

Spherical-code key-distribution protocols for qubits

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Recently spherical codes were introduced as potentially more capable ensembles for quantum key distribution. Here we develop specific key-creation protocols for the two qubit-based spherical codes, the trine and tetrahedron, and analyze them in the context of a suitably tailored intercept/resend attack, both in standard form, and in a “gentler” version whose back action on the quantum state is weaker. When compared to the standard unbiased basis protocols, Bennett-Brassard 1984 (BB84) and six-state, two distinct advantages are found. First, they offer improved tolerance of eavesdropping, the trine besting its counterpart BB84 and the tetrahedron the six-state protocol. Second, the key error rate may be computed from the sift rate of the protocol itself, removing the need to sacrifice key bits for this purpose. This simplifies the protocol and improves the overall key rate.

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Heretofore quantum key-distribution protocols have often been constructed using sets of unbiased bases, enabling key bit creation whenever the two parties Alice and Bob happen to send and measure the quantum system in the same basis. Alice randomly selects a basis and a state within that basis to send to Bob, who randomly chooses a basis in which to measure and decodes the bit according to their preestablished scheme. Should Bob choose the same basis as Alice, his outcome is perfectly correlated with hers. Each of the parties publicly announces the bases used, and for each instance they agree, they establish one letter of the key. The use of more than one basis prevents any would-be eavesdropper Eve from simply reading the encoded bit without Alice and Bob noticing. In two dimensions two sets of mutually unbiased bases exist, forming the Bennett-Brassard 1984 (BB84) [1] and six-state protocols [2].

Equiangular spherical codes can be used to construct a new scheme for key distribution [3]. Two such codes exist in two dimensions. In the Bloch-sphere representation we may picture these ensembles as three equally spaced coplanar states forming a trine or four equally spaced states forming a tetrahedron. Both Alice and Bob replace their use of unbiased bases with equiangular spherical codes; by arranging Bob’s code to be the dual of Alice’s, key creation becomes a process of elimination, as previously considered by Phoenix *et al.* [4]. Each of Bob’s measurement outcomes is orthogonal to one of Alice’s signals, and thus each outcome excludes one signal. Alice may then attempt to furnish the remaining information by announcing a certain number of signals that were not sent, a process known as sifting. By symmetry, Bob can also send the sifting information to Alice, in the form of outcomes not obtained; this convention will be followed here. The shared anticorrelation between signal and outcome allows them to remain one step ahead of an eavesdropper Eve, ensuring that unless she tampers with the quantum signal, she knows nothing of the created key.

Should Eve tamper with the signal, the disturbance can be

recognized by Alice and Bob in the statistics of their results. With this they can determine what she knows about their key, and they may either proceed to shorten their key string so as to remove Eve’s information of it, or else discard it entirely and begin anew. Unlike bases-based protocols, however, here Alice and Bob can determine the disturbance from the sifting rate directly, obviating the need to explicitly compare (and waste) portions of the key for this purpose.

The overarching questions in evaluating a key-distribution protocol are whether or not it is unconditionally secure, and if so, what the maximum tolerable error rate is. If, by granting Eve the ability to do anything consistent with the laws of physics, Alice and Bob can still share a key, the protocol is said to be unconditionally secure. This state of affairs persists up to the maximum tolerable error rate, at which point Alice and Bob must abandon their key-creation efforts. Establishing unconditional security is complicated and delicate, so here we restrict attention to more limited attacks, examining the intercept/resend attack and a “gentler” variant. In these settings we find that the spherical codes are more tolerant of noise than their basic counterparts. First, however, we must consider the protocols for the two spherical codes in detail.

Unlike the case of unbiased bases, in which Alice’s choice of signal or Bob’s outcome determines the key letter, for the trine and tetrahedron it is only the relation between Alice’s signal and Bob’s outcome that determines the bit. In the trine protocol, Alice’s choice of signal narrows Bob’s possible outcomes to the two lying 60° on either side. Each is equally likely, and they publicly agree beforehand that the one clockwise from Alice’s signal corresponds to 1 and the other 0. Alice hopes to determine which is the case when Bob announces one outcome that he *did not* receive. For any given outcome, he chooses randomly between the other two and publicly announces it. Half the time he announces that he did not receive the outcome which Alice already knew to be impossible. This tells Alice nothing new, and she announces that the protocol failed. In the other half of cases, Alice learns Bob’s outcome and announces success.

