{d\omega(k)} + \frac{k^2}{\mu_0}\right\}|\boldsymbol{\mathcal{A}}_s(\mathbf{k})|^2. \quad (12)$$

This step is both dangerous and useful. The danger comes from the apparent generality of Eq. (12), which might lead one to forget that it was derived for a quasimonochromatic field. The utility comes from the observation that this expression is also valid for a superposition of quasimonochromatic fields, provided that the differences between the carrier frequencies are large compared to the spectral widths of the individual wave packets. In this situation we shall say that the total field is "quasimultichromatic." With these caveats held firmly in mind, we use the relation $\epsilon(\omega(k))=\epsilon_0 n^2(\omega(k))$ to rewrite Eq. (12) as

$$\mathcal{U}_{em} = 2\epsilon_0\int \frac{d^3k}{(2\pi)^3}\sum_s \frac{\omega^2(k)n(k)}{v_{gr}(k)/c}|\mathcal{A}_s(\mathbf{k})|^2, \quad (13)$$

where

$$v_{gr}(k) = \frac{d\omega}{dk} = \frac{c}{n(k)+\omega(k)(dn/d\omega)_k} \quad (14)$$

is the group velocity and $v_{ph}(k)=c/n(k)$ is the phase velocity

The next step is to express the energy as the sum of energies $\hbar\omega(k)$ of radiation oscillators. To this end we define new amplitudes $\alpha_s(\mathbf{k})$ by the rule

$$\boldsymbol{\mathcal{A}}_s(\mathbf{k}) = \sqrt{\frac{\hbar[v_{gr}(k)/c]}{2\epsilon_0 n(k)\omega(k)}}\alpha_s(\mathbf{k}), \tag{15}$$

so that $|\alpha_s(\mathbf{k})|^2$ (with dimensions L^3) is a **k**-space density. The resulting expression,

$$\mathcal{U}_{em} = \int \frac{d^3k}{(2\pi)^3}\sum_s \hbar\omega(k)|\alpha_s(\mathbf{k})|^2, \tag{16}$$

for the total energy opens the way to the standard quantization rule

$$\alpha_s(\mathbf{k}) \to a_s(\mathbf{k}), \alpha_s^*(\mathbf{k}) \to a_s^\dagger(\mathbf{k}), \tag{17}$$

where the operators $a_s(\mathbf{k})$ and $a_s^\dagger(\mathbf{k})$ satisfy the canonical commutation relations,

$$[a_s(\mathbf{k}), a_{s'}^\dagger(\mathbf{k}')] = (2\pi)^3\delta_{ss'}\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{18}$$

In this scheme the Hamiltonian and the positive-frequency part of the field are respectively given by

$$H_{em} = \int \frac{d^3k}{(2\pi)^3}\sum_s \hbar\omega(k)a_s^\dagger(\mathbf{k})a_s(\mathbf{k}) \tag{19}$$

and

$$\mathbf{A}^{(+)}(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\sum_s \sqrt{\frac{\hbar v_{gr}(k)}{2\epsilon_0 n(k)\omega(k)c}}a_s(\mathbf{k})\mathbf{e}_s(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{20}$$

The excitations created by $a_s^\dagger(\mathbf{k})$ are quasiparticles that contain some admixture of electromagnetic and atomic degrees of freedom, i.e., they are "dressed" photons. This is in the spirit of Einstein's original model of light quanta in the vacuum, since each dressed photon carries energy $\hbar\omega(k)$ according to Eq. (19). Furthermore, one can show that the appearance of the group velocity in the normalization factor in Eq. (20) guarantees that a single-photon wave packet, propagating at the group velocity, carries the energy $\hbar\omega(k)$ associated with the carrier wave.

