Solitary-wave description of condensate micromotion in a time-averaged orbiting potential trap

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We present a detailed theoretical analysis of micromotion in a time-averaged orbiting potential trap. Our treatment is based on the Gross-Pitaevskii equation, with the full time-dependent behavior of the trap systematically approximated to reduce the trapping potential to its dominant terms. We show that within some well specified approximations, the dynamic trap has solitary-wave solutions, and we identify a moving frame of reference which provides the most natural description of the system. In that frame eigenstates of the time-averaged orbiting potential trap can be found, all of which must be solitary-wave solutions with identical, circular center of mass motion in the laboratory frame. The validity regime for our treatment is carefully defined, and is shown to be satisfied by existing experimental systems.

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I. INTRODUCTION

The time-averaged orbiting potential (TOP) trap [1] was an important tool in the realization of Bose-Einstein condensates, and it remains a common method for magnetically trapping atoms. Early theoretical descriptions of the TOP trap used two approximations: the adiabatic approximation, which assumes that the magnetic dipoles of the atoms align instantaneously to the magnetic field, and the *time-average approximation*, where the time dynamics of the trapping fields are neglected on the time scale of the motion of the trapped atoms. Under these assumptions, the TOP trap is represented by a static, harmonic potential and the condensate eigenstates are relatively easily calculated (usually by numerical means) and are stationary in space. However, condensates formed in a TOP trap undergo a spatial micromotion [2,3] due to the underlying dynamic nature of the TOP trap. This phenomenon has been studied theoretically under various levels of approximation, by partially lifting the time-average approximation [4,5] or by not applying the adiabatic approximation [6,7].

In this work we provide a detailed theoretical description of condensate micromotion in terms of TOP trap eigenstates, including condensate nonlinearity. Our approach applies the adiabatic approximation, but partially lifts the time-average approximation. Under these conditions, the TOP trap potential retains some time dependence and eigenstates of that potential cannot be found in the laboratory frame. However, system eigenstates do exist because a frame can be found in which the Hamiltonian for the system becomes time-independent. We have termed eigenstates of the system found in such a frame *dynamical eigenstates*, since these states are not stationary states in the laboratory frame. By calculating the dynamical eigenstates of the TOP trap, full characterization of condensate micromotion is possible. This is essential for an understanding of condensate growth and is also required for a description of velocity sensitive phenomena occurring in TOP traps, such as observed in Bragg scattering experiments [8,9].

In this paper we calculate dynamical eigenstates of the TOP trap potential in the *quadratic average approximation*. Within that approximation, the solutions are exact in both the linear *and* the nonlinear case. We begin, in Sec. II, by introducing the TOP trap potential, and various approximate forms of that potential. In Sec. III we derive the transformation to the *circularly translating frame* which we find to be the most natural frame in which to investigate the system. In Sec. IV we calculate solitary-wave solutions in the quadratic average approximation and show that dynamical eigenstates calculated using the circularly translating frame are a particular class of solitary-wave solutions in the laboratory frame. Identifying the dynamical eigenstates of the TOP trap allows us to characterize micromotion and specify the ground state of the system. In Sec. V, we assess the validity of the quadratic average approximation and demonstrate that for the typical parameter regime of the TOP trap, solitary-wave dynamical eigenstates provide accurate approximations to the dynamical eigenstates of the full TOP trap potential. In Sec. VI, we discuss laboratory frame solitary-wave solutions which are eigenstates of the TOP trap potential in the more commonly used rotating frame, and show that these are only a subset of the dynamical eigenstates found using the circularly translating frame. We conclude in Sec. VII.

II. APPROXIMATE FORMS OF THE TOP TRAP POTENTIAL

The TOP trap consists of a magnetic quadrupole trap [10,11] translated by a uniform bias field, whose direction rotates at frequency Ω [1]. For simplicity our discussion is presented in a set of dimensionless units defined by the position scale $x_0=\sqrt{\hbar/2m\omega_x}$ (a characteristic harmonic oscillator length, where m is the mass of an atom), and the time

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scale of the inverse of the time-averaged trap frequency ω_x, defined below. A key feature of the TOP trap is that the zero of the magnetic field follows a circular trajectory of radius r_0, and trapped atoms are confined well within that trajectory (the so-called "circle of death"), thereby reducing atom loss due to spin flips. Typically $r_0 \sim 1000-1300$ and $\Omega \sim 70-150$ [1,2,9].

A. The adiabatic approximation

The TOP trap potential in the adiabatic approximation is given by

$$V_{\mathrm{TOP}}(\boldsymbol{r},t) = r_0^2\left[1 + \frac{2(x\cos\Omega t + y\sin\Omega t)}{r_0} + \frac{x^2+y^2+4z^2}{r_0^2}\right]^{1/2}, \tag{1}$$

where $\boldsymbol{r}=(x,y,z)$. That approximation is valid when the bias field rotation frequency Ω is much smaller than the Lamor precession frequency [1].

B. The truncated TOP trap potential

Expanding the square root of Eq. (1) in a Taylor series, and neglecting terms above second order in the small parameter x_α/r_0, where x_α is one of x, y, or z, leads to the truncated TOP trap potential

$$V(\boldsymbol{r},t) = r_0^2 + r_0(x\cos\Omega t + y\sin\Omega t) + \frac{1}{2}(x^2+y^2+4z^2) - \frac{1}{2}(x\cos\Omega t + y\sin\Omega t)^2. \tag{2}$$

The evolution of the condensate wave function $\psi(\boldsymbol{r},t)$ is governed by the Gross-Pitaevskii equation

$$i\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} = \mathcal{L}(\boldsymbol{r},t)\psi(\boldsymbol{r},t), \tag{3}$$

where, for a TOP trap, the time evolution operator

$$\mathcal{L}(\boldsymbol{r},t) = -\nabla^2 + V(\boldsymbol{r},t) + C|\psi(\boldsymbol{r},t)|^2 \tag{4}$$

is time-dependent. The truncated TOP trap potential $V(\boldsymbol{r},t)$ is given by Eq. (2), and C is the dimensionless nonlinearity coefficient defined in terms of the number of atoms N, and the s-wave scattering length a, i.e.,

$$C = \frac{4\pi\hbar aN}{m\omega_x x_0^3}. \tag{5}$$

C. The time-average approximation

The most common treatment of condensate evolution in a TOP trap has also invoked the time-average approximation, whereby the potential of Eq. (2) is averaged over a period of the bias field rotation. This leads to the time-averaged, truncated form of the TOP trap potential

$$V_{\mathrm{H}}(\boldsymbol{r}) = r_0^2 + \frac{1}{4}(x^2+y^2+8z^2), \tag{6}$$

which is a static, harmonic potential, with frequency ω_x in the x-y plane (in SI units). In the time-average approximation, the potential in the time evolution operator of Eq. (4), is replaced by the time-independent trap $V_{\mathrm{H}}(\boldsymbol{r})$, of Eq. (6). This allows energy eigenstates of the system to be readily calculated.

The time-average approximation is normally assumed to be valid when the bias field rotation frequency is much larger than the frequency of the time-averaged harmonic trap, i.e., in our dimensionless units $\Omega \gg 1$. The time-averaged treatment neglects system dynamics occurring on the fast time scale of the bias field rotation, and it is this nonstationary behavior of a condensate in a TOP trap that we describe in this work.

D. The quadratic average approximation

Müller *et al.* [2] experimentally observed the dynamic effects of the TOP trap on condensate evolution, i.e., micromotion in a TOP trap. Their approach for calculating the condensate micromotion amplitude involved balancing the restoring force of the time-dependent terms of Eq. (2) that are linear in $\cos\Omega t$ or $\sin\Omega t$, with the centrifugal force. In line with that treatment, our work invokes what we shall refer to as the quadratic average approximation, where only the terms of Eq. (2) that are quadratic in $\cos\Omega t$ or $\sin\Omega t$ are time averaged. In that approximation the TOP trap potential is given by

$$V_{\mathrm{ap}}(\boldsymbol{r},t) = V_{\mathrm{H}}(\boldsymbol{r}) + r_0(x\cos\Omega t + y\sin\Omega t). \tag{7}$$

In the present paper, we calculate dynamical eigenstates of the TOP trap potential in the quadratic average approximation, where the trapping potential is given by Eq. (7). The accuracy of the quadratic average approximation is addressed in Sec. V.

