Silicon quantum computation based on magnetic dipolar coupling

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A dipolar gate alternative to the exchange-gate-based Kane quantum computer is proposed where the qubits are electron spins of shallow group V donors in silicon. Residual exchange coupling is treated as gate error amenable to quantum error correction, removing the stringent requirements on donor positioning characteristic of all silicon exchange-based implementations [Koiller *et al.*, Phys. Rev. Lett. **88**, 027903 (2002)]. Contrary to common speculation, such a scheme is scalable with no overhead in gating time even though it is based on long-range dipolar interqubit coupling.

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I. INTRODUCTION

Since the seminal exchange gate proposal of Loss and DiVincenzo, research on semiconductor spin quantum computation has focused on implementations based on the electron exchange interaction [1]. For silicon donor impurities the use of exchange coupling is problematic since the exchange energy depends sensitively on donor positioning due to the quantum interference arising from the sixfold degeneracy of Si conduction band [2]. This results in the necessity of donor positioning within one silicon bond (2.4 Å) otherwise severe tuning requirements will adversely affect the scalability of this implementation (in addition to many donor pairs having nearly zero exchange). This problem is attracting considerable attention [3] since Si spin quantum computer architecture is an active research area, and donor spins in nuclear-spin-free silicon (pure ^{28}Si) are expected to have very long coherence times [4–6] ($T_2 \sim T_1 \sim 10^3$ s [7]). Here we propose the magnetic dipolar interaction rather than the exchange interaction between well separated donor electron spins as a solution to this problem. The residual exchange interaction is treated as a source of imperfection in the dipolar gate, whose error probability can be kept below 10^{-4} per operation. Hence the exchange interaction can be ignored as long as error correction is applied, leading to no necessity of gate tuning. This is possible due to the long-range character of dipolar coupling (proportional to $1/d^3$, with d the interqubit separation) as opposed to the short-range nature of exchange $[J \propto d^{2.5}\exp(-d)]$ [8]. Nevertheless, this long-range character led to speculations that a dipolar quantum computer is not scalable [9]. We point out that this is not true, because only up to the fourth nearest neighbor (NN) couplings need to be considered, and highly efficient quantum gates can be constructed using the method of Leung *et al.* [10], which we develop further to avoid correlated error between any two qubits inside the error correction manifold. Similar pulsing sequences should be useful for a wide variety of solid state quantum computing architectures based on long-range interactions [11]. The resulting architecture takes advantage of g-factor manipulation and measurement at the single-spin level to avoid the scalability problems inherent to the ensemble NMR proposals (such as decreasing signal to noise ratio and overlapping resonances [12,13]).

II. DIPOLAR GATE FIDELITY IN THE PRESENCE OF EXCHANGE INTERACTION

The truncated magnetic coupling between two localized spins is given by

$$\mathcal{H}_{12} = \omega_1 S_{1z} + \omega_2 S_{2z} - [D_{12}(\theta,d) - J(a^*,d)]S_{1z}S_{2z}. \quad (1)$$

Here $\mathbf{S}_i$ are spin-1/2 operators, which couple to external magnetic fields through the Zeeman frequencies $\omega_i = \gamma_i B_i$. Equation (1) is valid provided we neglect terms proportional to $S_{i\pm}$, which amount to a correction quadratic in $[D_{12}/(\omega_1 - \omega_2)]$ [14]. Hence a strong inhomogeneous field is needed (or inhomogeneous gyromagnetic ratios γ_i), but since $D_{12}/\gamma_i \lesssim 0.01$ G (see below) field differences on the 10–100 G range are sufficient. The dipolar interaction is given by

$$D_{12}(\theta,d) = \frac{\gamma_1\gamma_2\hbar}{d^3}(3\cos^2\theta - 1), \quad (2)$$

where d is the interqubit distance and θ is the angle between the external magnetic field and the line joining the spins. The optimum dipolar architecture assumes $\theta = 0$, e.g., an array of spins directed along **B** (this optimal coupling is denoted D_{12} below). Equation (2) has a striking property: If $|\cos\theta| = 1/\sqrt{3}$, the interaction is exactly zero. Hence in an array of spin qubits dipolar coupling can be completely suppressed as long as $\pm\mathbf{B}$ makes one of the "magic" angles with the array: $\theta = 54.74°$, $125.26°$. Exchange-based proposals [1] usually require the donors to be pushed outside the array to switch on the exchange interaction. In addition, two- (2D) and three-dimensional (3D) arrangements have been considered, particularly to optimize quantum error correction [15,16]. In this case it may be impossible to find a geometry where all bonds are making a magic angle with the B field. Then if dipolar interaction is to be ignored, we will lose track of the spin state within 10 μs (this time should be considered short