The classical quasimultichromatic approximation implies that a plot of the power spectrum $|\alpha_s(\mathbf{k})|^2$ must consist of a set of narrow peaks centered on the carrier frequencies of the wave packets making up the classical field, but this condition makes no sense when applied to the operator $a_s^\dagger(\mathbf{k})a_s(\mathbf{k})$. In the quantum theory this kind of information is carried by the states, so we need to choose a subspace $\mathfrak{H}_{qm}$ of the total electromagnetic Fock space that corresponds to the classical quasimultichromatic field [21]. The number states

$$|\underline{n}\rangle \equiv |n_{s_1}(\mathbf{k}_1), n_{s_2}(\mathbf{k}_2), \cdots\rangle \tag{21}$$

that satisfy

$$a_{s_j}^\dagger(\mathbf{k}_j)a_{s_j}(\mathbf{k}_j)|\underline{n}\rangle = n_{s_j}(\mathbf{k}_j)|\underline{n}\rangle, \tag{22}$$

provide a basis for the entire Fock space, so the subspace $\mathfrak{H}_{qm}$ can be defined as the set of all linear combinations of number states satisfying the condition that $n_s(\mathbf{k})=0$ unless $\omega(k)$ lies in a narrow band centered on one of the carrier frequencies. The operator expressions (19) and (20) are valid only when applied to state vectors in $\mathfrak{H}_{qm}$.

III. CANONICAL MOMENTUM

The quantization scheme presented in Sec. II involves the following assumptions: (a) The medium can only respond through the electronic polarization of the atoms; no center-of-mass motion is allowed. (b) The material response is spatially homogeneous, at least on the scale of optical wavelengths. (c) The medium is isotropic. Assumption (c) (which is valid for vapors, liquids, and glasses) justifies the use of a scalar dielectric function. The quantization scheme can be generalized to crystals by using a dielectric tensor instead.

The combination of assumptions (a) and (b) implies that the positional and inertial degrees of freedom of the constituent atoms are irrelevant in this model. As a consequence of these assumptions, the generator, $\mathbf{P}_{can}$, of spatial translations is completely defined by its action on the field operators,

$$[A_j^{(+)}(\mathbf{r}), \mathbf{P}_{can}] = \frac{\hbar}{i}\boldsymbol{\nabla} A_j^{(+)}(\mathbf{r}). \tag{23}$$

Using the expansion (20) to evaluate both sides leads to

$$[a_s(\mathbf{k}), \mathbf{P}_{can}] = \hbar\mathbf{k}a_s(\mathbf{k}). \tag{24}$$

The operator

$$\mathbf{P}_{can} = \int \frac{d^3k}{(2\pi)^3}\sum_s \hbar\mathbf{k}a_s^\dagger(\mathbf{k})a_s(\mathbf{k}) \tag{25}$$

obviously satisfies this condition. Any alternative form $\mathbf{P}'_{can}$ would have to satisfy $[a_s(\mathbf{k}), \mathbf{P}'_{can}-\mathbf{P}_{can}]=0$ for all modes $\mathbf{k}s$, which is only possible if the operator $\mathbf{Z}\equiv\mathbf{P}'_{can}-\mathbf{P}_{can}$ is actually a c-number. In this case $\mathbf{Z}$ can be set to zero, for example by assuming that the vacuum state is an eigenstate of $\mathbf{P}_{can}$ with zero eigenvalue, or equivalently that the vacuum state is invariant under spatial translations. By analogy with Eq. (2) we will call $\mathbf{P}_{can}$ the "canonical momentum" of the field. From Eq. (25) we then see that a photon with wave vector **k** propagating in a dispersive medium is assigned the momentum $\hbar\mathbf{k}$, just as in the vacuum.

One physical justification for the interpretation of $\mathbf{P}_{can}$ as a form of electromagnetic momentum is provided by the empirical fact that this $\hbar\mathbf{k}$-type of momentum is conserved in nonlinear optical processes, such as spontaneous parametric down-conversion. In this process an initial photon with energy and momentum $(\hbar\omega_0, \hbar\mathbf{k}_0)$ spontaneously decays into two down-converted photons with energies and momenta $(\hbar\omega_1, \hbar\mathbf{k}_1)$ and $(\hbar\omega_2, \hbar\mathbf{k}_2)$, respectively, so as to conserve energy and canonical momentum through the well-verified phase-matching conditions [24]

$$\hbar\omega_0 = \hbar\omega_1 + \hbar\omega_2, \hbar\mathbf{k}_0 = \hbar\mathbf{k}_1 + \hbar\mathbf{k}_2. \tag{26}$$

Further pieces of evidence are that the spontaneous emission of a photon with wave vector **k** in the medium results in an atomic recoil momentum $\mathbf{p}_{rec}=\hbar\mathbf{k}$, and that the Cerenkov and Doppler effects are also simply explained by the assignment of a momentum $\hbar\mathbf{k}$ to each emitted photon [11,25].