III. THE CIRCULARLY TRANSLATING FRAME

In the laboratory frame, the TOP trap potential in the quadratic average approximation, given by Eq. (7), is time-dependent and eigenstates of the Gross-Pitaevskii equation cannot be found. By transforming to a frame that translates in a circular trajectory with radius γ (whose value is to be determined) and with angular frequency Ω about the origin of the laboratory frame, we can remove this time dependence. We refer to that frame as the "circularly translating frame" and we shall see that it is the natural frame in which to describe the TOP trap system.

The translation in coordinate space is defined by

$$\boldsymbol{R} = \boldsymbol{r} - \gamma(\cos\Omega t, \sin\Omega t, 0), \tag{8}$$

as illustrated in Fig. 1. The momentum in the circularly translating frame is derived by differentiating Eq. (8), yielding

$$\boldsymbol{P} = \boldsymbol{p} + \frac{1}{2}\gamma\Omega(\sin\Omega t, -\cos\Omega t, 0), \tag{9}$$

where we have used the fact that in our dimensionless units $\boldsymbol{p}=\boldsymbol{v}/2$.

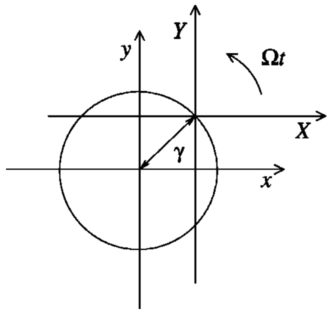

FIG. 1. The circularly translating frame, defined by coordinates $\boldsymbol{R}=(X,Y,Z)$ and Eq. (8).

A. Unitary transformation to the circularly translating frame

We now derive the quantum mechanical transformation to the circularly translating frame. For clarity, we shall denote quantum mechanical operators by $\hat{O}$, and begin with the linear case.

The Schrödinger equation for a single particle state in the TOP trap, in the quadratic average approximation, is given in the laboratory frame by

$$i\frac{d|\psi\rangle}{dt} = \hat{H}_{\rm ap}|\psi\rangle, \tag{10}$$

where

$$\hat{H}_{\rm ap} = \hat{\boldsymbol{p}}^2 + \hat{V}_{\rm ap}(\hat{\boldsymbol{r}},t). \tag{11}$$

The transformation to the circularly translating frame is achieved by the unitary transformation

$$\hat{U}(t) \equiv \hat{U}_{\boldsymbol{p}}(\boldsymbol{b}(t))\hat{U}_{\boldsymbol{r}}(\boldsymbol{a}(t)), \tag{12}$$

where

$$\hat{U}_{\boldsymbol{r}}(\boldsymbol{a}(t)) = e^{i\hat{\boldsymbol{p}}\cdot\boldsymbol{a}(t)} \tag{13}$$

translates position by $\boldsymbol{a}(t), \hat{U}_{\boldsymbol{r}}(\boldsymbol{a}(t))|\boldsymbol{r}\rangle=|\boldsymbol{r}-\boldsymbol{a}(t)\rangle$, and

$$\hat{U}_{\boldsymbol{p}}(\boldsymbol{b}(t)) = e^{-i\hat{\boldsymbol{r}}\cdot\boldsymbol{b}(t)} \tag{14}$$

translates momentum by $\boldsymbol{b}(t), \hat{U}_{\boldsymbol{p}}(\boldsymbol{b}(t))|\boldsymbol{p}\rangle=|\boldsymbol{p}-\boldsymbol{b}(t)\rangle$. In a comparison with Eqs. (8) and (9) we find that

$$\boldsymbol{a}(t) = \gamma(\cos\Omega t, \sin\Omega t, 0) \tag{15}$$

and

$$\boldsymbol{b}(t) = -\frac{1}{2}\gamma\Omega(\sin\Omega t, -\cos\Omega t, 0). \tag{16}$$

In the transformation to the circularly translating frame $\hat{U}_{\boldsymbol{r}}(\boldsymbol{a}(t))$ and $\hat{U}_{\boldsymbol{p}}(\boldsymbol{b}(t))$ commute, since $\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=0$. Defining the transformed state vector to be

$$|\psi\rangle^{\rm t} \equiv \hat{U}(t)|\psi\rangle, \tag{17}$$

Eq. (10) becomes

$$i\frac{d|\psi\rangle^{\rm t}}{dt} = \hat{H}^{\rm t}_{\rm ap}|\psi\rangle^{\rm t}, \tag{18}$$

where

$$\hat{H}^{\rm t}_{\rm ap} = \hat{U}_{\boldsymbol{p}}(\boldsymbol{b}(t))\hat{U}_{\boldsymbol{r}}(\boldsymbol{a}(t))\hat{H}_{\rm ap}\hat{U}^{\dagger}_{\boldsymbol{r}}(\boldsymbol{a}(t))\hat{U}^{\dagger}_{\boldsymbol{p}}(\boldsymbol{b}(t)) - [\hat{\boldsymbol{p}}+\boldsymbol{b}(t)]\cdot\frac{d\boldsymbol{a}(t)}{dt} + \hat{\boldsymbol{r}}\cdot\frac{d\boldsymbol{b}(t)}{dt}. \tag{19}$$

The Schrödinger equation in the coordinate representation can be determined by projecting Eq. (18) onto state $|\boldsymbol{R}\rangle$. Using the identities

$$\langle\boldsymbol{R}|\psi\rangle = \psi(\boldsymbol{R},t) \tag{20}$$

and

$$\langle\boldsymbol{R}|\hat{\boldsymbol{p}}|\psi\rangle = -i\nabla_{\boldsymbol{R}}\psi(\boldsymbol{R},t), \tag{21}$$

where we have denoted

$$\nabla_{\boldsymbol{R}} = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right), \tag{22}$$

yields

$$i\frac{\partial\psi^{\rm t}(\boldsymbol{R},t)}{\partial t} = H^{\rm t}_{\rm ap}(\boldsymbol{R},t)\psi^{\rm t}(\boldsymbol{R},t), \tag{23}$$

where

$$H^{\rm t}_{\rm ap}(\boldsymbol{R},t) = [-i\nabla_{\boldsymbol{R}}+\boldsymbol{b}(t)]^2 + V_{\rm ap}(\boldsymbol{R}+\boldsymbol{a}(t)) + [i\nabla_{\boldsymbol{R}}-\boldsymbol{b}(t)]\cdot\frac{d\boldsymbol{a}(t)}{dt} + \boldsymbol{R}\cdot\frac{d\boldsymbol{b}(t)}{dt}. \tag{24}$$

The wave functions in the laboratory frame and the circularly translating frame are related by

$$\psi^{\rm t}(\boldsymbol{R},t) = e^{-i\boldsymbol{R}\cdot\boldsymbol{b}(t)}\psi(\boldsymbol{R}+\boldsymbol{a}(t),t). \tag{25}$$

B. Application to the Gross-Pitaevskii equation

The above derivation, for the Schrödinger equation, may also be adapted to the Gross-Pitaevskii equation, since the nonlinear term transforms simply under substitution of Eq. (25). Thus, by substituting $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ from Eqs. (15) and (16) into the Hamiltonian of Eq. (24), and including the nonlinear term (which is described in terms of the new density $|\psi^{\rm t}(\boldsymbol{R},t)|^2$), the Gross-Pitaevskii equation in the circularly translating frame is

$$i\frac{\partial\psi^{\rm t}(\boldsymbol{R},t)}{\partial t} = \mathcal{L}^{\rm t}_{\rm ap}(\boldsymbol{R},t)\psi^{\rm t}(\boldsymbol{R},t), \tag{26}$$

where

$$\mathcal{L}^{\rm t}_{\rm ap}(\boldsymbol{R},t) = H^{\rm t}_{\rm ap}(\boldsymbol{R},t) + C|\psi^{\rm t}(\boldsymbol{R},t)|^2 \tag{27}$$

and

$$H^{\mathrm{t}}_{\mathrm{ap}}(\boldsymbol{R},t) = -\nabla^2_{\boldsymbol{R}} + V_{\mathrm{H}}(\boldsymbol{R}) + \frac{1}{2}(\gamma + 2r_0 - \gamma\Omega^2)(X\cos\Omega t + Y\sin\Omega t) + \gamma r_0 - \frac{1}{4}\gamma^2(\Omega^2 - 1). \quad (28)$$

The single particle Hamiltonian of Eq. (28) is identical to the Hamiltonian derived using a classical frame transformation to a noninertial frame of reference [15], applied to the circularly translating frame. Choosing $\gamma = \gamma_{\mathrm{t}}$, where

$$\gamma_{\mathrm{t}} = \frac{2r_0}{\Omega^2 - 1}, \quad (29)$$

the evolution operator of Eq. (27) simplifies to

$$\mathcal{L}^{\mathrm{t}}_{\mathrm{ap}}(\boldsymbol{R},t) = -\nabla^2_{\boldsymbol{R}} + V_{\mathrm{H}}(\boldsymbol{R}) + C|\psi^{\mathrm{t}}(\boldsymbol{R},t)|^2 + \varepsilon, \quad (30)$$

where

$$\varepsilon = \frac{1}{4}\gamma_{\mathrm{t}}^2(\Omega^2 - 1). \quad (31)$$

The energy offset ε can be interpreted by expressing Eq. (31) in the form

$$\varepsilon = V_{\mathrm{ap}}([\gamma_{\mathrm{t}},0,0],t=0) - V_{\mathrm{H}}(\mathbf{0}) - E_{\Omega}. \quad (32)$$

The first two terms represent the additional potential energy due to the displacement of a point body from the trap center to radius γ_{t}. The remaining term $E_{\Omega} = \gamma_{\mathrm{t}}^2\Omega^2/4$ represents the energy of a point body rotating about the origin of the laboratory frame, at a radius γ_{t} with frequency Ω, which is simply the expected energy shift associated with the transformation to the circularly translating frame [15].

