Weakly entangled states are dense and robust

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Motivated by the mathematical definition of entanglement we undertake a rigorous analysis of the separability and nondistillability properties in the neighborhood of those three-qubit mixed states which are entangled and completely biseparable. Our results are not only restricted to this class of quantum states, since they rest upon very general properties of mixed states and unextendible product bases for any possible number of parties. Robustness against noise of the relevant properties of these states implies the significance of their possible experimental realization, therefore being of physical—and not exclusively mathematical—interest.

Quantum mechanics has focused much of the attention of the physics community during the last century, both for its extraordinary predictive power and its apparent paradoxes. It has been a turning point in our conception of the fundamental laws of nature, and it still leads today to important discussion about its foundations. A major step forward was realizing that the laws of quantum mechanics can be of practical use. Exploiting quantum correlations—or *entanglement*—[1,2] as a fundamental resource (such as energy) has been proven to be an outstanding success with applications such as quantum teleportation, quantum cryptography, or quantum computation [3]. Characterization of entanglement is therefore one of the most relevant problems in quantum information science.

It was initially Werner [4] who introduced the current mathematical definition of a "mixed entangled state": a mixed state ρ of n parties is entangled if and only if it cannot be decomposed as $\rho = \Sigma_{i=1}^{r} p_i \rho_i^1 \otimes \cdots \otimes \rho_i^n$, $\{p_i\}_{i=1}^{r}$ being a certain probability distribution. According to this definition, any quantum state ρ that is not entangled can always be created by means of local operations on each separate party together with classical communication between them (LOCC). A different (but related) concept, the *distillability* of quantum states, was introduced by Bennett *et al.* [5]: given M copies of a bipartite quantum state ρ, we say that it is distillable if and only if we can obtain N copies of a maximally entangled pure state of the two parties by means of LOCC (the generalization to the n-partite case is straightforward, by separating the n parties into two different sets and applying again the same definition). This naturally led to the concept of *distillable entanglement* of state ρ, which corresponds to the ratio N/M in the limit of infinite number of copies.

The existence of entangled quantum states that are nondistillable, the so-called *bound entangled* states, was soon proved by Horodecki *et al.* [6]. The discovered states had positive partial transposition (PPT) with respect to one of the two subsystems considered for the distillation protocol. Despite that recent results have found particular practical applications for some of these states [7–9], it still seems that most of them (e.g., those mixed states of three qubits which, despite being entangled, are separable with respect to all the possible bipartitions of the system [10,11]) are actually so

weakly entangled that their quantum correlations are not known to be of any practical use for quantum information tasks. This situation leads us to think that, perhaps, our definition of entanglement is excessively mathematical, and that a more physical definition of what a quantum-correlated state is, might be of interest. The importance of this problem hinges on whether it is actually possible to realize bound entangled states, which posses this "useless" character, in the laboratory. For this to be achievable we must demand as a necessary condition that the action of small and unavoidable decoherence effects leave the essential properties of the quantum state unchanged. If, for instance, an *a priori* interesting property were only to hold for pure states, it would actually not be of physical interest, as pure states are a mathematical idealization of physically realizable states. But this experimental condition must be considered for mixed states in general.

In this paper we prove that mixed quantum states that are entangled and are biseparable with respect to any set of bipartitions of the system form a set of nonzero measure in Hilbert space. In particular this implies that those three-qubit density matrices that are bound entangled and completely biseparable [10,11] have real physical significance. Our results have been obtained by comparing the properties of the neighborhood in Hilbert space around these states with those of the original state. Motivated by this class of three-qubit states we also undertake a more general analysis of the distillability properties of the neighborhood of those bound entangled states associated with an unextendible product basis.

Our study begins by considering the following theorem.

Theorem 1. n-qubit density matrices ρ which are separable with respect to some particular bipartition of the system form a set of nonzero measure in Hilbert space (it is a *dense* set).