An isotropic medium is invariant under continuous rotations, so an extension of the above argument shows that the rotation generator $\mathbf{J}_{can}$ is again entirely defined by its action on the fields

$$[A_j^{(+)}(\mathbf{r}),(\mathbf{J}_{can})_i] = \left(\mathbf{r}\times\frac{\hbar}{i}\boldsymbol{\nabla}\right)_i A_j^{(+)}(\mathbf{r}) + i\hbar\epsilon_{jik}A_k^{(+)}(\mathbf{r}). \tag{27}$$

Substituting Eq. (20) into this condition yields the commutator

$$[a_s(\mathbf{k}),(\mathbf{J}_{can})_i] = \frac{\hbar}{i}e_{sj}^*(\mathbf{k})\left(\mathbf{k}\times\frac{\partial}{\partial\mathbf{k}}\right)_i\left\{\sum_r a_r(\mathbf{k})e_{rj}(\mathbf{k})\right\} + \hbar s\frac{k_i}{k}a_s(\mathbf{k}), \tag{28}$$

and inspection shows that $\mathbf{J}_{can}$ is given by the standard form [26]

$$\mathbf{J}_{can} = \int\frac{d^3k}{(2\pi)^3}\left\{\mathbf{a}_j^\dagger(\mathbf{k})\left(\hbar\mathbf{k}\times\frac{1}{i}\frac{\partial}{\partial\mathbf{k}}\right)_i\mathbf{a}_j(\mathbf{k}) + \frac{\mathbf{k}}{k}\sum_s\hbar s a_s^\dagger(\mathbf{k})a_s(\mathbf{k})\right\}, \tag{29}$$

where

$$\mathbf{a}(\mathbf{k}) = \sum_s a_s(\mathbf{k})\mathbf{e}_s(\mathbf{k}). \tag{30}$$

IV. KINETIC MOMENTA

The quantization scheme we are using starts with the standard classical expression for the electromagnetic energy in a dispersive dielectric, so it would seem natural to construct the operators for momentum and angular momentum by applying the same quantization rule (17) to the appropriate classical expressions. Since it is precisely the identification of the appropriate expressions that is disputed in the Abraham *vs* Minkowski controversy, we must consider both possibilities. Integrating Eqs. (5) and (6) over all space leads to the rival expressions

$$\boldsymbol{\mathcal{P}}_A = \int d^3r\,\mathbf{g}_A(\mathbf{r},t) = \int\frac{d^3k}{(2\pi)^3}\frac{S(\mathbf{k})}{c^2}\frac{\mathbf{k}}{k} \tag{31}$$

and

$$\boldsymbol{\mathcal{P}}_M = \int d^3r\,\mathbf{g}_M(\mathbf{r},t) = \int\frac{d^3k}{(2\pi)^3}\frac{S(\mathbf{k})}{v_{ph}^2(k)}\frac{\mathbf{k}}{k}, \tag{32}$$

for the total momentum, where

$$S(\mathbf{k}) = 2\epsilon_0 c^2 k\omega(k)\sum_s|\boldsymbol{\mathcal{A}}_s(\mathbf{k})|^2 \tag{33}$$

is the time-averaged magnitude of the Poynting flux. Applying the quantization rule (17) to $\boldsymbol{\mathcal{P}}_A$ and $\boldsymbol{\mathcal{P}}_M$ produces the operators

$$\mathbf{P}_A = \int\frac{d^3k}{(2\pi)^3}\sum_s\hbar\omega(k)\frac{v_{gr}(k)}{c^2}\frac{\mathbf{k}}{k}a_s^\dagger(\mathbf{k})a_s(\mathbf{k}),$$