Upon hearing his message was a success, Bob can determine the signal Alice sent. For any outcome Bob receives, he

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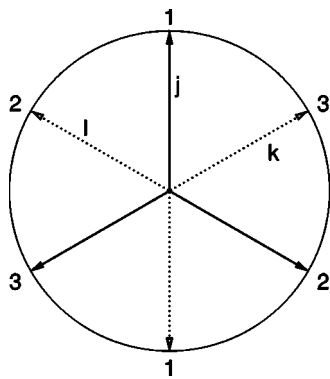


FIG. 1. Bloch-sphere representation of the trine-based protocol by which Alice and Bob create a secret key bit, shown here creating a 1. Alice's three possible signal states are shown in black and Bob's measurement outcomes in dotted lines; antipodal points are orthogonal. Without loss of generality we may assume that Alice sends the state $j=1$. The antipodal point is the impossible outcome for Bob; here he obtains the outcome $k=3$. Of the two outcomes he did not get, he picks one at random and announces this to Alice. Here he announces the outcome $l=2$, and Alice infers the value of k . Had Bob announced the other outcome, the protocol would fail, as this tells Alice nothing she does not already know. Here she announces that she is satisfied with Bob's message, and Bob infers the value of j , since Alice's signal could not have been l . Now they compute the bit $(1 - \epsilon_{jkl})/2 = 1$. The announcement only reveals l , so the bit is completely secret.

immediately knows one signal Alice could not have sent, and the message that his announcement was successful indicates to him that she also did not send the signal orthogonal to his message. Had she sent that signal, she would have announced failure; thus Bob learns the identity of Alice's signal. Each knowing the relative position of signal and outcome, they can each generate the same requisite bit. This round of communication is the analog of sifting in the protocols utilizing unbiased bases: a follow-up classical communication referencing the quantum signals with which Alice and Bob establish the key.

Mathematically, we might consider the protocol as follows. Alice sends signal j , and Bob necessarily obtains $k=j+1$ or $k=j+2$. He announces that he did not receive $l \neq k$. If $l=j$, Alice announces failure. Otherwise each party knows the identity of j , k , and l , and they compute the key bit as $(1 - \epsilon_{jkl})/2$. Figure 1 shows the case that they agree on a 1.

Though Eve may listen to the messages on the classical channel, she still has no knowledge of the bit value, for all she knows is one outcome Bob did not receive and the corresponding antipodal state not sent by Alice. Of the two remaining equally likely alternatives, one corresponds to a 0 and the other a 1. Hence the protocol establishes one fully secret bit half the time, analogous to the BB84 protocol.

The strategy for the tetrahedron is entirely similar, except that Bob must now reveal two outcomes not obtained. As depicted in Fig. 2, Alice uses four tetrahedral states in the Bloch-sphere picture, and as before Bob uses the dual of Alice's tetrahedron for measurement. Alice sends signal j and Bob receives $k \neq j$. He then randomly chooses two outcomes l and m he did not obtain and announces them. One-

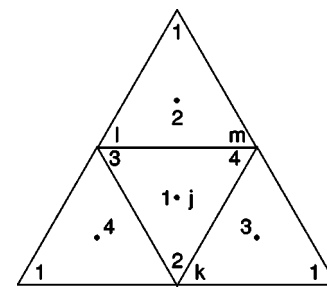


FIG. 2. Unfolded view of the Bloch-sphere tetrahedron states. Vertices of triangles correspond to Bob's outcomes, their centers Alice's signals; all three vertices of the large triangle represent the same point antipodal to its center. Suppose Alice sends signal j ; Bob necessarily receives $k \neq j$. Here we suppose $j=1$ and $k=2$. Bob then announces two outcomes not obtained, here shown as $l=3$ and $m=4$. Had either message equaled j , which happens $2/3$ of the time, Alice would announce failure. Otherwise, as here, she accepts. Thus Alice determines k , and Bob finds out j . They compute the bit $(1 + \epsilon_{jklm})/2 = 1$. The announcement reveals only l and m , so the bit is secret.

third of the time this is successful, in that $l \neq j$ and $m \neq j$. This allows Alice to infer k , and her message of satisfaction allows Bob to infer j , just as for the trine. They then each compute the bit $(1 + \epsilon_{jklm})/2$.

Again they stay one step ahead of Eve as she listens to the messages, as she can only narrow Alice's signal down to two possibilities. Given the order of Bob's messages, one of these corresponds to 0 and the other to 1, so Eve is ignorant of the bit's identity. Using the tetrahedron allows Alice and Bob to establish one fully secret bit one-third of the time, analogous to the six-state protocol.

In the two protocols, the dual arrangement of signals and measurements allows Alice and Bob to proceed by elimination to establish a putative key. To ensure security of the protocols, however, the arrangement must also disallow Eve from reading the signal without Alice and Bob noticing. Analyzing the intercept/resend attack provides evidence of how well the protocols based on spherical codes measure up to this task.