The time evolution operator in the circularly translating frame, as given by Eq. (30), substituted into Eq. (26) yields the Gross-Pitaevskii equation for a time-independent harmonic trap, with an energy offset of ε. Thus, eigenstates of the TOP trap in the circularly translating frame exist in the quadratic average approximation. For clarity we write these as

$$\psi^{\mathrm{t}}(\boldsymbol{R},t) = \psi_{\mathrm{H}}(\boldsymbol{R})e^{-i(\mu_{\mathrm{H}}+\varepsilon)t}, \quad (33)$$

where $\psi_{\mathrm{H}}(\boldsymbol{R})$ are the well-known solutions to the time-independent Gross-Pitaevskii equation for a time-independent harmonic trap, i.e.,

$$\mu_{\mathrm{H}}\psi_{\mathrm{H}}(\boldsymbol{R}) = [-\nabla^2_{\boldsymbol{R}} + V_{\mathrm{H}}(\boldsymbol{R}) + C|\psi_{\mathrm{H}}(\boldsymbol{R})|^2]\psi_{\mathrm{H}}(\boldsymbol{R}). \quad (34)$$

C. Generalization to quantum field theory

The transformation given by Eq. (25) can be applied to the operator Heisenberg equations of motion for the full quantum field operator $\hat{\Psi}(\boldsymbol{r},t)$. In much the same way as our discussion above, this yields the equation of motion in the circularly translating frame

$$i\frac{\partial\hat{\Psi}_{\mathrm{t}}(\boldsymbol{R},t)}{\partial t} = \left[-\nabla^2_{\boldsymbol{R}} + V_{\mathrm{H}}(\boldsymbol{R}) + C\hat{\Psi}^{\dagger}_{\mathrm{t}}(\boldsymbol{R},t)\hat{\Psi}_{\mathrm{t}}(\boldsymbol{R},t) + \varepsilon\right]\hat{\Psi}_{\mathrm{t}}(\boldsymbol{R},t). \quad (35)$$

Since this represents the full quantum field theory, the motion of uncondensed particles is also correctly treated in the circularly translating frame.

IV. SOLITARY-WAVE SOLUTIONS

Solitary-wave solutions, where the wave function evolves without changing shape, can be found for the TOP trap in the quadratic average approximation. Morgan *et al.* [12] have shown that the Gross-Pitaevskii equation, with particular forms of potential, has solitary-wave solutions which propagate in one dimension of a multidimensional space. That work was extended by Margetis [13] where solitary-wave solutions may have center of mass motion in any of the space dimensions. Also, Japha and Band [14] have shown that in a moving harmonic trap the motion of the condensate center of mass can be entirely decoupled from the evolution of the condensate shape. We have extended the derivation by Morgan *et al.* [12] to include the case where solitary-wave solutions can propagate in three dimensions, as was indicated to be possible by Margetis [13]. In the following we present a brief summary of the results of our derivation.

We begin by postulating that solitary-wave solutions to the TOP trap will have the form

$$\psi_{\mathrm{SW}}(\boldsymbol{r},t) = \psi_{\mathrm{H}}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t))e^{-i\mu_{\mathrm{H}}t + iS(\boldsymbol{r},t)}, \quad (36)$$

where the envelope wave function $\psi_{\mathrm{H}}(\boldsymbol{r})$ is an eigenstate of the time-independent Gross-Pitaevskii equation for the TOP trap potential in the time-average approximation with chemical potential μ_{H}, i.e., defined by Eq. (34). The position offset in the envelope wave function is

$$\overline{\boldsymbol{r}}(t) = \int \psi^*_{\mathrm{SW}}(\boldsymbol{r},t)\boldsymbol{r}\psi_{\mathrm{SW}}(\boldsymbol{r},t)d\boldsymbol{r} - \int \psi^*_{\mathrm{H}}(\boldsymbol{r})\boldsymbol{r}\psi_{\mathrm{H}}(\boldsymbol{r})d\boldsymbol{r}, \quad (37)$$

which can be interpreted as the time-dependent position of the center of mass of the solitary wave since the second integral is zero due to the particular form of $V_{\mathrm{H}}(\boldsymbol{r})$. The phase $S(\boldsymbol{r},t)$ is determined by substituting the solitary-wave solution (36) into the time-dependent Gross-Pitaevskii equation (3), where $\mathcal{L}(\boldsymbol{r},t)$ is replaced by $\mathcal{L}_{\mathrm{ap}}(\boldsymbol{r},t)$, in which the quadratic average approximation is used, i.e.,

$$\mathcal{L}_{\mathrm{ap}}(\boldsymbol{r},t) = -\nabla^2 + V_{\mathrm{ap}}(\boldsymbol{r},t) + C|\psi(\boldsymbol{r},t)|^2, \quad (38)$$

where $V_{\mathrm{ap}}(\boldsymbol{r},t)$ is given by Eq. (7). Taking a similar approach to that of Morgan *et al.* [12] the Gross-Pitaevskii equation can be separated into real and imaginary parts yielding two equations. The equation derived from the imaginary part can be simplified by writing

$$S(\boldsymbol{r},t) = \frac{1}{2}\boldsymbol{r}\cdot\frac{d\overline{\boldsymbol{r}}(t)}{dt} + K(\boldsymbol{r},t). \quad (39)$$

We choose the trivial solution $K(\boldsymbol{r},t) = K(t)$ which is the only possible solution in the one-dimensional case [12] and has also been suggested as the unique solution in the general case [13]. Substituting the trivial solution into the equation derived from the real part of the Gross-Pitaevskii equation we find that

$$K(t) = \frac{1}{4}\int \left[\overline{\boldsymbol{r}}^2(t) - \left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)^2 \right] dt. \tag{40}$$

By equating mixed differentials of $S(\boldsymbol{r},t)$, the center of mass motion of the solitary-wave solutions can be found to obey

$$\frac{1}{2}\frac{\partial^2 \overline{\boldsymbol{r}}(t)}{\partial t^2} = -\nabla F(\boldsymbol{r},t), \tag{41}$$

which is a form of Ehrenfest's theorem, where

$$F(\boldsymbol{r},t) = V_{\mathrm{ap}}(\boldsymbol{r},t) - V_{\mathrm{H}}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t)). \tag{42}$$

For solitary-wave solutions to exist, in the form that we have discussed, both sides of Eq. (41) must be independent of $\boldsymbol{r}$ and therefore the function $F(\boldsymbol{r},t)$ must be at most linear in $\boldsymbol{r}$. The TOP trap potential in the quadratic average approximation obeys that criterion and thus solitary-wave solutions exist. It is possible to solve for $\overline{\boldsymbol{r}}(t)$ which has six constants of integration, given by the initial values of the center of mass position and momentum of the particular solitary-wave solution [see Eq. (A1)].