$$= \int\frac{d^3k}{(2\pi)^3}\sum_s\frac{v_{gr}(k)v_{ph}(k)}{c^2}\hbar\mathbf{k}\;a_s^\dagger(\mathbf{k})a_s(\mathbf{k}), \tag{34}$$

and

$$\mathbf{P}_M = \int\frac{d^3k}{(2\pi)^3}\sum_s\hbar\omega(k)n^2(k)\frac{v_{gr}(k)}{c^2}\frac{\mathbf{k}}{k}a_s^\dagger(\mathbf{k})a_s(\mathbf{k}),$$

$$= \int\frac{d^3k}{(2\pi)^3}\sum_s\frac{v_{gr}(k)}{v_{ph}(k)}\hbar\mathbf{k}\;a_s^\dagger(\mathbf{k})a_s(\mathbf{k}). \tag{35}$$

Conversely, the classical limit of $\mathbf{P}_A(\mathbf{P}_M)$ is $\boldsymbol{\mathcal{P}}_A(\boldsymbol{\mathcal{P}}_M)$. Comparing these expressions to the canonical momentum (25) shows that—just as for the kinetic momentum in Eq. (3)—neither of these kinetic momentum operators is the generator of spatial translations.

Since $a_s^\dagger(\mathbf{k})a_s(\mathbf{k})$ is the number operator for photons in the $\mathbf{k}s$ mode, the expressions (34) and (35) imply that a single dressed photon in a dispersive dielectric has the momentum

$$\mathbf{p}_A = \frac{v_{gr}(k)}{cn(k)}\hbar\mathbf{k} = \frac{v_{gr}(k)v_{ph}(k)}{c^2}\hbar\mathbf{k}, \tag{36}$$

for the Abraham form, and

$$\mathbf{p}_M = \frac{n(k)v_{gr}(k)}{c}\hbar\mathbf{k} = \frac{v_{gr}(k)}{v_{ph}(k)}\hbar\mathbf{k} \tag{37}$$

for the Minkowski form. It has been experimentally verified [22] that a single-photon wave packet propagates at the group velocity $v_{gr}(k) < c$ in a passive, transparent medium, such as glass. Roughly speaking, the peak of the wave packet indicates the most likely "position" of the photon, when it is regarded as a particle. This suggests that the dressed photon might be regarded as a relativistic particle with velocity $v_{gr}(k)$, and this would in turn lead to the definition of an effective mass as $|\mathbf{p}|/v_{gr}$.

In the Abraham picture, a dressed photon in a dispersive medium has the effective mass

$$m_A^{eff} = \frac{p_A}{v_{gr}(k)} = \frac{\hbar\omega(k)}{c^2}. \tag{38}$$

This is what is sometimes called the relativistic mass, and should not be confused with the rest mass. Thus the single-quantum energy $\hbar\omega(k)$ determines the relativistic inertial mass of the dressed photon. This is consistent with Planck's law of inertia for electromagnetic energy [27], which states that for any closed system containing a dielectric, the ratio of the momentum density to the energy flux is given by $1/c^2$. Planck's law of inertia was formulated classically for nondispersive media, but this definition of the effective mass generalizes it to the quantum level, and includes dispersive dielectrics. Thus we interpret Planck's law of inertia to mean that each dressed photon contributes an inertial mass, given by Eq. (38), to a blackbody cavity which is filled with a uniform dielectric, and which is undergoing rigid-body acceleration. We should also note that Planck's law of inertia is automatically satisfied by the Obukhov-Hehl form of the energy-momentum tensor.

In the Minkowski picture, the dressed photon propagating inside the dielectric medium possesses an effective mass

$$m_M^{eff} = \frac{p_M}{v_{gr}(k)} = \frac{\hbar\omega(k)}{v_{ph}^2(k)} = n^2(k)\frac{\hbar\omega(k)}{c^2}, \qquad (39)$$

which differs from the Abraham expression by the extra factor $n^2(k)$ in the numerator.