TABLE I. A group V donor electron spin quantum computer, where free evolution of the spin-spin dipolar interaction implements CZ gates. Here we show donor electron ground state energies E_d [20], Bohr radius a^* [17], optimum interqubit distance d_{opt} (for the exchange interaction to be ignored within 10^{-4} error probability), interqubit distance d_0 (such that $D_{12}=J$), and the CZ gate times. Fastest gate times are obtained for bismuth donors.

Donor	E_d (meV)	a^* (Å)	d_{opt} (Å)	d_0 (Å)	τ_{CZ} (μs)
Sb	43	18.6	315	263	150
P	45	18.2	307	256	140
As	54	16.6	279	232	105
Bi	71	14.5	241	200	68

if compared with other mechanisms such as nuclear spectral diffusion [5,6]). Hence dipolar coupling may have to be taken into account even in exchange gate quantum computing architectures. For the exchange interaction between two hydrogenic donors we use the asymptotic expression [8]

$$J(a^*,d) \approx \frac{1.6}{\hbar\varepsilon}\frac{e^2}{a^*}\left(\frac{d}{a^*}\right)^{5/2}\exp\left(-2\frac{d}{a^*}\right), \tag{3}$$

valid for interdonor distances d much larger than the Bohr radius a^*. Equation (3) is to be regarded as an envelope for the strong oscillations of the exchange energy stemming from conduction band degeneracy [2]. The Bohr radius a^* is related to the experimental donor ground state energy E_d; see Table I [17].

We will now show how a silicon donor quantum computer can be implemented with the dipolar interaction and single-spin rotations. The effect of the exchange interaction will be treated as an error, leading to a lower bound on qubit separation. Assuming $J=0$ in Eq. (1), a controlled-Z (CZ) gate is obtained by free evolution during the time interval $\tau_{\mathrm{CZ}}=\pi/D_{12}$ together with g-factor shifts [18],

$$U_{\mathrm{CZ}} = e^{-(3\pi/4)i}e^{(3\pi/2)iS_{1z}}e^{-(\pi/2)iS_{2z}}\exp\left(-i\frac{\pi}{D_{12}}\mathcal{H}_{12}\right). \tag{4}$$

Below we show how to correct for the Zeeman frequencies ω_i.

We now search for the minimum interqubit distance d so that J can be ignored. A residual exchange interaction J will add an additional evolution operator to Eq. (4):

$$U(\alpha) = \exp(-i\alpha S_{1z}S_{2z}), \tag{5}$$

with $\alpha=\pi J/D_{12}$. This causes phase error in the CZ gate, which is better evaluated by looking at two input states orthogonal to each other. Equivalently we look at the controlled-NOT (CNOT) gate (obtained by a basis change on the CZ gate, $U_{\mathrm{CNOT}}=e^{-i(\pi/2)S_{2y}}U_{\mathrm{CZ}}e^{i(\pi/2)S_{2y}}$). Therefore the "erroneous" evolution is given by $\tilde{U}(\alpha)U_{\mathrm{CNOT}}$, where

$$\tilde{U}(\alpha) = e^{-i(\pi/2)S_{2y}}U(\alpha)e^{i(\pi/2)S_{2y}} \tag{6}$$

is a 4×4 matrix with elements equal to $\cos(\alpha/4)$, $\sin(\alpha/4)$, and 0. The error due to a finite α can be evaluated by calculating the fidelity functions