A. Solitary-wave solutions which are eigenstates in the circularly translating frame

The dynamical eigenstates calculated using the circularly translating frame, given by Eq. (33), are a particular class of solitary-wave solutions in the laboratory frame. This can be confirmed by transforming the solitary-wave solutions, as given by Eq. (36), into the circularly translating frame, and requiring that these solutions satisfy the time-independent Gross-Pitaevskii equation in that frame, i.e.,

$$\mu_{\mathrm{SW}}^{\mathrm{t}}\psi_{\mathrm{SW}}^{\mathrm{t}}(\boldsymbol{R},t) = \mathcal{L}_{\mathrm{ap}}^{\mathrm{t}}(\boldsymbol{R},t)\psi_{\mathrm{SW}}^{\mathrm{t}}(\boldsymbol{R},t), \tag{43}$$

where

$$\psi_{\mathrm{SW}}^{\mathrm{t}}(\boldsymbol{R},t) = \psi_{\mathrm{H}}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t))e^{-i\mu_{\mathrm{H}}t + iS(\boldsymbol{r},t) + i\gamma_{\mathrm{t}}\Omega\boldsymbol{r}\cdot(\sin\Omega t, -\cos\Omega t, 0)/2} \tag{44}$$

and

$$\mu_{\mathrm{SW}}^{\mathrm{t}} = \mu_{\mathrm{H}} + \varepsilon. \tag{45}$$

In order to satisfy Eq. (43), the solitary-wave solutions of Eq. (44) have a restriction on $\overline{\boldsymbol{r}}(t)$, as derived in Appendix A. Solitary-wave solutions of the laboratory frame are eigenstates of the TOP trap in the circularly translating frame if and only if the initial conditions of the center of mass motion of the solitary-wave solutions have particular values such that

$$\overline{\boldsymbol{r}}(t) = \gamma_{\mathrm{t}}(\cos\Omega t, \sin\Omega t, 0) \tag{46}$$

and, therefore,

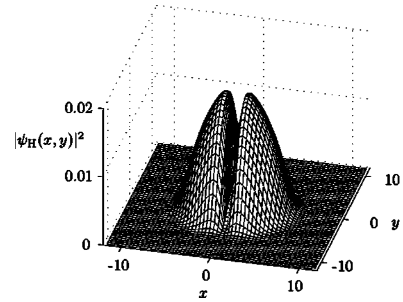

FIG. 2. A particular eigenstate of the two-dimensional equivalent of Eq. (34), calculated numerically. The two-dimensional nonlinear strength is $C_{2\mathrm{D}}=600$ and the chemical potential is $\mu_{\mathrm{H}} = 10.56$.

$$\overline{\boldsymbol{p}}(t) = -\frac{1}{2}\gamma_{\mathrm{t}}\Omega(\sin\Omega t, -\cos\Omega t, 0). \tag{47}$$

These equations represent circular motion at the TOP trap frequency Ω, with radius γ_{t}. Previously [2,9] the micromotion position amplitude has been determined to be $2r_0/\Omega^2$, which is in agreement with our result [see Eq. (29)] in the limit $\Omega \gg 1$.

With the center of mass motion for solitary-wave dynamical eigenstates of the TOP trap given by Eqs. (46) and (47), we find that $\boldsymbol{R}=\boldsymbol{r}-\overline{\boldsymbol{r}}(t)$ and the phase $S(\boldsymbol{r},t)$ becomes

$$S(\boldsymbol{r},t) = -\frac{1}{2}\gamma_{\mathrm{t}}\Omega\boldsymbol{r}\cdot(\sin\Omega t, -\cos\Omega t, 0) - \varepsilon t. \tag{48}$$

Making these substitutions, Eq. (44) simplifies to Eq. (33) so that all dynamical eigenstates of the TOP trap, calculated using the circularly translating frame, are a particular class of solitary-wave solutions in the laboratory frame with center of mass motion given by Eqs. (46) and (47). This shows that the origin of the circularly translating frame [refer to Eqs. (8) and (9) with $\gamma=\gamma_{\mathrm{t}}$] moves with the center of mass motion of the solitary-wave dynamical eigenstates of the TOP trap, therefore justifying our choice of the circularly translating frame for describing the TOP trap.

B. Dynamical eigenstates

All the dynamical eigenstates of the TOP trap follow the same circular trajectory in the laboratory frame, as given by Eqs. (46) and (47). This motion is independent of both the chemical potential of the state and the nonlinear strength of the system. Furthermore, the solitary-wave dynamical eigenstates retain their orientation with the laboratory frame throughout their trajectory.

As an example, a two-dimensional excited state of the envelope wave function $\psi_{\mathrm{H}}(x,y)$, with a nodal line along the y axis, is presented in Fig. 2. In the laboratory frame, the solitary-wave dynamical eigenstate of the TOP trap, corresponding to the envelope wave function in Fig. 2, consists of the envelope wave function moving in a circular trajectory, while maintaining its orientation with the laboratory frame, and the orientation of the nodal line along the y axis.

In the linear case $(C=0)$ the solitary-wave dynamical eigenstates form a complete basis for the TOP trap in the circularly translating frame. This is because these states, in the circularly translating frame, are eigenstates of the harmonic oscillator equation with an additional energy offset [see Eqs. (30) and (33)].

C. Condensation and the ground state of the TOP trap

The ground state of the TOP trap system is the solitary-wave dynamical eigenstate with the lowest chemical potential in the circularly translating frame. This occurs when μ_{H}, the chemical potential of the envelope wave function, takes its lowest possible value [see Eq. (45)]. Since, as noted in Sec. III C, the uncondensed atoms experience the same potential as the condensate, in the circularly translating frame, these thermalize during evaporation into the usual Bose-Einstein distribution, and hence condensation from the vapor will be into the TOP trap ground state, as determined using the circularly translating frame. Therefore, the state into which bosons condense, in the quadratic average approximation is the solitary-wave dynamical eigenstate given by Eq. (44) with the envelope wave function being the ground state of Eq (34) and $\overline{\boldsymbol{r}}(t)$ given by Eq. (46).

V. VALIDITY OF APPROXIMATIONS

A. Corrections to the quadratic average approximation

Throughout this work we have used the quadratic average approximation without assessing its validity. Here we give a systematic assessment of the validity regime of the quadratic average approximation for the linear case (where the mean-field interaction is neglected). This allows simple analytic results to be obtained.

The single particle Hamiltonian, with the truncated time dependent TOP trap potential of Eq. (2), takes the form (in the circularly translating frame)

$$\hat{H}^{t} = \hat{H}^{\mathrm{t}}_{\mathrm{ap}} + \hat{W}(\hat{\boldsymbol{R}}), \tag{49}$$

where $\hat{H}^{\mathrm{t}}_{\mathrm{ap}}$ is the single particle Hamiltonian in the quadratic average approximation, i.e.,

$$\hat{H}^{\mathrm{t}}_{\mathrm{ap}} = \hat{P}_X^2 + \hat{P}_Y^2 + \hat{P}_Z^2 + \hat{V}_{\mathrm{H}}(\hat{\boldsymbol{R}}) + \varepsilon. \tag{50}$$

In Eqs. (49) and (50) we have used an operator formalism where the position and momentum component operators are denoted by $\hat{\boldsymbol{R}}$ and $\hat{P}_J$, respectively. The perturbative potential $\hat{W}(\hat{\boldsymbol{R}})$ accounts for the remaining terms of the TOP trap potential of Eq. (2) that are not retained in the quadratic average approximation. In the circularly translated frame, these terms are given by

$$\hat{W}(\hat{\boldsymbol{R}}) = -\frac{1}{4}(\hat{X}^2 - \hat{Y}^2)(\cos^2\Omega t - \sin^2\Omega t) - \frac{1}{2}\gamma_{\mathrm{t}}(\hat{X}\cos\Omega t + \hat{Y}\sin\Omega t) - \hat{X}\hat{Y}\sin\Omega t\cos\Omega t - \frac{1}{4}\gamma_{\mathrm{t}}^2. \tag{51}$$

The harmonic oscillator creation and annihilation operators in the circularly translating frame are defined, in our dimensionless units, as

$$a_X = \frac{1}{2}\hat{X} + i\hat{P}_X, \tag{52}$$

$$a_Y = \frac{1}{2}\hat{Y} + i\hat{P}_Y, \tag{53}$$

$$a_Z = \sqrt{2}\hat{Z} + i\hat{P}_Z, \tag{54}$$

where $[a_J, a_K^\dagger] = \delta_{JK}$, and J and K are one of X, Y, or Z. Making these substitutions we find that

$$\hat{H}^{\mathrm{t}}_{\mathrm{ap}} = a_X^\dagger a_X + a_Y^\dagger a_Y + a_Z^\dagger a_Z + \varepsilon + r_0^2 + 1 + \sqrt{2} \tag{55}$$

and

$$\hat{W}(\hat{\boldsymbol{R}}) = -\frac{1}{4}(a_X^{\dagger 2} + a_X^2 + 2a_X^\dagger a_X)(\cos^2\Omega t - \sin^2\Omega t) + \frac{1}{4}(a_Y^{\dagger 2} + a_Y^2 + 2a_Y^\dagger a_Y)(\cos^2\Omega t - \sin^2\Omega t) - \frac{1}{2}\gamma_{\mathrm{t}}(a_X^\dagger + a_X)\cos\Omega t - \frac{1}{2}\gamma_{\mathrm{t}}(a_Y^\dagger + a_Y)\sin\Omega t - (a_X^\dagger + a_X)(a_Y^\dagger + a_Y)\sin\Omega t\cos\Omega t - \frac{1}{4}\gamma_{\mathrm{t}}^2. \tag{56}$$