Comparing the three expressions (34), (35), and (25) for $\mathbf{P}_A$, $\mathbf{P}_M$, and $\mathbf{P}_{can}$, respectively, shows that $\mathbf{P}_A = \mathbf{P}_M$ can only hold if $n^2(k)=1$, i.e., for the vacuum, and that $\mathbf{P}_A = \mathbf{P}_{can}$ is only possible in the unlikely special case that $v_{gr}(k)v_{ph}(k) = c^2$. On the other hand, equality between $\mathbf{P}_{can}$ and $\mathbf{P}_M$ occurs for any nondispersive medium, i.e., whenever there is a range of frequencies for which $\omega dn(\omega)/d\omega \ll n(\omega)$. In this case the phase and group velocities coincide, and $\mathbf{P}_{can} = \mathbf{P}_M$. This situation occurs automatically in the low-frequency or static limit $\omega \to 0$, since Eq. (14) shows that $v_{gr}(0) = v_{ph}(0)$. This is a good approximation for dielectrics in the low-frequency limit, as was pointed out by Gordon [16]. Thus, in the low-frequency limit the Minkowski momentum should be identified with Gordon's pseudo-momentum, or in the language of this paper, with the canonical momentum.

Most of the previous treatments of the Minkowski momentum have been restricted to nondispersive media [23], so an alternative procedure would be to interpret the canonical momentum as the appropriate generalization of the Minkowski momentum to dispersive media. In this approach, the definition (35) of the Minkowski momentum would be dropped and replaced by the classical limit of the definition (25) of the canonical momentum.

Similar results follow from the alternative classical expressions of the total angular momentum. The classical angular momentum defined by the Abraham momentum density is

$$\mathcal{J}_A = \int d^3r\ \mathbf{r} \times \mathbf{g}_A(\mathbf{r},t), \qquad (40)$$

so the corresponding quantum operator is

$$\mathbf{J}_A = \int \frac{d^3k}{(2\pi)^3}\frac{v_{gr}(k)v_{ph}(k)}{c^2}\left\{\mathbf{a}_j^\dagger(\mathbf{k})\left(\hbar\mathbf{k}\times\frac{1}{i}\frac{\partial}{\partial\mathbf{k}}\right)_i \mathbf{a}_j(\mathbf{k}) + \frac{\mathbf{k}}{k}\sum_s \hbar s a_s^\dagger(\mathbf{k})a_s(\mathbf{k})\right\}. \qquad (41)$$

Similarly the Minkowski angular momentum

$$\mathcal{J}_M = \int d^3r\ \mathbf{r} \times \mathbf{g}_M(\mathbf{r},t) \qquad (42)$$

leads to the operator

$$\mathbf{J}_M = \int \frac{d^3k}{(2\pi)^3}\frac{v_{gr}(k)}{v_{ph}(k)}\left\{\mathbf{a}_j^\dagger(\mathbf{k})\left(\hbar\mathbf{k}\times\frac{1}{i}\frac{\partial}{\partial\mathbf{k}}\right)_i \mathbf{a}_j(\mathbf{k}) + \frac{\mathbf{k}}{k}\sum_s \hbar s a_s^\dagger(\mathbf{k})a_s(\mathbf{k})\right\}. \qquad (43)$$

V. EXPERIMENTAL TESTS

A. Radiation-pressure experiment of Jones and Leslie

An important experiment which bears on the question of the momentum of light in dielectric media was carried out by Jones and Leslie [28]. In this work the radiation pressure of a light beam striking a mirror immersed in various optically dense liquids was measured with high accuracy. Each measurement was compared to the radiation pressure of the same light beam striking the same mirror in air. The experimental data showed that the mechanical momentum imparted to the mirror is directly proportional to the index of refraction $n(\omega)$ of the medium to within ±0.05%. Several alternative hypotheses, such as proportionality to the "group index"

$$n_{gr}(\omega) = n(\omega) + \omega dn/d\omega \qquad (44)$$

or inverse proportionality to $n(\omega)$, were excluded by many standard deviations.

