Pure-state informationally complete and "really" complete measurements

J. Finkelstein*
Department of Physics, San José State University, San José, California 95192, USA
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I construct a positive-operator-valued measure (POVM) which has $2d$ rank-1 elements and which is informationally complete for generic pure states in d dimensions, thus confirming a conjecture made by Flammia, Silberfarb, and Caves (e-print quant-ph/0404137). I show that if a rank-1 POVM is required to be informationally complete for *all* pure states in d dimensions, it must have at least $3d-2$ elements. I also show that, in a POVM which is informationally complete for all pure states in d dimensions, for any vector there must be at least $2d-1$ POVM elements which do not annihilate that vector.

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I. INTRODUCTION

Consider the following situation: you are given many copies of a quantum system; you know they are all in the same state, but you do not know which state that is, and you want to perform measurements in order to find out. If the statistics of the outcome of these measurements are sufficient to uniquely identify the state, the measurements are called "informationally complete" [1] (I-complete). In this paper I will present some results for a special case of this situation, in which you know that the system is in a pure state, but you do not know in which pure state.

A set of measurements can be considered equivalent to a single "generalized" measurement, which is described by a positive operator-valued measure (POVM) [2]. I will denote elements of a POVM as E_i; they are positive operators which satisfy $\Sigma_i E_i = I$, and if the state of the system is denoted as ρ, then the probability of the ith outcome is given by $\mathrm{Tr}(\rho E_i)$. For a pure state $\rho = |\psi\rangle\langle\psi|$, that probability is the expectation value $\langle\psi|E_i|\psi\rangle$.

Pure state I-complete POVM's have been discussed in a recent article by Flammia, Silberfarb, and Caves (FSC) [3]. I will adopt their definition, which is as follows.

Definition (*PSI-completeness*). A pure-state informationally complete (PSI-complete) POVM on a finite-dimensional quantum system is a POVM whose outcome probabilities are sufficient to determine any pure state (up to a global phase), except for a set of pure states that is dense only on a set of measure zero.

Let d denote the (finite) dimension of the Hilbert space for our quantum system. FSC show that any PSI-complete POVM must have at least $2d$ elements; this, together with their construction of an example (see also Ref. [4]) that does in fact have $2d$ elements, shows that the minimal number of elements of a PSI-complete POVM for a system with d dimensions is indeed $2d$. FSC also conjecture that, for a PSI-complete POVM whose elements are all of rank 1, the minimal number would be "close to or even equal to $2d$." In the next section of this paper I will confirm that conjecture by displaying a rank-1 PSI-complete POVM with exactly $2d$ elements.

The definition of PSI-completeness given above allows there to be pure states which cannot be identified uniquely by the expectation values of the POVM elements, but it does demand that any such states be confined to a set of measure zero. This means that, if a pure state were selected at random, then with probability 1 it *would* be uniquely identified. Of course, in practice we could never measure those expectation values with infinite precision, which means that we should not expect to identify the state with infinite precision. One might hope that, if we knew the expectation values to a good approximation, we would then be able, with probability 1, to identify the state to a good approximation, in the sense that (outside of a set of measure zero) any two pure states which were compatible with the same imprecisely known set of expectation values would necessarily be close together (in, say, the Hilbert-space norm). However, PSI-completeness does not guarantee this. Consider two distinct states $|\psi_a\rangle$ and $|\psi_b\rangle$ which were both compatible with the same precisely given set of expectation values; then imprecisely known expectation values would be compatible with states sufficiently close to $|\psi_a\rangle$ and also to states sufficiently close to $|\psi_b\rangle$. Since the set of states sufficiently close to $|\psi_a\rangle$ or to $|\psi_b\rangle$ has finite measure, there would be a (small but) finite probability that we would not be able to know if the state was close to $|\psi_a\rangle$ or to $|\psi_b\rangle$.

We could strengthen the definition of PSI-completeness by insisting that *all* pure states be uniquely identified by the expectation values of the POVM elements. I will say that such a POVM is PSI *really* complete:

Definition (*PSI really-completeness*). A pure-state informationally *really* complete (PSIR-complete) POVM on a finite-dimensional quantum system is a POVM whose outcome probabilities are sufficient to determine any pure state (up to a global phase).

In the third section of this paper I will prove two theorems about the number of elements necessary for a POVM to be PSIR-complete.

Theorem I. Let $\{E_i\}$ be the elements of a PSIR-complete POVM for a system of dimension d. Then for any nonzero vector $|\phi\rangle$ in the Hilbert space, there are at least $2d-1$ elements which do not annihilate $|\phi\rangle$ (that is, with $E_i|\phi\rangle \neq 0$).