$$F\{|\psi\rangle,\alpha\} = |\langle\psi|U_{\mathrm{CNOT}}^{\dagger}\tilde{U}(\alpha)U_{\mathrm{CNOT}}|\psi\rangle|, \tag{7}$$

which are simply given by $|\cos(\alpha/4)|$, leading to an error probability of $\alpha^2/16$ for small α (the error is given by $E=1-F^2$). If one wants to ignore exchange interaction, all that needs to be done is to keep $\alpha^2/16$ less than some critical bound p, for example the seven-qubit encoding threshold $p=10^{-4}$ [19]. Hence we have $J/D_{12}\leqslant\sqrt{p}$, or

$$\frac{J}{D} \approx \left(\frac{a^*}{0.02\ \text{Å}}\right)^2\left(\frac{d}{a^*}\right)^{11/2}\exp\left(-2\frac{d}{a^*}\right) \leqslant 10^{-2}, \tag{8}$$

for $p=10^{-4}$. The length scale for the prefactor in this expression is given by $\sqrt{2\varepsilon/1.6}\hbar\gamma/e\approx0.02$ Å. The range where this inequality is satisfied is approximately given by $0<d\lesssim0.03a^*$ and $d\gtrsim17a^*$. The first condition arises due to the divergence of the dipolar interaction, and is not useful here [also Eq. (3) is appropriate only for $d\gg a^*$]. The physical solution is the second one, which is optimal (fastest gate) for $d_{\mathrm{opt}}\approx17a^*$. Table I shows d_{opt} for various donors together with their CZ gate time [π/D_{12}; see Eq. (4)].

III. A SCHEME FOR DECOUPLING LONG-RANGE INTERACTIONS

Up to now we have shown that dipolar coupling between two donors can generate precise two-qubit evolution, i.e., a dipolar coupled-qubit Si gate can be constructed. However, the situation becomes complicated when we consider an array of many donors. Particularly the long-range nature of the dipolar interaction implies every spin in the array will be coupled to each other, raising questions about the scalability of this proposal (This was one of the original motivations for introducing the exchange gate since exchange can be exponentially suppressed by electrically controlling wave function overlap.) For example, it is possible that the complexity of the pulsing sequences (leading to the desired quantum algorithm) might scale exponentially with the number of qubits, effectively making the problem of determining the evolution as hard as any mathematical problem a quantum algorithm is constructed to solve [9]. Nevertheless, this is not true for the case considered here, because using the same argument leading to the discard of the exchange interaction we can neglect (within the 10^{-4} threshold) dipolar coupling between any spin and its fifth or higher NN [by Eq. (8), $D_{1k}/D_{12}=1/k^3$, which is less than 10^{-2} for $k\geqslant5$]. Hence Eq. (1) generalized to a 1D spin array is

$$\mathcal{H} = \sum_i \omega_i S_{iz} - \frac{1}{2}\sum_{i;j=i-4}^{i+4} D_{ij}S_{iz}S_{jz}, \tag{9}$$

where i is an integer labeling the location of each donor (i is assumed positive as well as negative). The finite coupling range allows us to develop quantum gates using a sequence of π pulses applied to subsets of the spins [each π pulse is given by $X^{(i)}=\exp(i\pi S_{ix})$]. The key point is that the interaction between any two spins can be canceled using two π pulses [14],

$$\exp\left(-i\frac{\tau}{2}DS_{1z}S_{2z}\right)X^{(2)}\exp\left(-i\frac{\tau}{2}DS_{1z}S_{2z}\right)X^{(2)}=I, \quad (10)$$

where I is the identity operator. Our task is now to find the "decoupling" scheme which completely refocuses Eq. (9) after some time interval τ (therefore enabling single-qubit rotation on any spin) and also to produce sequences for "selective recoupling," which provide CZ evolution for any NN pair. For this task we use the method of Ref. [10] which consists in constructing sign matrices S_n representing the π pulses. An $n\times m$ sign matrix has each element equal to ± 1 (denoted simply by $\pm$), and correspond to a system of n spins where evolution during a time τ is divided into m time intervals. If spin i has its interaction reversed in any lth time interval (by application of $X^{(i)}$ before and after this time interval), then $(S_n)_{il}=-1$; otherwise $(S_n)_{il}=+1$. For example, Eq. (10) corresponds to