At the heart of this experiment is a "radiation-pressure mirror," fabricated from multilayer dielectric coatings with high reflectivity and low absorption at the 632.8 nm wavelength of the helium-neon laser used in the experiment. This mirror is located near the bottom of the apparatus, where it is attached by epoxy to a thin, central vertical wire. The mirror and wire can be immersed in a variety of dielectric liquids. A high-intensity, 15 mW helium-neon laser beam is directed near normal incidence towards this lower mirror, and the radiation pressure exerted by the laser beam generates a torque upon the wire. In the experiment, the resulting torque is measured both before and after a dielectric liquid is poured into the space surrounding the mirror.

A second, "twist-detecting" mirror (called an "optical lever") is attached to the same wire near the top of the apparatus, and is also immersed in the liquid. In this way, the central wire connecting the two mirrors transmits the mechanical torque generated by the radiation pressure from the lower to the upper mirror. The wire is wrapped around the upper mirror many times so as to form a current-carrying coil which, in the presence of a uniform magnetic field, exerts a torque on the upper mirror. The reflected light signal from the upper mirror is detected by a pair of balanced photodiodes, and is used as the primary input into a feedback circuit that controls the current in the coil, so that the torque generated by its interaction with the magnetic field exactly cancels the torque arising from the radiation pressure exerted by the laser beam on the lower mirror. (The radiation pressure exerted upon the upper mirror by the low-intensity light beam for monitoring the angular displacement of the "optical lever" is negligible.) The central wire is grounded at the bottom of the metallic apparatus, and is insulated from the top, in order for a current to be fed through the wire.

The use of a counterbalancing torque generated in the upper mirror guarantees that no mechanical motion of the lower mirror, or of the fluid, ever occurs during a measurement, i.e., these are *null measurements*. Nonlinearities in the system do not affect the position of the null, and also there is no need to include any hydrodynamic effects (including electrostrictive pressure effects) in the calculation of the radiation pressure. After the system has been balanced and comes

TABLE I. Ratios of radiation pressure in liquid to that in air (data from [28]).

Liquid	R_{expt}	R_{can}	R_M	R_A
methanol	$1.3281\pm\sigma(\sigma=0.0018)$	$1.3275(-0.3\sigma)$	$1.3134(-8.2\sigma)$	$0.7453(-324\sigma)$
acetone	$1.3553\pm\sigma(\sigma=0.0018)$	$1.3563(+0.6\sigma)$	$1.3359(-10.8\sigma)$	$0.7262(-350\sigma)$
ethanol	$1.3594\pm\sigma(\sigma=0.0022)$	$1.3606(+0.5\sigma)$	$1.3437(-7.1\sigma)$	$0.7259(-288\sigma)$
isopropanol	$1.3762\pm\sigma(\sigma=0.0020)$	$1.3756(-0.3\sigma)$	$1.3577(-9.3\sigma)$	$0.7175(-329\sigma)$
CCl_4	$1.4614\pm\sigma(\sigma=0.0021)$	$1.4581(-1.6\sigma)$	$1.4313(-14.3\sigma)$	$0.6732(-375\sigma)$
toluene	$1.4898\pm\sigma(\sigma=0.0018)$	$1.4921(+1.3\sigma)$	$1.4528(-20.5\sigma)$	$0.6525(-465\sigma)$
benzene	$1.4970\pm\sigma(\sigma=0.0021)$	$1.4974(+0.2\sigma)$	$1.4518(-21.5\sigma)$	$0.6475(-405\sigma)$

into mechanical equilibrium, a measurement of the current passing through the coil around the upper mirror is a direct measure of the radiation pressure exerted by the laser beam on the lower mirror.

The experiment employs synchronous detection to cancel out systematic errors. The laser beam is periodically translated from the left side to the right side of the radiation-pressure mirror with respect to the central wire. This is done symmetrically, so that the radiation-pressure-generated torque periodically reverses sign. The electronic feedback system is designed so that the current sent to the coil wrapped around the upper mirror is also reversed in sign in synchronism with the periodic switching of the laser beam. Derivative feedback to the coil around the upper mirror is used to achieve critical damping of this torsional-oscillator system.