Since a POVM which is I-complete for all states, whether pure or mixed, is *a fortiori* PSIR-complete, the conclusion of Theorem I holds for those POVM's also.

Theorem II. A PSIR-complete rank-1 POVM for a system

*Electronic address: JLFINKELSTEIN@lbl.gov

of dimension d must have at least $3d-2$ elements.

Together with the rank-1 PSI-complete POVM with $2d$ elements presented in the second section, Theorem II shows that allowing failure of state identification on a set of measure zero does decrease the minimum number of elements in a rank-1 POVM, for all $d>2$.

There has been some interest in discussing I-complete measurements utilizing "mutually unbiased bases" (MUB's) [5–7]. Two orthonormal bases $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ for a space of dimension d are mutually unbiased if, for all i and j, $|\langle e_i|f_j\rangle|^2=1/d$. It is known that, for some values of d, there exist $d+1$ MUB's and that in those cases the set of projectors on all of those basis elements is I-complete [5]. That would be a total of $(d+1)d=d^2+d$ projectors, but because not all of the expectation values of these projectors are independent, this set can be related to a rank-1 POVM with d^2 elements, which is the minimum number of elements of a I-complete POVM [8]. It is also known [9] that for any value of d there does exist at least 3 MUB's (and it is conjectured [10,11] that for some values of d no more than that), which can be related to a rank-1 POVM with $3d-2$ elements. This is the smallest number not ruled out by Theorem II, and so one might hope that this POVM could be PSIR-complete. However, this would certainly not be true if there were a fourth basis mutually unbiased with respect to the other three, since the expectation values for any two elements of this fourth basis would coincide, and hence those two elements could not be uniquely identified. Furthermore, while Theorem II establishes that any rank-1 PSIR-complete POVM has at least $3d-2$ elements, it does not assert that a greater number might not in fact be required. To my knowledge, the minimum number of elements of a rank-1 PSIR-complete POVM is at present unknown.

II. RANK-1 PSI-COMPLETE POVM WITH 2*d* ELEMENTS

In this section I will show that, for any dimension d, there exists a rank-1 PSI-complete POVM with $2d$ elements, thus confirming the conjecture made by FSC.

Let $\{|e_i\rangle | i=0,\ldots,d-1\}$ be an orthonormal basis for a Hilbert space of dimension d. Write a vector in this space as

$$|\psi\rangle=\sum_{i=0}^{d-1}c_i|e_i\rangle. \tag{1}$$

I will use the global phase freedom, and the indifference to sets of measure zero in the definition of PSI-completeness, to assert that c_0 is real and strictly positive. Now consider the following set of operators.

The d operators $|e_i\rangle\langle e_i|$, for $i=0,\ldots,d-1$.

The $(d-1)$ operators $(|e_0\rangle+i|e_i\rangle)(\langle e_0|-i\langle e_i|)$ for $i=1,\ldots,d-1$.

The single operator $(\sum_{i=0}^{d-1}|e_i\rangle)(\sum_{j=0}^{d-1}\langle e_j|)$.

This is a set of $2d$ operators, which I will show is PSI-complete; that is, for any vector $|\psi\rangle$ outside of a set of measure zero, knowledge of the expectation values of these operators would enable one to calculate the values of the coefficients c_i.

We have the expectation values

$$\langle\psi|(|e_i\rangle\langle e_i|)|\psi\rangle=|c_i|^2; \tag{2}$$

this gives us the value of $|c_i|$ for $i=1,\ldots,d-1$ and (since $c_0>0$) the value of c_0. For $i=1,\ldots,d-1$, we also have the expectation values

$$\langle\psi|(|e_0\rangle+i|e_i\rangle)(\langle e_0|-i\langle e_i|)|\psi\rangle=c_0^2+|c_i|^2+2c_0\,\mathrm{Im}\,c_i; \tag{3}$$

together with the values of c_0 and of $|c_i|$, this gives us the value of $\mathrm{Im}\,c_i$. And since $(\mathrm{Re}\,c_i)^2=|c_i|^2-(\mathrm{Im}\,c_i)^2$, at this point we know everything except for the signs of $(\mathrm{Re}\,c_i)$ for $i=1,\ldots,d-1$.