If Eve tampers with the signals in order to learn their identity, the inevitable disturbance allows Alice and Bob to infer how much Eve knows about the raw key. They can then proceed to use error correction and privacy amplification procedures to distill a shorter key which, with high probability, is identical for Alice and Bob and which Eve has low probability of knowing anything about. Instead of delving into the details of error correction and privacy amplification, we may instead use a lower bound on the optimal rate of the distilled key, i.e., its length as a fraction of the raw key [5]. This provides a reasonable guess as to what may be achieved in practice and is known to be achievable using one-way communication. Given $N \rightarrow \infty$ samples from a tripartite distribution $p(a, b, e)$, Alice and Bob can construct a protocol to distill with high probability a length RN string about which Eve has asymptotically zero information for

$$R = I(A:B) - \min\{I(A:E), I(B:E)\}. \tag{1}$$

Here the tripartite distribution refers to Alice's and Bob's bit values a and b , and Eve's best guess e from the eavesdrop-

ping. The quantity $I(A:B)$ is the mutual information between two parties, quantifying how much knowledge of one's outcome implies about the other's.

Here we are assuming that Eve simply intercepts a fraction q of the signals, measures them, and sends a new state on to Bob. The first task is then to determine R as a function of q and then to relate q to the statistics compiled in the course of the protocol. As it happens, Eve's best attack in the intercept/resend context is to use *both* Alice's and Bob's trines for measurement, half the time pretending to be Alice and the other half Bob. This holds for the tetrahedron as well and is due to the minimum in Eq. (1), which gives the equation a symmetry between Alice and Bob with respect to Eve. Choosing only one of the trines (or tetrahedra) to measure breaks this symmetry, leading the minimum to pick the smaller information quantity and yield a consequently larger key. By mixing the two strategies, Eve restores the symmetry and increases the minimum knowledge she has about either party's bit string. Phoenix *et al.* [4] note that the scheme in which Eve pretends to be Bob maximizes her mutual information with Alice; however, as the analysis stops there and does not proceed to consider either Eve's information about the key bits or any secret key rate bounds, it is insufficient as a cryptographic analysis.

To determine the mutual information quantities as functions of q , it suffices to consider first the case in which Eve intercepts every signal and uses Alice's ensemble for measurement. With these quantities in hand, we can mix Eve's two measurement strategies appropriately and then include her probability of interception. We begin with the trine. Given a signal state from Alice, there are two cases to consider. Either Eve measures and gets the same state, which happens with probability $2/3$, or she obtains one of the other two results, with probability $1/6$ for each. Whatever her outcome, she passes the corresponding state along to Bob and guesses that it was the state sent by Alice, *unless the subsequent exchange of classical messages eliminates this possibility*, at which point she reserves judgment about the key bit.

Suppose Eve's outcome corresponds to Alice's signal, and thus no disturbance is caused. Naturally, Alice and Bob go on to establish a bit half the time, a bit known to Eve. On the other hand, should her outcome not coincide with Alice's signal, there are two further possibilities. Half the time Bob obtains a result consistent with Alice's signal, i.e., not the orthogonal state, and a further half of the time the sifting succeeds. However, the required sifting messages will eliminate Eve's outcome as Alice's signal, thus forcing Eve to abandon her guess. In the remaining case, Bob's result is orthogonal to Alice's signal, which guarantees successful sifting, but also different bit values for Alice and Bob. Half of Eve's guesses are excluded while the remainder agree with Bob's.

Putting all this together, one obtains that the protocol fails with probability $5/12$. All three agree one-third of the time, and Alice's bit is different from that shared by Bob and Eve one-twelfth of the time. In the remaining one-sixth of events, Eve does not field a guess, as the messages exchanged by Alice and Bob contradict her measurement results; better to abstain than to introduce a purely random guess. In this sub-

set of events, Alice and Bob agree a further half the time.

Of the key bits created, Bob and Alice agree with probability $5/7$, while Eve and Alice agree with probability $4/7$. Eve only fields a guess with probability $5/7$, and always agrees with Bob when she does. These numbers are obtained by considering the raw probabilities of agreement and renormalizing by $12/7$. Should Eve instead measure the signals using Bob's trine ensemble, her agreement probabilities with Alice and Bob are swapped. Mixing the two strategies then yields Eve a no-guess probability of $2/7$, an agreement probability with either party of $9/14$, and an error probability of $1/14$.

To interpolate between the end points of no interception and full interception, note that to condition on the cases of successful bit creation, the probability of bit agreement must be renormalized by the probability of sifting success. This probability depends linearly on q : $p_{\text{sift}}=(6+q)/12$. All probabilities must therefore contain $6+q$ in the denominator, whence we may derive pairwise probabilities that the parties' bit values agree: $p_{ab}=(6-q)/(6+q)$, and $p_{ae}=9q/2(6+q)$, respectively. Eve's probability to not guess at all is $2(3-2q)/(6+q)$. Determining the relevant mutual informations from these expressions is straightforward; for expressions involving Eve, simply treat the "no guess" as another outcome which has no correlation at all to the other party.