We will analyze this experiment by assuming that each photon in the beam carries momentum $\mathbf{p}$ that is normal to the mirror. Let us call the rate of arrival of photons normally incident at the mirror $\dot{N}_{inc}$. For a perfectly reflective mirror the momentum transfer per photon at normal incidence is $2\mathbf{p}$, so the magnitude $F_{rad}=|\mathbf{F}_{rad}|$ of the force due to the flux of photons striking the mirror at normal incidence is

$$F_{rad} = \dot{N}_{inc} 2|\mathbf{p}|. \tag{45}$$

The entrance window to the apparatus is antireflection coated, and there is negligible absorption in the liquid; therefore the rate of arrival of laser photons at the mirror is the same as the rate of arrival of laser photons at the entrance window. If the entire laser output is focused through the entrance window onto the surface of the mirror, $\dot{N}_{inc}$ is closely approximated by

$$\dot{N}_{inc} = \frac{P_{laser}}{\hbar\omega_L}, \tag{46}$$

where P_{laser} is the output power of the laser and $\hbar\omega_L$ is the energy per laser photon.

There are three possible choices for $\mathbf{p}$. For $\mathbf{p}=\mathbf{p}_{can}=\hbar\mathbf{k}$, the force on the mirror is

$$(F_{rad})_{can} = \dot{N}_{inc} 2\hbar k = n(\omega_L)\left[\dot{N}_{inc} 2\frac{\hbar\omega_L}{c}\right] = n(\omega_L)\left[2\frac{P_{laser}}{c}\right], \tag{47}$$

where we have used the dispersion relation $k=n(\omega_L)\omega_L/c$. For $\mathbf{p}=\mathbf{p}_M$ or $\mathbf{p}=\mathbf{p}_A$ the relations (37) and (36) yield the corresponding forces

$$(F_{rad})_M = \frac{n^2}{n_{gr}}\left[2\frac{P_{laser}}{c}\right] \tag{48}$$

and

$$(F_{rad})_A = \frac{1}{n_{gr}}\left[2\frac{P_{laser}}{c}\right], \tag{49}$$

where n_{gr} is the group index defined in Eq. (44).

In each case we want to calculate the ratio

$$R = \frac{F_{rad}(\text{dielectric})}{F_{rad}(\text{air})} \tag{50}$$

of the radiation-pressure forces on the mirror with and without the liquid. Since $n=n_{gr}=1$ in air, the three alternative values are

$$R_{can} = n(\omega_L), \tag{51}$$

$$R_M = \frac{n^2(\omega_L)}{n_{gr}(\omega_L)}, \tag{52}$$

and

$$R_A = \frac{1}{n_{gr}(\omega_L)}, \tag{53}$$

for the canonical, Minkowski, and Abraham momenta, respectively.

The results of evaluating the alternative values (51)–(53) of the ratio R using the data provided by Jones and Leslie are presented in Table I. For each dielectric we show the average experimental value R_{exp} and the corresponding standard deviation σ, together with the predicted values and their differences from the experimental value expressed as a multiple of σ. For example, in the case of benzene the observed ratio differs from the Minkowski prediction (52) by 22 standard deviations and from the Abraham prediction (53) by 405 standard deviations.

Therefore the Jones and Leslie experiment demonstrates that near normal incidence the radiation pressure on a mirror immersed in a dielectric liquid is given by the rate of transfer of the *canonical* momentum $\hbar\mathbf{k}$ per photon within an accuracy of ±0.05%. In this connection it is important to note that the theories of Gordon [16] and Loudon [6] both predict that the radiation pressure force on a mirror immersed in a dispersionless dielectric will be determined by the Minkowski, rather than the Abraham, momentum. As we have noted above, the Minkowski and canonical momenta agree for dispersionless materials, but we have further demonstrated in Table I that the experimental results for optical frequency radiation in a dispersive medium decisively favor the canonical momentum over the Minkowski momentum, as well as over the Abraham momentum.