Utilizing the number operator kets, which satisfy $a_J^\dagger a_J|n_J\rangle = n_J|n_J\rangle$, the eigenket of the single particle Hamiltonian is $|n_X, n_Y, n_Z\rangle^{\mathrm{t}}$, i.e.,

$$\hat{H}^{\mathrm{t}}_{\mathrm{ap}}|n_X, n_Y, n_Z\rangle^{\mathrm{t}} = E^{\mathrm{t}}|n_X, n_Y, n_Z\rangle^{\mathrm{t}}. \tag{57}$$

The energy spectrum is given by

$$E^{\mathrm{t}} = n_X + n_Y + n_Z + \varepsilon + r_0^2 + 1 + \sqrt{2}, \tag{58}$$

in agreement with Eq. (45). The energy spectrum of the harmonic oscillator terminates at the ground state $|0,0,0\rangle^{\mathrm{t}}$, which has energy $E_g^{\mathrm{t}} = \varepsilon + r_0^2 + 1 + \sqrt{2}$.

Using time-dependent perturbation theory, the evolution of the ground state to first order in the perturbation $\hat{W}(\hat{\boldsymbol{R}})$ is given by

$$|\psi\rangle^{\mathrm{t}} = A(t)e^{-i(E_g^{\mathrm{t}} - \gamma_{\mathrm{t}}^2/4)t}\Bigg[|0,0,0\rangle^{\mathrm{t}} + \frac{\gamma_{\mathrm{t}}}{4}\left(\frac{2e^{-it}}{\Omega^2 - 1} + \frac{e^{i\Omega t}}{\Omega + 1} - \frac{e^{-i\Omega t}}{\Omega - 1}\right)|1,0,0\rangle^{\mathrm{t}} + \frac{i\gamma_{\mathrm{t}}}{4}\left(\frac{2\Omega e^{-it}}{\Omega^2 - 1} - \frac{e^{i\Omega t}}{\Omega + 1} - \frac{e^{-i\Omega t}}{\Omega - 1}\right) \times |0,1,0\rangle^{\mathrm{t}} + \frac{i}{8}\left(\frac{2\Omega e^{-2it}}{\Omega^2 - 1} - \frac{e^{2i\Omega t}}{\Omega + 1} - \frac{e^{-2i\Omega t}}{\Omega - 1}\right)|1,1,0\rangle^{\mathrm{t}} + \frac{1}{8\sqrt{2}}\left(\frac{2e^{-2it}}{\Omega^2 - 1} + \frac{e^{2i\Omega t}}{\Omega + 1} - \frac{e^{-2i\Omega t}}{\Omega - 1}\right)(|2,0,0\rangle^{\mathrm{t}} - |0,2,0\rangle^{\mathrm{t}})\Bigg], \tag{59}$$

where $A(t)$ is a constant of normalization. From this expression we can deduce that the quadratic average approximation is valid in the linear case, within the parameter regime where $\gamma_{\mathrm{t}} \ll \Omega$ and $1 \ll \Omega$.

Nonlinear case. It is clear that a perturbative two time scale asymptotic expansion in powers of $1/\Omega$ could be made for the nonlinear case. Thus, we expect that for the nonlinear case, the quadratic average approximation is also valid within the regime derived above for the linear case.

We have carried out two-dimensional numerial calculations for the nonlinear case which verify this. For example, for a typical TOP trap system where $r_0=1241$ and $\Omega=153$, we have propagated the Gross-Pitaevskii equation using both the truncated TOP trap potential of Eq. (2) and the potential in the quadratic average approximation, given by Eq. (7). An appropriate value for the two-dimensional nonlinear strength is $C_{2D}=600$ which corresponds to $N\sim 2\times 10^4$ in the Otago TOP trap [9]. The initial state was chosen to be the ground state of the TOP trap in the quadratic average approximation, i.e., the ground state of Eq. (34) calculated numerically using optimization methods and shifted in position and momentum according to Eqs. (46) and (47) (at $t=0$). That state was propagated by the Gross-Pitaevskii equation for one period of the bias field rotation for two cases: (i) with the truncated TOP trap potential, giving $\psi^{\text{trunc}}(x,y,t=2\pi/\Omega)$ and (ii) with the potential in the quadratic average approximation, yielding $\psi^{\text{quad}}(x,y,t=2\pi/\Omega)$. The method used was an accurate fourth order algorithm, with a grid of 512×512 points over a 60×60 range in position, and 20 000 time steps. The deviation between the two solutions was found to be $\int|\psi^{\text{trunc}}(x,y,t=2\pi/\Omega)-\psi^{\text{quad}}(x,y,t=2\pi/\Omega)|^2dxdy=4.44\times 10^{-8}$.

B. Validity of solitary-wave dynamical eigenstates

The validity of our solitary-wave dynamical eigenstates as dynamical eigenstates of the full TOP trap depends on three validity conditions: (i) the adiabatic approximation, (ii) the truncation of the TOP trap potential from Eq. (1) to Eq. (2), and (iii) the quadratic average approximation. The quadratic average approximation was found above to be valid in the regime where $\gamma_t \ll \Omega$ and $1\ll\Omega$. We note that the condition $\gamma_t\ll\Omega$ can be rewritten as $2r_0\ll\Omega^3$. The truncation of the TOP trap potential to yield $V(\boldsymbol{r},t)$ of Eq. (2) is valid provided $x_\alpha \ll r_0$ where x_α is one of x, y, or z. A useful estimate of x_α is given by the sum of the Thomas-Fermi radius of the solitary-wave dynamical eigenstate envelope wave function with the lowest chemical potential, and the radius of the dynamical eigenstates trajectory, γ_t. This yields

$$x_\alpha \approx \left(\frac{30C}{\sqrt{2}\pi}\right)^{1/5} + \gamma_t, \tag{60}$$

where C is given by Eq. (5). The adiabatic approximation is valid when the bias field rotation frequency Ω is much smaller than the Larmor precession frequency, given in our dimensionless units by the potential. As an estimate of the Larmor precession frequency we use the magnitude of $V_{\text{TOP}}(\boldsymbol{r},t)$ which, assuming that $x_\alpha\ll r_0$, is of the order of r_0^2. Thus, the adiabatic approximation is valid provided that $\Omega \ll r_0^2$.

Finally, collating the validity regimes we find that our solitary-wave dynamical eigenstates, calculated using the circularly translating frame, are an accurate description of the dynamical eigenstates of the full TOP trap system within the parameter regime where

$$2r_0 \ll \Omega^3 \ll r_0^6,$$
$$1\ll\Omega, \tag{61}$$
$$30C \ll \sqrt{2}\pi r_0^5.$$

Typical experimental parameters are well within these criteria. As an example, the Otago TOP trap system of ^{87}Rb, where $a=55\times 10^{-10}$ m and $\omega_x=18$ Hz [9], leads to the third validity condition from Eq. (61) becoming $N\ll r_0^5$ so all three conditions are easily satisfied.

VI. THE ROTATING FRAME

In previous theoretical work the *rotating frame* has been used to calculate eigenstates of the TOP trap system under various levels of approximation [4,5]. However, it can be shown that using the rotating frame to describe the TOP trap allows only a limited set of dynamical eigenstates to be found. For completeness, we present this calculation in Appendix B where, using the same methods as in Secs. III and IV, we show that a particular class of solitary-wave solutions in the laboratory frame are eigenstates of the time-independent Gross-Pitaevskii equation in the rotating frame [Eq. (B6)]. As before, these solitary-wave dynamical eigenstates follow a circular trajectory in the laboratory frame, described by Eqs. (46) and (47), but unlike solitary-wave dynamical eigenstates calculated using the circularly translating frame, the solitary-wave solutions which are eigenstates of Eq. (B6) must also obey an additional symmetry, which is that the envelope wave function must be an eigenstate of the z component of angular momentum [see Eq. (B8)]. This condition requires solitary-wave dynamical eigenstates calculated using the rotating frame to be cylindrically symmetric about their center of mass. Figure 2 shows an example of a dynamical eigenstate envelope wave function with a corresponding solitary-wave dynamical eigenstate which does not satisfy the time-independent Gross-Pitaevskii equation in the rotating frame. Physically we can see why: the nodal line of that solitary-wave dynamical eigenstate, which remains oriented along the y axis in the laboratory frame, will appear to rotate in the rotating frame so that the dynamical eigenstate is not stationary in that frame.