$$S_2=\begin{pmatrix}+ & +\\ + & -\end{pmatrix}. \quad (11)$$

The interaction between two spins i,j is decoupled if the rows i and j of S_n disagree in sign for half of the m time intervals. Equivalently, the inner product between these rows is zero. This property leads to a connection with the theory of Hadamard matrices: $H_{\bar{n}}$ is an $\bar{n}\times\bar{n}$ Hadamard matrix if and only if $H_{\bar{n}}\cdot(H_{\bar{n}})^T=\bar{n}I$. Hence a possible solution for the decoupling problem of n spins is to construct S_n from n rows out of a Hadamard matrix $H_{\bar{n}}$ where $\bar{n}\geqslant n$. Actually such a solution turns out to be the most efficient one (the smallest $\bar{n}$ satisfying $\bar{n}\geqslant n$ is the minimum number of intervals m in the set of possible S_n) because one can show that it is impossible to add an additional row orthogonal to $H_{\bar{n}}$. Hadamard matrices exist for $\bar{n}=1$, 2 [Eq. (11)], 4, 8, 12, … (see Ref. [10] and references therein). The finite coupling range of Eq. (9) suggests $\bar{n}=12$ as a convenient solution. S_n can be assembled as an $n\times 12$ matrix composed of seven-ordered rows from H_{12} (identical rows are seven rows apart). For the particular case of $n=14$ (general n is obtained by row repetition),

$$S_{14}=\left(\begin{array}{c}
+\,+\,+\,+\,+\,+\,-\,-\,-\,-\,-\,-\\
+\,+\,+\,-\,-\,-\,+\,+\,+\,-\,-\,-\\
+\,-\,-\,+\,+\,-\,-\,+\,+\,-\,-\,+\\
+\,+\,-\,+\,-\,-\,+\,-\,-\,+\,-\,+\\
+\,+\,-\,-\,+\,-\,-\,+\,-\,+\,+\,-\\
+\,+\,-\,-\,-\,+\,-\,-\,+\,-\,+\,+\\
+\,-\,+\,+\,-\,-\,-\,-\,+\,+\,+\,-\\
\hline
+\,+\,+\,+\,+\,+\,-\,-\,-\,-\,-\,-\\
+\,+\,+\,-\,-\,-\,+\,+\,+\,-\,-\,-\\
+\,-\,-\,+\,+\,-\,-\,+\,+\,-\,-\,+\\
+\,+\,-\,+\,-\,-\,+\,-\,-\,+\,-\,+\\
+\,+\,-\,-\,+\,-\,-\,+\,-\,+\,+\,-\\
+\,+\,-\,-\,-\,+\,-\,-\,+\,-\,+\,+\\
+\,-\,+\,+\,-\,-\,-\,-\,+\,+\,+\,-
\end{array}\right). \quad (12)$$

Here we extract the first (and last) seven rows of S_{14} from H_{12} with the first row $(++\cdots)$ excluded so that Zeeman splitting is also canceled. S_{14} requires a total of 80 π pulses which are applied in 12 sets (fewer than 14 pulses are applied in each set because $X^{(i)2}=I$—hence no rotations need to be applied when the sign is the same for neighboring time intervals). An array of n spins will require fewer than $6n$ pulses. Selective recoupling is achieved by choosing identical rows for the spins which are to be coupled. These rows are chosen from the four remaining rows of H_{12}, for example,