B. Experimental relevance of the Abraham momentum

Of the three momenta we have studied, only the canonical momentum is required to explain atomic recoil in spontaneous emission, the Cerenkov and Doppler effects, and all conventional nonlinear and quantum optics experiments involving the phase-matching relations. In addition, the radiation-pressure experiment of Jones and Leslie is consistent with the choice of the canonical momentum for the dressed photons. When, if ever, are the Abraham or Minkowski forms of momentum needed? In this connection, there have been important experiments demonstrating the relevance of the Abraham momentum by James [29] and by Walker *et al.* [30]. (For a review of these experiments, see Brevik [18].) These experiments, which were first proposed by Marx and Györgyi [31], involve toroidal or annular, dielectric-filled regions subjected to crossed electric and magnetic fields, with low-frequency time variations. In particular, in the experiment of Walker *et al.*, the Abraham force due to the time-varying polarization current crossed into the magnetic field was verified to within an accuracy of ±5%. This implies that the Minkowski theory is in disagreement with the experimental data of Walker *et al.*, by 20 standard deviations.

Note that these toroidal experiments involved "closed" systems, in the sense that the dielectric medium and electromagnetic fields are entirely enclosed, for example, within the toroidal torsional bob of the torsional oscillator used by Walker *et al.* Thus in these experiments the dielectric medium experiences *accelerated* motion during measurements. No external forces are present, and the whole enclosed system of fields and dielectric rotates together as a rigid body. By contrast, the Jones and Leslie configuration involves an "open" system, in which an external torque is used in feedback to prevent any accelerated motions of the mirror and the dielectric liquid during measurements.

Furthermore, two papers by Lai [14,15] have convincingly demonstrated theoretically that in the low-frequency or static limit, the Minkowski momentum density would give unphysical results for the measurement of the total angular momentum in all such closed-system experiments in which acceleration of the dielectric is allowed. Thus the experiments by James and by Walker *et al.*, and the papers by Lai, all provide strong evidence that the Abraham, rather than the Minkowski momentum, is required for a correct description of all such closed systems that undergo accelerated motions. This is consistent with Planck's law of inertia for electromagnetic energy. Since the canonical momentum is identical to the Minkowski momentum in the static limit, these results also rule out the canonical momentum as being physically relevant in these kinds of experiments. However, one of the assumptions of the Milonni theory is that center-of-mass motions of atoms of the medium are not allowed. Hence it is not surprising the canonical momentum derived from this theory does not apply to these experiments.

VI. CONCLUSIONS

The *ad hoc* quantization scheme employed above leads in a natural way to several forms of momentum for the electromagnetic field in a dispersive medium. The first is the canonical momentum which is uniquely defined as the generator of spatial translations. The conservation law for the canonical momentum is validated by the atomic recoil in spontaneous emission, the Cerenkov and Doppler effects, and the phase-matching conditions in nonlinear optics. Furthermore, the canonical momentum correctly predicts the results of the Jones and Leslie radiation-pressure experiment. The explicit appearance of the group velocity in the *ad hoc* scheme suggests that experiments to measure quantum fluctuations of the electromagnetic field in a variety of dielectric media would be of great interest.

The second form, the kinetic momentum, is not unique, since the operators are derived by quantizing the classical expressions of the Abraham and Minkowski momenta. The experiments discussed in Sec. V B demonstrate the experimental relevance of the Abraham, as opposed to the canonical, momentum for closed systems. Since these experiments have all been carried out for classical, low-frequency fields, they do not provide direct evidence for the meaning of the operators $\mathbf{P}_A$ or $\mathbf{P}_M$. Investigating the quantum significance of the Abraham or Minkowski momenta would again require experiments sensitive to quantum fluctuations.

In addition to these experimental questions, there are also issues of theoretical consistency that have to be faced. The conjectured form (38) of the Abraham effective photon mass is based on the implicit assumption that the dressed photon model can be applied to accelerated media. This is inconsistent with the basic assumption in the quantization scheme that no center-of-mass acceleration of the atoms occurs. One possible way to resolve this contradiction would be to imitate Milonni's scheme by starting with a classical expression for the electromagnetic energy in an accelerated medium.

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