We still have one more expectation value—namely,

$$\langle\psi|\left(\sum_{i=0}^{d-1}|e_i\rangle\right)\left(\sum_{j=0}^{d-1}\langle e_j|\right)|\psi\rangle = \left|\sum_{i=0}^{d-1}c_i\right|^2=\left(\sum_{i=0}^{d-1}\mathrm{Re}c_i\right)^2+\left(\sum_{i=0}^{d-1}\mathrm{Im}c_i\right)^2, \tag{4}$$

and so we know the value of $|(\sum_{i=0}^{d-1}\mathrm{Re}c_i)|$. I will show that, in the generic case, this is enough to tell us the sign of each $\mathrm{Re}c_i$ and hence to uniquely identify $|\psi\rangle$. Suppose for example that we knew that $c_0=+5$, $|\mathrm{Re}c_1|=8$, $|\mathrm{Re}c_2|=4$, and that $|c_0+\mathrm{Re}c_1+\mathrm{Re}c_2|=7$; this would tell us that $\mathrm{Re}c_1=-8$ and that $\mathrm{Re}c_2=-4$. To see in general what ambiguities are allowed by all of the expectation values, suppose that a given set of expectation values was compatible with two distinct vectors $|\psi\rangle$ and $|\psi'\rangle$, with coordinates c_i and c_i', respectively. It would then be true that $c_0=c_0'$, that $|\mathrm{Re}c_i|=|\mathrm{Re}c_i'|$ for $i=1,\ldots,d-1$, and that $|(\sum_{i=0}^{d-1}\mathrm{Re}c_i)|=|(\sum_{i=0}^{d-1}\mathrm{Re}c_i')|$. I will divide the coordinates into two sets, according to whether $\mathrm{Re}c_i=+\mathrm{Re}c_i'$ or $\mathrm{Re}c_i=-\mathrm{Re}c_i'$. Define $E:=\{i|\mathrm{Re}c_i=\mathrm{Re}c_i'\}$; since $\mathrm{Re}c_0=\mathrm{Re}c_0'$, E is not empty. Also define $U:=\{i|\mathrm{Re}c_i\neq\mathrm{Re}c_i'\}$; note that $\mathrm{Re}c_i=-\mathrm{Re}c_i'$ for $i\in U$ and that, since $|\psi\rangle$ and $|\psi'\rangle$ were assumed to be distinct, U is not empty. We know that either $(\sum_{i=0}^{d-1}\mathrm{Re}c_i)=+(\sum_{i=0}^{d-1}\mathrm{Re}c_i')$ or $(\sum_{i=0}^{d-1}\mathrm{Re}c_i)=-(\sum_{i=0}^{d-1}\mathrm{Re}c_i')$; if $(\sum_{i=0}^{d-1}\mathrm{Re}c_i)=+(\sum_{i=0}^{d-1}\mathrm{Re}c_i')$, then

$$\sum_{i\in E}\mathrm{Re}c_i+\sum_{i\in U}\mathrm{Re}c_i=\sum_{i\in E}\mathrm{Re}c_i'+\sum_{i\in U}\mathrm{Re}c_i', \tag{5}$$

which gives

$$\sum_{i\in E}\mathrm{Re}c_i+\sum_{i\in U}\mathrm{Re}c_i=\sum_{i\in E}\mathrm{Re}c_i-\sum_{i\in U}\mathrm{Re}c_i \tag{6}$$

and so

$$\sum_{i\in U}\mathrm{Re}c_i=0. \tag{7}$$

On the other hand, if $(\sum_{i=0}^{d-1}\mathrm{Re}c_i)=-(\sum_{i=0}^{d-1}\mathrm{Re}c_i')$, then

$$\sum_{i\in E}\mathrm{Re}c_i+\sum_{i\in U}\mathrm{Re}c_i=-\left(\sum_{i\in E}\mathrm{Re}c_i'+\sum_{i\in U}\mathrm{Re}c_i'\right), \tag{8}$$

which gives

$$\sum_{i\in E}\mathrm{Re}c_i+\sum_{i\in U}\mathrm{Re}c_i=-\left(\sum_{i\in E}\mathrm{Re}c_i-\sum_{i\in U}\mathrm{Re}c_i\right) \tag{9}$$

and so

$$\sum_{i\in E} \mathrm{Re} c_i = 0. \tag{10}$$

So either Eq. (7) or (10) must be correct. For fixed sets E and U, states satisfying either of these equations are confined to closed sets of measure zero. And since there are only a finite number of possibilities for E and U, all states outside of a closed set of measure zero can be unambiguously identified, and so our operators are PSI-complete.

Given these rank-1 PSI-complete operators, we can form a rank-1 PSI-complete POVM with $2d$ elements with the same procedure used by FSC: denoting our operators by P_i, with $i=1,\ldots,2d$, define $G=\Sigma_{i=1}^{2d}P_i$; this is nonsingular since the P_i are PSI-complete, so we can define $E_i := G^{-1/2}P_iG^{-1/2}$. These operators are then the $2d$ elements of a rank-1 PSI-complete POVM.