Solitary-wave dynamical eigenstates of the TOP trap retain their orientation with respect to the laboratory frame as they move. Consequently, the rotating frame is not an appropriate choice for describing dynamical eigenstates of the TOP trap system, because the eigenvalue equation in the rotating frame incorporates the angular momentum operator, and places additional symmetry constraints on dynamical eigenstates of the system that are not in general necessary.

VII. DISCUSSION

We have carried out a detailed characterization of condensate micromotion in a TOP trap, under some well-defined approximations. Those approximations, which are well justified for typical TOP traps are (i) the adiabatic approximation (which neglects spin precession effects), (ii) the assumption that the condensate is located well within the circle of death, and (iii) the quadratic average approximation (which time averages quadratically oscillating terms in the potential). Our treatment allows for condensate nonlinearity and we have shown that within these approximations, solitary-wave solutions of the nonlinear Gross-Pitaevskii equation exist. We have identified the circularly translating frame as the most appropriate frame for describing the system, and have shown that eigenstates can be found in that frame, and that they must all be solitary-wave solutions of a certain type. In particular, all of the solitary-wave dynamical eigenstates have identical center of mass motion, which in the laboratory frame is a circular trajectory with radius γ_t and momentum magnitude $\gamma_t\Omega/2$.

Previous theoretical discussions of dynamical eigenstates of the TOP trap have been given within similar approximations, but with the additional restriction that the nonlinearity due to the atomic interactions is either approximated or neglected. Kuklov *et al.* [4] have obtained exact eigenstates for the linear Schrödinger equation within the adiabatic approximation using the truncated TOP trap potential of Eq. (2) and those authors have also presented an approximate many-body treatment. Their exact single particle solutions are obtained using numerous transformations and the form of eigenstate micromotion is not readily evident. Their method also employs the rotating frame which, as we have shown within the quadratic average approximation, limits the possible dynamical eigenstates that can be found. Minogin *et al.* [5] have used an approximate interaction picture method which provides information about the atomic momentum modulation in a TOP trap, but does not describe the micromotion in the position coordinates.

Our choice of the circularly translating frame allows solitary-wave dynamical eigenstates, which retain their orientation relative to the laboratory frame, to be readily identified for the TOP trap system. These dynamical eigenstates have no restriction on the z component of angular momentum of the envelope wave function. By contrast, we have shown that the dynamical eigenstates calculated using the rotating frame constitute only a subset of the dynamical eigenstates calculated using the circularly translating frame, and are required to be cylindrically symmetric about their center of mass.

Finally, we have shown that the validity regime for the quadratic average approximation is defined by the conditions $1 \ll \Omega$ and $2r_0 \ll \Omega^3$. These criteria are well satisfied by existing TOP trap systems.

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APPENDIX A: RESTRICTIONS ON SOLITARY-WAVE SOLUTIONS WHICH ARE EIGENSTATES IN THE CIRCULARLY TRANSLATING FRAME

We seek solutions to Eq. (43) which have the form of Eq. (44). We will derive the particular form for the center of mass motion $\overline{\boldsymbol{r}}(t)$ for which such solutions exist. The center of mass motion can be solved in general using Eq. (41), which gives

$$\overline{x}(t) = (x_1 - \gamma_t)\cos t + v_1 \sin t + \gamma_t \cos \Omega t,$$
$$\overline{y}(t) = x_2 \cos t + (v_2 - \gamma_t\Omega)\sin t + \gamma_t \sin \Omega t, \tag{A1}$$
$$\overline{z}(t) = x_3 \cos 2\sqrt{2}t + \frac{v_3}{2\sqrt{2}}\sin 2\sqrt{2}t,$$

where we have defined $\overline{\boldsymbol{r}}|_{(t=0)} = (x_1, x_2, x_3)$ and $d\overline{\boldsymbol{r}}(t)/dt|_{(t=0)} = (v_1, v_2, v_3)$.

Eigenstates calculated using the circularly translating frame must satisfy both Eq. (43) and

$$i\frac{\partial \psi^t_{SW}(\boldsymbol{R},t)}{\partial t} = \mu^t_{SW}\psi^t_{SW}(\boldsymbol{R},t). \tag{A2}$$

Substituting the solitary-wave solution of Eq. (44) into Eqs. (43) and (A2), and making the change of variables $\boldsymbol{s} = \boldsymbol{r} - \overline{\boldsymbol{r}}(t)$, yields, respectively,

$$\begin{aligned}\mu^t_{SW} = {} & \mu_H + \frac{1}{2}\gamma_t^2\Omega^2 + \frac{1}{4}\overline{\boldsymbol{r}}^2(t) + \frac{7}{4}\overline{z}^2(t) + \frac{1}{4}\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)^2 \\ & - \frac{1}{2}\gamma_t\overline{\boldsymbol{r}}(t)\cdot(\cos \Omega t, \sin \Omega t, 0) + \frac{1}{2}\boldsymbol{s}\cdot\overline{\boldsymbol{r}}(t) \\ & + \frac{1}{2}\gamma_t\Omega\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)\cdot(\sin \Omega t, -\cos \Omega t, 0) + \frac{7}{2}\boldsymbol{s}\cdot(0,0,\overline{z}(t)) \\ & - \frac{1}{2}\gamma_t\boldsymbol{s}\cdot(\cos \Omega t, \sin \Omega t, 0) \\ & - i\left[\frac{d\overline{\boldsymbol{r}}(t)}{dt} + \gamma_t\Omega(\sin \Omega t, -\cos \Omega t, 0)\right]\cdot\boldsymbol{g}(\boldsymbol{s})\end{aligned} \tag{A3}$$

and

$$\begin{aligned}\mu^t_{SW} = {} & \mu_H + \frac{1}{2}\gamma_t^2\Omega^2 - \frac{1}{4}\overline{\boldsymbol{r}}^2(t) - \frac{1}{2}\overline{\boldsymbol{r}}(t)\cdot\frac{d^2\overline{\boldsymbol{r}}(t)}{dt^2} + \frac{1}{4}\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)^2 \\ & - \frac{1}{2}\gamma_t\Omega^2\overline{\boldsymbol{r}}(t)\cdot(\cos \Omega t, \sin \Omega t, 0) \\ & + \frac{1}{2}\gamma_t\Omega\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)\cdot(\sin \Omega t, -\cos \Omega t, 0) - \frac{1}{2}\boldsymbol{s}\cdot\frac{d^2\overline{\boldsymbol{r}}(t)}{dt^2} \\ & - \frac{1}{2}\gamma_t\Omega^2\boldsymbol{s}\cdot(\cos \Omega t, \sin \Omega t, 0) \\ & - i\left[\frac{d\overline{\boldsymbol{r}}(t)}{dt} + \gamma_t\Omega(\sin \Omega t, -\cos \Omega t, 0)\right]\cdot\boldsymbol{g}(\boldsymbol{s}),\end{aligned} \tag{A4}$$

where

$$\boldsymbol{g}(\boldsymbol{s}) = \frac{\nabla_s\psi_H(\boldsymbol{s})}{\psi_H(\boldsymbol{s})}. \tag{A5}$$

Equating Eqs. (A3) and (A4), and substituting the general form of $\overline{\boldsymbol{r}}(t)$, given by Eq. (A1), we find that equality requires $x_3 = 0$ and $v_3 = 0$ and therefore $\overline{z}(t) = 0$. Inserting this result, and the general expressions for $\overline{x}(t)$ and $\overline{y}(t)$ from Eq. (A1), Eqs. (A3) and (A4) both simplify to

$$\begin{aligned}\mu_{\mathrm{SW}}^{\mathrm{t}} = \mu_{\mathrm{H}} + \frac{1}{2}\gamma_{\mathrm{t}}^2\Omega^2 + \frac{1}{4}[x_1^2 + x_2^2 + v_1^2 + v_2^2 - 2\gamma_{\mathrm{t}}(x_1 + v_2\Omega)] \\ + i\boldsymbol{g}(\boldsymbol{s})\cdot[(x_1 - \gamma_{\mathrm{t}})\sin t - v_1\cos t, x_2\sin t \\ - (v_2 - \gamma_{\mathrm{t}}\Omega)\cos t, 0] \\ + \frac{1}{2}\boldsymbol{s}\cdot[(x_1 - \gamma_{\mathrm{t}})\cos t + v_1\sin t, x_2\cos t \\ + (v_2 - \gamma_{\mathrm{t}}\Omega)\sin t, 0].\end{aligned} \tag{A6}$$