$$S'_{14}=\left(\begin{array}{c}
+\,+\,+\,+\,+\,+\,-\,-\,-\,-\,-\,-\\
+\,+\,+\,-\,-\,-\,+\,+\,+\,-\,-\,-\\
+\,-\,-\,+\,+\,-\,-\,+\,+\,-\,-\,+\\
+\,+\,-\,+\,-\,-\,+\,-\,-\,+\,-\,+\\
+\,+\,-\,-\,+\,-\,-\,+\,-\,+\,+\,-\\
+\,+\,-\,-\,-\,+\,-\,-\,+\,-\,+\,+\\
+\,-\,+\,+\,-\,-\,-\,-\,+\,+\,+\,-\\
\hline
+\,+\,+\,+\,+\,+\,-\,-\,-\,-\,-\,-\\
+\,+\,+\,-\,-\,-\,+\,+\,+\,-\,-\,-\\
+\,-\,-\,+\,+\,-\,-\,+\,+\,-\,-\,+\\
+\,+\,-\,+\,-\,-\,+\,-\,-\,+\,-\,+\\
+\,+\,-\,-\,+\,-\,-\,+\,-\,+\,+\,-\\
+\,+\,-\,-\,-\,+\,-\,-\,+\,-\,+\,+\\
+\,-\,+\,+\,-\,-\,-\,-\,+\,+\,+\,-
\end{array}\right) \quad (13)$$

implements CZ operations between spins 6, 7 and 10, 11 in parallel (bold). We point out that each seven-qubit structure in Eqs. (12) and (13) forms an error correction block for the Steane code [19]. Note that residual dipolar interaction couples qubits in different blocks. This is important for the assumption of uncorrelated errors within each block and the validity of the 10^{-4} threshold (see Sec. IV). The spurious couplings lead to error of the order of $7^{-6}\sim 10^{-5}$ in Eq. (12) and $22^{-6}\sim 10^{-8}$ for selective recoupling when all blocks execute a CZ gate in parallel (massive parallelization of the CZ gate is needed for efficient computation and quantum error correction).

Therefore the complete gate time for a large 1D array is the same as for two donors (approximately 100 μs, being optimal for bismuth—see Table I). This shows that a dipolar donor electron spin quantum computer is reliable: If the silicon lattice is isotopically purified (free of ^{29}Si nuclear spins), the coherence time will be limited by the spurious exchange and dipolar couplings, with a quality factor of the order of 10^4. A key advantage of this architecture is the interqubit distance, which is three times larger than other proposals for donors [1]. Also there is no need for an interqubit "J" gate, or any electrical control over wave function overlap [1,3]. This should make gate lithography much simpler (one needs to incorporate g-factor control [21] and single-spin measurement/initialization electrodes [22] on top of each donor).

The considerations above can be generalized to any long-range coupling $D\propto 1/d^r$. The number of NN's which need to be decoupled is given by $\max(k)\leqslant p^{-1/2r}$, where p is the desired error probability. Hence τ_{CZ} needs to be broken into

$\bar{n} \sim p^{-1/2r}$ time intervals. For example, $r \gtrsim 1$ and $p = 10^{-4}$ lead to $\bar{n} \sim 100$. Implementation of any quantum gate is possible as long as the time for single-spin rotation is much less than $\tau_{CZ}/\bar{n}$. The dipolar case considered here clearly satisfies this criterion, since $\tau_{CZ}/\bar{n} \sim 10$ μs (rotation times of the order of 0.1 μs are easily achievable). Finally, notice that this approach for decoupling can also be applied to general anisotropic exchange interactions, since these can be transformed into the $S_{iz}S_{jz}$ form by appropriate spin rotations.

IV. ERROR CORRECTION OF RESIDUAL LONG-RANGE COUPLING

Here we show how imperfections arising from spurious long-range couplings connecting qubits in *distinct quantum error correction blocks* can be corrected by the usual syndrome diagnosis (projective measurement on each block) [23]. The proof presented here is based on the simplest error correction code, the "three-bit-flip code" (Sec. 10.1.1 of Ref. [23]). However, we emphasize that these results are easily extended to the complete seven-bit Steane code [19] which corrects for any type of continuous error on each qubit within its block. The essence of our proof is that the syndrome measurement on each block effectively destroys error correlation between qubits belonging to different blocks.