By determining the probability of error in Alice's and Bob's bit strings as a function of q , we may compare to other protocols. For the trine, errors occur in the key string with probability $2q/(6+q)$. Using the calculated agreement probabilities in the rate bound, one obtains that $R=0$ corresponds to a maximum tolerable bit error rate of 20.4%. This compares favorably with the BB84 protocol's maximum tolerable bit error rate of 17.1% under the same attack [6]. In terms of *channel* error rate these figures double, if we consider the quantum channel to be a depolarizing channel instead of arising from Eve's interference. If Bob receives the maximally mixed state instead of Alice's signal, the probability of error given successful sifting is $1/2$. Hence a fully depolarizing channel leads to a bit error rate of 50% for either protocol.

Analysis of the tetrahedron protocol proceeds similarly by examining the various cases. In this case, when $q=1$ the failure rate of the protocol drops to $5/9$, while Alice and Bob agree with probability $5/8$, Eve has probability $7/16$ of knowing Alice's or Bob's bit value, and she reserves judgment half the time. As the successful sifting rate of the protocol varies as $(3+q)/9$, we may determine the form of the probabilities using the same method to be $p_{ab}=p_{eb}=(6-q)/2(3+q)$ and $p_{ae}=7q/4(3+q)$, while the error rate in the key string is $3q/2(3+q)$ and Eve's probability of not guessing is $(3-q)/(3+q)$. Using these probabilities in the rate bound yields a maximum error rate of 26.7%. As before, this compares favorably to the maximum tolerable error rate in the six-state protocol of 22.7%.

Eve's attack could be gentler, however. In the version already considered, her positive-operator-valued measure (POVM) consists of subnormalized projectors onto the code states in addition to an element proportional to the identity operator, corresponding to the case in which Eve opts not to intercept the signal. A similar POVM can be created by dis-

tributing a piece of identity operator to all the other elements. The crucial difference is that the state Eve sends on to Bob after her measurement is different. Using the square root of each POVM element in the formula for the postmeasurement state, the resulting measurement yields Eve more information for the same amount of disturbance. Note that in the context of the BB84 protocol, this attack was determined to be optimal when Eve does not wait to hear in which basis the signal was prepared [7].

Enlisting the aid of MATHEMATICA to carry out the book-keeping yields the following results. Since the attack is stronger, the maximum tolerable error decreases; in particular the trine can create secret keys up to a 16.6% bit error rate, as opposed to 15.3% for its cousin BB84. The tetrahedron remains the most robust, sustaining key creation up to a maximum error rate of 22.6%, as compared to 21.0% for the six-state protocol.

Framing the key rate in terms of the error rate is solely for ease of comparison, as it is not necessary for Alice and Bob to sacrifice key bits in order to obtain an estimate of q when using spherical codes, in contrast to the situation for the unbiased bases. For spherical codes, the sifting rate of the protocol itself determines q ; as the channel becomes noisier and Bob's outcome less correlated with Alice's signal, the sifting rate increases. Of course, not all of this increase provides useful key: most of it leads to errors. But Eve cannot substitute signals solely for the purpose of modifying the sift rate, as her signals will be uncorrelated with Alice's and will therefore also lead to an increase in the sift rate. Hence she is precluded from masking her interceptions, and Alice and

Bob can determine q from the sifting rate itself.

Finally, a word on the feasibility of implementing such protocols. Generation of trine or tetrahedral codewords as polarization states of (near) single-photon sources is not difficult. The generalized measurements accompanying the ensembles can be performed by using polarizing beam splitters and wave plates to map polarization states into different propagation modes and proceeding from there with linear optical elements to produce the appropriate interference. Such measurements have indeed been performed with rms errors in observed statistical distributions of a few percent [8]. The physical implementation need not be identical to the logical construction of the protocol, however. For instance, three states constructed from two pairs of singlets together with ordinary photodetectors can implement the trine protocol [9].

Two advantages of using spherical codes have been established herein. First and foremost is the strong possibility of improved eavesdropping resistance. Subsequent analyses either of stronger attacks, such as use of an asymmetric cloning machine [10], or of the use of error-correcting codes to beat back noise [11] are required to demonstrate this fact in the setting of unconditional security, although the intercept/resent attacks are indicative of the trend [12]. Beyond security is the ability to directly estimate the error rate from the sift rate itself, obviating any need to sacrifice raw key bits.

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