For the solitary-wave solutions of Eq. (44) to be eigenstates of the TOP trap in the circularly translating frame, the chemical potential $\mu_{\mathrm{SW}}^{\mathrm{t}}$ must be independent of $\boldsymbol{s}$ and t. Solving Eq. (A6) for $\boldsymbol{g}(\boldsymbol{s})$, at $t=0$ and $t=\pi/2$, we find that

$$\begin{aligned}\ln\psi_{\mathrm{H}}(\boldsymbol{s}) = -\frac{1}{2}is_1s_2\frac{x_2^2 + (v_2 - \gamma_{\mathrm{t}}\Omega)^2}{v_1x_2 - (x_1 - \gamma_{\mathrm{t}})(v_2 - \gamma_{\mathrm{t}}\Omega)} + C_1(s_3) \\ = \frac{1}{2}is_1s_2\frac{(x_1 - \gamma_{\mathrm{t}})^2 + v_1^2}{v_1x_2 - (x_1 - \gamma_{\mathrm{t}})(v_2 - \gamma_{\mathrm{t}}\Omega)} + C_2(s_3),\end{aligned} \tag{A7}$$

where $\boldsymbol{s}=(s_1,s_2,s_3)$, and $C_1(s_3)$ and $C_2(s_3)$ are constants of integration. The only possible solution is therefore $(x_1,x_2,v_1,v_2)=(\gamma_{\mathrm{t}},0,0,\gamma_{\mathrm{t}}\Omega)$, which eliminates $\boldsymbol{g}(\boldsymbol{s})$ and $\boldsymbol{s}$ from Eq. (A6) yielding

$$\mu_{\mathrm{SW}}^{\mathrm{t}} = \mu_{\mathrm{H}} + \frac{1}{4}\gamma_{\mathrm{t}}^2(\Omega^2 - 1), \tag{A8}$$

which is in agreement with Eq. (45). Concluding then, solitary-wave solutions described by Eq. (44) which are eigenstates of the TOP trap in the circularly translating frame exist if and only if $\overline{\boldsymbol{r}}(t)|_{(t=0)}=(\gamma_{\mathrm{t}},0,0)$ and $d\overline{\boldsymbol{r}}(t)/dt|_{(t=0)}=(0,\gamma_{\mathrm{t}}\Omega,0)$, and therefore the center of mass motion of these states in the laboratory frame, given in general by Eq. (A1), simplifies to

$$\overline{\boldsymbol{r}}(t) = \gamma_t(\cos\Omega t, \sin\Omega t, 0). \tag{A9}$$

APPENDIX B: SOLITARY-WAVE DYNAMICAL EIGENSTATES DERIVED USING THE ROTATING FRAME

In this appendix we calculate dynamical eigenstates of the TOP trap potential in the quadratic average approximation using the rotating frame. That frame, with coordinates $\boldsymbol{r}'=(x',y',z')$, rotates at the frequency of the bias field, and is defined by the coordinate transformation

$$\begin{aligned}x' = x\cos\Omega t + y\sin\Omega t, \\ y' = -x\sin\Omega t + y\cos\Omega t, \\ z' = z.\end{aligned} \tag{B1}$$

In the rotating frame the Gross-Pitaevskii equation becomes

$$i\frac{\partial\psi^{\mathrm{r}}(\boldsymbol{r}',t)}{\partial t} = \mathcal{L}_{\mathrm{ap}}^{\mathrm{r}}(\boldsymbol{r}',t)\psi^{\mathrm{r}}(\boldsymbol{r}',t), \tag{B2}$$

where the evolution operator in the rotating frame is time-independent and is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{ap}}^{\mathrm{r}}(\boldsymbol{r}',t) = \mathcal{L}_{\mathrm{ap}}(\boldsymbol{r},t) - \Omega\hat{L}_z(\boldsymbol{r}') \\ = -\nabla_{\boldsymbol{r}'}^2 + V_{\mathrm{ap}}^{\mathrm{r}}(\boldsymbol{r}') - \Omega\hat{L}_z(\boldsymbol{r}') + C|\psi^{\mathrm{r}}(\boldsymbol{r}',t)|^2,\end{aligned} \tag{B3}$$

and the wave function in the rotating frame is $\psi^{\mathrm{r}}(\boldsymbol{r}',t)=\psi(\boldsymbol{r},t)$. The angular momentum in the rotating frame has a component in the z direction given by

$$\hat{L}_z(\boldsymbol{r}') = i\left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right), \tag{B4}$$

and we note that $\hat{L}_z(\boldsymbol{r}')=\hat{L}_z(\boldsymbol{r})$ [15]. Finally, the TOP trap potential of Eq. (7) becomes, in the rotating frame,

$$V_{\mathrm{ap}}^{\mathrm{r}}(\boldsymbol{r}') = \frac{1}{4}(x' + 2r_0)^2 + \frac{1}{4}(y'^2 + 8z'^2), \tag{B5}$$

which is a stationary harmonic potential shifted from the origin.

Eigenstates of the TOP trap in the rotating frame obey the time-independent Gross-Pitaevskii equation in that frame, i.e.,

$$\mu^{\mathrm{r}}\psi^{\mathrm{r}}(\boldsymbol{r}',t) = \mathcal{L}_{\mathrm{ap}}^{\mathrm{r}}(\boldsymbol{r}',t)\psi^{\mathrm{r}}(\boldsymbol{r}',t). \tag{B6}$$

Here we show that a particular class of solitary-wave solutions obey Eq. (B6). We denote solitary-wave solutions which are TOP trap eigenstates in the rotating frame by $\psi_{\mathrm{SW}}^{\mathrm{r}}(\boldsymbol{r}',t)$, with chemical potential $\mu_{\mathrm{SW}}^{\mathrm{r}}$. Transforming the solitary-wave solution of Eq. (36) into the rotating frame yields

$$\psi_{\mathrm{SW}}^{\mathrm{r}}(\boldsymbol{r}',t) = \psi_{\mathrm{H}}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t))e^{-i\mu_{\mathrm{H}}t + iK(t) + ix'\{[d\overline{x}(t)/dt]\cos\Omega t + [d\overline{y}(t)/dt]\sin\Omega t\}/2 + iy'\{[d\overline{y}(t)/dt]\cos\Omega t - [d\overline{x}(t)/dt]\sin\Omega t\}/2 + iz'[d\overline{z}(t)/dt]/2}. \tag{B7}$$

The detail of substituting Eq. (B7) into Eq. (B6) and enforcing μ^{r} to be independent of spatial and temporal coordinates is given later in this appendix. The results are discussed here.

Solitary-wave dynamical eigenstates of the TOP trap potential, in the quadratic average approximation, as calculated in the rotating frame, have two restrictions. The first is that the center of mass motion of the dynamical eigenstates must be given by Eqs. (46) and (47). This is not suprising since we also found this restriction on solitary-wave dynamical eigenstates calculated using the circularly translating frame. The second restriction is that

$$\hat{L}_z(\boldsymbol{r})\psi_{\mathrm{H}}(\boldsymbol{r}) = l_z\psi_{\mathrm{H}}(\boldsymbol{r}), \tag{B8}$$

enforcing the envelope wave function to be an eigenstate of the z component of angular momentum. This means that dynamical eigenstates, as calculated using the rotating frame, must have a cylindrically symmetric density about their center of mass. We found in our discussion of dynamical eigenstates in Sec. IV B, that nonsymmetric dynamical eigenstates do exist for the TOP trap and that they are solitary-wave solutions in the laboratory frame retaining their orientation to that frame. Solitary-wave dynamical eigenstates which also satisfy Eq. (B8) are only a subset of the dynamical eigenstates found using the circularly translating frame.

Substituting Eqs. (46) and (B8) into the solitary-wave solution in the rotating frame, Eq. (B7), the solitary-wave dynamical eigenstates calculated using the rotating frame, i.e., satisfying Eq. (B6), have the form

$$\psi^{\mathrm{r}}_{\mathrm{SW}}(\boldsymbol{r}',t) = \phi(\boldsymbol{r}')e^{i\gamma_{\mathrm{t}}\Omega y'/2 - i\mu^{\mathrm{r}}_{\mathrm{SW}}t}, \tag{B9}$$

where we have explicitly extracted the time dependence in the envelope wave function by writing

$$\psi_{\mathrm{H}}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t)) = \phi(\boldsymbol{r}')e^{i\Omega l_z t}. \tag{B10}$$

The chemical potential spectrum in the rotating frame is

$$\mu^{\mathrm{r}}_{\mathrm{SW}} = \mu_{\mathrm{H}} + \varepsilon - \Omega l_z, \tag{B11}$$

where l_z is defined by Eq. (B8). The wave function phase $\gamma_{\mathrm{t}}\Omega y'/2$, in Eq. (B9), is derived from the coordinate dependent phase of Eq. (B7) and accounts for the center of mass momentum of the eigenstates, given in the laboratory frame by Eq. (47). The chemical potential spectrum of the rotating frame, given by Eq. (B11), can be decomposed into three parts: the energy of the state that forms the envelope, the additional energy offset ε, and an angular momentum term arising from the rotating frame, as expected [15].