Consider two error correction blocks constituted by qubits 1,2,3 (first block) and 4,5,6 (second block). The residual coupling Hamiltonian is

$$\mathcal{H} = -4c'(S_{1z}S_{4z} + S_{2z}S_{5z} + S_{3z}S_{6z}), \tag{14}$$

and the evolution operator after one "clock time" τ is

$$\begin{aligned}\mathcal{U}(\tau) = \exp(-i\mathcal{H}\tau) &= \cos^3(c)I + i\sin(c)\cos^2(c)\\ &\times[\sigma_{1z}\sigma_{4z} + \sigma_{2z}\sigma_{5z} + \sigma_{3z}\sigma_{6z}] - \cos(c)\sin^2(c)\\ &\times[\sigma_{1z}\sigma_{2z}\sigma_{4z}\sigma_{5z} + \sigma_{1z}\sigma_{3z}\sigma_{4z}\sigma_{6z} + \sigma_{2z}\sigma_{3z}\sigma_{5z}\sigma_{6z}]\\ &- i\sin^3(c)[\sigma_{1z}\sigma_{2z}\sigma_{3z}\sigma_{4z}\sigma_{5z}\sigma_{6z}].\end{aligned} \tag{15}$$

Here $c = c'\tau = \pi D_{14}/D_{12}$ is much less than 1 (in the case of seven-qubit blocks $c \sim 7^{-3}$). To map this problem into the bit-flip code we use the y basis for our spin qubits:

$$|0\rangle = |+y\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle), \tag{16}$$

$$|1\rangle = |-y\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle). \tag{17}$$

In this basis the Pauli matrices of Eq. (15) act as a bit-flip operator ($\sigma_z|0\rangle = |1\rangle, \sigma_z|1\rangle = |0\rangle$). Equation (15) contains three contributions. (1) Simultaneous bit flips of one spin in block 1 and another in block 2. This leads to error probability of $\sim c^2$ for each spin in each block [square of the amplitude; see Eq. (7)]. Note that error is correlated between blocks. (2) Simultaneous double bit flip in both blocks, with probability equal to the square of the single-bit-flip probability ($\sim c^4$). This process has the same order of magnitude of two independent single-bit-flip errors occurring at the same time. This type of error is only corrected after two concatenations of error correction are in place. (3) Three-bit-flip error in both blocks; the probability is the cube of a single bit flip, equivalent to three simultaneous independent bit flips.

We start by assuming blocks 1 and 2 store the state

$$|\psi(0)\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle. \tag{18}$$

For fault tolerant quantum computing, we add four additional ancilla qubits which encode the state as

$$\begin{aligned}|\psi(0)\rangle = a_1|000,000\rangle + a_2|000,111\rangle + a_3|111,000\rangle\\ + a_4|111,111\rangle.\end{aligned} \tag{19}$$

Time evolution under the spurious coupling Hamiltonian [Eq. (15)] yields

$$\begin{aligned}|\Psi(\tau)\rangle = \cos^3(c)|\Psi(0)\rangle - i\sin^3(c)|\phi\rangle + i\sin(c)\cos^2(c)\\ \times\{a_1[|100,100\rangle + i\tan(c)|011,011\rangle]\\ + a_2[|100,011\rangle + i\tan(c)|011,100\rangle]\\ + a_3[|011,100\rangle + i\tan(c)|100,011\rangle]\\ + a_4[|011,011\rangle + i\tan(c)|100,100\rangle]\} + \cdots,\end{aligned} \tag{20}$$

where the swapped state $|\phi\rangle$ is given by

$$\begin{aligned}|\phi\rangle = a_1|111,111\rangle + a_2|111,000\rangle + a_3|000,111\rangle\\ + a_4|000,000\rangle.\end{aligned} \tag{21}$$

Error correction proceeds with projection measurements over the syndromes 0, 1, 2, and 3 in each block $k = 1, 2$:

$$P_0^{(k)} = |000\rangle\langle 000| + |111\rangle\langle 111|, \tag{22}$$

$$P_1^{(k)} = |100\rangle\langle 100| + |011\rangle\langle 011|, \tag{23}$$

$$P_2^{(k)} = |010\rangle\langle 010| + |101\rangle\langle 101|, \tag{24}$$

$$P_3^{(k)} = |001\rangle\langle 001| + |110\rangle\langle 110|. \tag{25}$$

Depending on the outcome of the measurement we apply the corresponding correction operator $U_i^{(k)}$ (for example, $U_0^{(1)} = I$, $U_1^{(1)} = \sigma_{1z}$, $U_2^{(1)} = \sigma_{2z}$, $U_3^{(1)} = \sigma_{3z}$). The final corrected density matrix is an incoherent superposition of each possible error:

$$\rho_c = \sum_{i,j} U_i^{(1)}P_i^{(1)}U_j^{(2)}P_j^{(2)}|\Psi(\tau)\rangle \times \langle\Psi(\tau)|P_i^{(1)\dagger}U_i^{(1)\dagger}P_j^{(2)\dagger}U_j^{(2)\dagger}. \tag{26}$$