III. PROOF OF THE THEOREMS

In this section I will prove the two theorems stated in the Introduction.

Theorem I. Let $\{E_i\}$ be the elements of a PSIR-complete POVM for a system of dimension d. Then for any nonzero vector $|\phi\rangle$ in the Hilbert space, there are at least $2d-1$ elements which do not annihilate $|\phi\rangle$ (that is, with $E_i|\phi\rangle \neq 0$).

Proof. Let $\{E_i\}$ be the elements of a POVM for a system of dimension d. For any vector $|\phi\rangle$, let K_ϕ be the number of POVM elements with $E_i|\phi\rangle \neq 0$. I will show that if there is a nonzero vector with $K_\phi < 2d-1$, then this POVM is not PSIR-complete.

Let $|\phi\rangle$ be a nonzero vector with $K_\phi < 2d-1$, and let $|\chi\rangle$ denote a nonzero vector orthogonal to $|\phi\rangle$. I will show that $|\chi\rangle$ can be chosen so that

$$\mathrm{Re}\langle\phi|E_i|\chi\rangle = 0 \tag{11}$$

for all elements E_i. For those E_i which annihilate $|\phi\rangle$, Eq. (11) is valid for any $|\chi\rangle$, so Eq. (11) represents K_ϕ conditions which $|\chi\rangle$ must satisfy. These conditions are not all independent; since $\Sigma_i E_i = I$ implies that

$$\sum_i \mathrm{Re}\langle\phi|E_i|\chi\rangle = \mathrm{Re}\langle\phi|\chi\rangle = 0, \tag{12}$$

there are not more than $K_\phi - 1$ independent conditions. Now let $\{|e_j\rangle \,|\, j=1,\ldots,d-1\}$ be an orthonormal basis for the subspace orthogonal to $|\phi\rangle$; then, for each value of i, Eq. (11) can be written

$$\sum_{j=1}^{d-1} [\mathrm{Re}\langle\phi|E_i|e_j\rangle \mathrm{Re}\langle e_j|\chi\rangle - \mathrm{Im}\langle\phi|E_i|e_j\rangle \mathrm{Im}\langle e_j|\chi\rangle] = 0. \tag{13}$$

For each value of i, this is a real, linear homogeneous equation for the $2d-2$ real parameters $\mathrm{Re}\langle e_j|\chi\rangle$ and $\mathrm{Im}\langle e_j|\chi\rangle$. Since no more than $K_\phi - 1$ of these equations are independent and since $K_\phi - 1 < 2d-2$, there must be a nontrivial solution.

So we can choose $|\chi\rangle$ to satisfy Eq. (11) and then define

$$|\psi_\pm\rangle = |\phi\rangle \pm |\chi\rangle. \tag{14}$$

The expectation values are

$$\langle\psi_\pm|E_i|\psi_\pm\rangle = \langle\phi|E_i|\phi\rangle + \langle\chi|E_i|\chi\rangle \pm 2\,\mathrm{Re}\langle\phi|E_i|\chi\rangle. \tag{15}$$

Equations (11) and (15) together imply that

$$\langle\psi_+|E_i|\psi_+\rangle = \langle\psi_-|E_i|\psi_-\rangle \tag{16}$$

for all i. Finally, let N be the (common) norm of $|\psi_+\rangle$ and of $|\psi_-\rangle$; then, the normalized vectors $(N^{-1})|\psi_\pm\rangle$ represent two distinct states whose expectation values for all of the POVM elements agree, and so the POVM is not PSIR-complete.

Theorem II. A rank-1 PSIR-complete POVM for a system of dimension d must have at least $3d-2$ elements.

Proof. Let $\{E_i\}$ be the elements of a rank-1 PSIR-complete POVM for a system of dimension d. Define $F := \Sigma_{i=1}^{d-1}E_i$; since each E_i has rank 1, the rank of F is no greater than $d-1$. This means that there must be (at least) a one-dimensional subspace annihilated by F, and hence, since the POVM elements are positive, by each E_i for $i=1,\ldots,d-1$. Let $|\phi\rangle$ be any nonzero vector in that subspace. $|\phi\rangle$ is annihilated by each E_i for $i=1,\ldots,d-1$; in addition, according to Theorem I there must be at least $2d-1$ elements which do *not* annihilate $|\phi\rangle$. Therefore the POVM must contain at least $(d-1)+(2d-1)=3d-2$ elements.

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