Restrictions on solitary-wave solutions which are eigenstates in the rotating frame. We seek solutions to Eq. (B6) which have the form of Eq. (B7). The conditions required for such solitary-wave solutions to be eigenstates of the TOP trap in the rotating frame can be found following a similar structure to that used in Appendix A for the circularly translating frame.

Solitary-wave solutions which are to be TOP trap eigenstates in the rotating frame must satisfy both Eq. (B6) and

$$i\frac{\partial \psi^{\mathrm{r}}_{\mathrm{SW}}(\boldsymbol{r}',t)}{\partial t} = \mu^{\mathrm{r}}_{\mathrm{SW}}\psi^{\mathrm{r}}_{\mathrm{SW}}(\boldsymbol{r}',t). \tag{B12}$$

Substituting the solitary wave solution of Eq. (B7) into Eqs. (B6) and (B12), and making the change of variables $\boldsymbol{s}=\boldsymbol{r} - \overline{\boldsymbol{r}}(t)$, yields, respectively,

$$\begin{aligned}\mu^{\mathrm{r}}_{\mathrm{SW}} = \mu_{\mathrm{H}} &- \frac{1}{2}\overline{\boldsymbol{r}}(t)\cdot\frac{d^2\overline{\boldsymbol{r}}(t)}{dt^2} + \frac{1}{4}\overline{\boldsymbol{r}}^2(t) + \frac{7}{4}\overline{z}^2(t)\\ &+ \overline{\boldsymbol{r}}(t)\cdot r_0(\cos\Omega t, \sin\Omega t, 0) + \frac{7}{2}\boldsymbol{s}\cdot(0,0,\overline{z}(t))\\ &+ \frac{1}{2}\boldsymbol{s}\cdot[\overline{\boldsymbol{r}}(t) + 2r_0(\cos\Omega t,\sin\Omega t,0)]\\ &+ \frac{1}{2}\Omega\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)\cdot(\overline{y}(t),-\overline{x}(t),0) + \frac{1}{4}\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)^2\\ &- \frac{1}{2}\Omega\boldsymbol{s}\cdot\left(\frac{d\overline{y}(t)}{dt},-\frac{d\overline{x}(t)}{dt},0\right) - \Omega l_z(\boldsymbol{s})\\ &- i\left[\frac{d\overline{\boldsymbol{r}}(t)}{dt} + \Omega(\overline{y}(t),-\overline{x}(t),0)\right]\cdot\boldsymbol{g}(\boldsymbol{s})\end{aligned} \tag{B13}$$

and

$$\begin{aligned}\mu^{\mathrm{r}}_{\mathrm{SW}} = \mu_{\mathrm{H}} &- \frac{1}{2}\overline{\boldsymbol{r}}(t)\cdot\frac{d^2\overline{\boldsymbol{r}}(t)}{dt^2} - \frac{1}{4}\overline{\boldsymbol{r}}^2(t) - \frac{1}{2}\boldsymbol{s}\cdot\frac{d^2\overline{\boldsymbol{r}}(t)}{dt^2}\\ &+ \frac{1}{2}\Omega\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)\cdot(\overline{y}(t),-\overline{x}(t),0) + \frac{1}{4}\left(\frac{d\overline{\boldsymbol{r}}(t)}{dt}\right)^2\\ &- \frac{1}{2}\Omega\boldsymbol{s}\cdot\left(\frac{d\overline{y}(t)}{dt},-\frac{d\overline{x}(t)}{dt},0\right) - \Omega l_z(\boldsymbol{s})\\ &- i\left[\frac{d\overline{\boldsymbol{r}}(t)}{dt} + \Omega(\overline{y}(t),-\overline{x}(t),0)\right]\cdot\boldsymbol{g}(\boldsymbol{s}),\end{aligned} \tag{B14}$$

where $\boldsymbol{g}(\boldsymbol{s})$ is given by Eq. (A5) and

$$l_z(\boldsymbol{s}) = \frac{\hat{L}_z(\boldsymbol{s})\psi_{\mathrm{H}}(\boldsymbol{s})}{\psi_{\mathrm{H}}(\boldsymbol{s})}. \tag{B15}$$

Equating Eqs. (B13) and (B14), and substituting the general form of $\overline{\boldsymbol{r}}(t)$, given by Eq. (A1), we find that equality requires $x_3=0$ and $v_3=0$ and therefore $\overline{z}(t)=0$. Inserting this result, and the general expressions for $\overline{x}(t)$ and $\overline{y}(t)$ from Eq. (A1), Eqs. (B13) and (B14) both simplify to

$$\begin{aligned}\mu^{\mathrm{r}}_{\mathrm{SW}} = \mu_{\mathrm{H}} - \Omega l_z(\boldsymbol{s}) &+ \frac{1}{4}(x_1^2 + x_2^2 + v_1^2 + v_2^2)\\ &+ \frac{1}{2}[\Omega(v_1x_2 - v_2x_1) + \gamma_{\mathrm{t}}x_1(\Omega^2-1)]\\ &- i\boldsymbol{g}(\boldsymbol{s})\cdot(c_1\cos t + c_2\sin t, c_3\cos t - c_4\sin t, 0)\\ &- \frac{1}{2}\boldsymbol{s}\cdot(c_2\cos t - c_1\sin t, -c_4\cos t - c_3\sin t, 0),\end{aligned} \tag{B16}$$

where $c_1=v_1+\Omega x_2$, $c_2=v_2\Omega - x_1 - \gamma_{\mathrm{t}}(\Omega^2-1)$, $c_3=v_2-\Omega x_1$, and $c_4=v_1\Omega+x_2$. For the solitary-wave solutions in the laboratory frame to be eigenstates of the TOP trap in the rotating frame, the chemical potential $\mu^{\mathrm{r}}_{\mathrm{SW}}$ must be independent of $\boldsymbol{s}$ and t. Solving for $\boldsymbol{g}(\boldsymbol{s})$, at $t=0$ and $t=\pi/2$, we find that the only possible solution occurs when $c_1=c_2$ and $c_3=-c_4$. Substituting back into Eq. (B16) we find that $(c_1,c_2,c_3,c_4)=(0,0,0,0)$, or rather $(x_1,x_2,v_1,v_2)=(\gamma_{\mathrm{t}},0,0,\gamma_{\mathrm{t}}\Omega)$. Furthermore, we find that $l_z(\boldsymbol{s})$ must be independent of $\boldsymbol{s}$, and the chemical potential in the rotating frame is

$$\mu_{\rm SW}^{\rm r} = \mu_{\rm H} + \frac{1}{4}\gamma_{\rm t}^2(\Omega^2 - 1) - \Omega l_z, \tag{B17}$$

which is in agreement with Eq. (B11). Concluding then, solitary-wave solutions which are eigenstates of the TOP trap potential in the rotating frame exist if and only if $\overline{\boldsymbol{r}}(t)|_{(t=0)} = (\gamma_{\rm t}, 0, 0)$, $d\overline{\boldsymbol{r}}(t)/dt|_{(t=0)} = (0, \gamma_{\rm t}\Omega, 0)$, and

$$\hat{L}_z(\boldsymbol{s})\psi_{\rm H}(\boldsymbol{s}) = l_z\psi_{\rm H}(\boldsymbol{s}). \tag{B18}$$

The center of mass motion in the laboratory frame of solitary-wave dynamical eigenstates calculated using the rotating frame is identical to that of solitary-wave dynamcial eigenstates found using the circularly translating frame. Equation (B7) can now be written as

$$\psi_{\rm SW}^{\rm r}(\boldsymbol{r}', t) = \psi_{\rm H}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t))e^{i\gamma_{\rm t}\Omega y'/2 - i(\mu_H+\varepsilon)t}, \tag{B19}$$

and using Eqs. (B12) and (B18) it can be shown that

$$\psi_{\rm H}(\boldsymbol{r} - \overline{\boldsymbol{r}}(t)) = \phi(\boldsymbol{r}')e^{i\Omega l_z t}. \tag{B20}$$

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