Because error is correlated between blocks the projection $P_i^{(1)}P_j^{(2)}$ onto state $|\Psi(\tau)\rangle$ is zero unless $i = j$. For example,

$$\prod_{k=1}^{2} U_0^{(k)}P_0^{(k)}|\Psi(\tau)\rangle = \cos^3(c)|\Psi(0)\rangle - i\sin^3(c)|\phi\rangle,$$

$$\prod_{k=1}^{2} U_l^{(k)} P_l^{(k)} |\Psi(\tau)\rangle = i \sin(c)\cos^2(c) \times [|\Psi(0)\rangle + i \tan(c)|\phi\rangle] \tag{27}$$

for $l=1, 2, 3$ [note that Eq. (20) omitted the syndrome subspaces 2,3]. Finally, it is straightforward to calculate the fidelity squared,

$$|F|^2 = \langle\Psi(0)|\rho_c||\Psi(0)\rangle = \cos^4(c)[1 + 2\sin^2(c)] + \sin^4(c)[1 + 2\cos^2(c)]|\langle\Psi(0)|\phi\rangle|^2. \tag{28}$$

Maximum error occurs when the second term of Eq. (28) is zero. This leads to

$$\max(E) = 1 - \min(|F|^2) \approx 3c^4 + O(c^6), \tag{29}$$

which is the square of the error without error correction. Hence neglecting small dipolar coupling between different error correction blocks is for all practical purposes equivalent to having a source of independent uncorrelated error identical to the one assumed in the quantum error correction literature.

V. DISCUSSION

We now consider the feasibility of our dipolar quantum computer proposal for III-V semiconductor donor impurities and quantum dots. Although these materials have a small effective mass (implying higher a^* and d_{opt}), some of the narrow gap semiconductors have quite large bulk g factors, enhancing dipolar coupling. A simple estimation is obtained from the relation $\tau_{\mathrm{CZ}} \sim (0.3/m^*)^3(2/g)^2 \times 100\ \mu$s. Using the parameters of Ref. [24] we get $\tau_{\mathrm{CZ}} \sim 0.1$ s for GaAs and $\tau_{\mathrm{CZ}} \sim 1$ ms for GaSb, InAs, and InSb donor impurities (quantum dots have dipolar gate times higher by approximately a factor of 10 due to larger Bohr radii). Hence our proposal is not feasible for GaAs, but might work for the narrow gap III-V materials as long as decoherence due to nuclear spectral diffusion is suppressed by nuclear polarization [5]. In this case spin flip followed by phonon emission will be the dominant decohering process. Adjusting the external magnetic field, coherence times of the order of a few seconds are achievable [24], suggesting the possibility of quality factors greater than 10^3 in a narrow gap donor dipolar quantum computer, which does not require exchange interaction control and can be constructed with current lithography techniques.

In conclusion we consider a quantum computer architecture based on dipolar-coupled donors in silicon. Although gate times are considerably longer than in exchange-based implementations (albeit the same time scales as in the solid state NMR proposals [12]), one does not need atomic precision donor implantation or electrical control of two-qubit couplings. In particular, "top-down" construction schemes based on ion implantation should benefit from our proposal, because these lack precision in donor positioning in addition to creating interstitial defects [25] (dipolar coupling is nearly insensitive to electronic structure). Our proposal for decoupling of short-range "always on" interactions together with error correction of the remaining long-range couplings applies equally well to any solid state implementation based on other types of long-range interactions (as long as the coupling is bilinear) [11,12], opening the way to implementations which do not have severe lithography requirements.

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