Proof. We consider a density matrix ρ of n qubits, which is separable with respect to some particular bipartition. Let us introduce independent perturbations in the parameter space of n-qubit density matrices in such a way that the perturbed operator is still separable with respect to that bipartition. For this purpose, we consider the following set of $2^n \times 2^n$ independent and completely separable projectors:

$$E(j_1,\ldots,j_n) \equiv |j_1,\ldots,j_n\rangle\langle j_1,\ldots,j_n|, \quad j_\alpha = \{0,1,\phi_1,\phi_2\} \tag{1}$$

for $\alpha = 1,\ldots,n$, $|\phi_1\rangle = (1/\sqrt{2})|0+1\rangle$, and $|\phi_2\rangle = (1/\sqrt{2})|0+i1\rangle$. We can now write any perturbation by using this set of projectors as $\rho(\epsilon_\mu) = (1/C)[\rho + \Sigma_\mu \epsilon_\mu E(\mu)]$, ϵ_μ being a set of real parameters, $\mu \equiv (j_1,\ldots,j_n)$ and C the normalization constant. For the perturbed density matrix to be physical the set of parameters ϵ_μ must be such that the condition of positive semidefiniteness holds for the perturbed operator. Because the projectors $E(\mu)$, which form a basis of the $2^n \times 2^n$ density matrices, correspond to separable pure states, if we restrict ourselves to the region $\epsilon_\mu \geq 0 \; \forall \mu$ (which physically corresponds to a local noise), these states share at least the same separability properties as those of the unperturbed state ρ. It is then proven that these states form a set of nonzero measure in Hilbert space, since the number of independent perturbation parameters is maximal, and the separability property is robust against local noise. ∎

Our next claim is that given an entangled state ρ, its entanglement is preserved in an infinitesimal neighborhood. This is proven in our second theorem.

Theorem 2. If the n-qubit state ρ is entangled and the real parameters ϵ_μ are infinitesimal, then the state $\rho(\epsilon_\mu) = (1/C) \times [\rho + \Sigma_\mu \epsilon_\mu E(\mu)]$ is entangled (in particular, entanglement is a *robust* property).

Proof. The proof is based on witness operators. Let us briefly recall their definition: given two convex subsets S_1 and S_2 such that S_1 is included in S_2, a Hermitian operator W is a witness operator if and only if (i) $\forall \sigma \in S_1, \mathrm{tr}(W\sigma) \geq 0$, (ii) there is at least one $\rho \in S_2$ such that $\mathrm{tr}(W\rho) < 0$, and (iii) $\mathrm{tr}(W) = 1$. In the quantum case, S_1 is the set of separable states, S_2 is the set of all quantum states, and W is an observable that has an expectation value greater than or equal to 0 for all the separable states and less than 0 for some entangled state. It is a well-known fact [12–14] that a quantum state ρ is entangled if and only if there exists a witness operator W that "detects" ρ, i.e., $\mathrm{tr}(W\rho) < 0$ and $\mathrm{tr}(W\sigma) \geq 0 \; \forall \sigma$ separable (this is a consequence of the Hahn-Banach's theorem for convex sets). Given the perturbed state $\rho(\epsilon_\mu)$, it will be entangled if and only if there exists a witness operator W such that $\mathrm{tr}[W\rho(\epsilon_\mu)] < 0$. Assuming that ρ is entangled, we choose to work with the particular witness operator W_ρ that "detects" it [i.e., $\mathrm{tr}(W_\rho \rho) < 0$]. For $\rho(\epsilon_\mu)$ to be detected by W_ρ we impose that

$$\mathrm{tr}[W_\rho \rho(\epsilon_\mu)] = \frac{1}{C}\left(\mathrm{tr}(W_\rho \rho) + \sum_\mu \epsilon_\mu \mathrm{tr}[W_\rho E(\mu)]\right) < 0. \tag{2}$$

Because $\mathrm{tr}(W_\rho \rho) < 0$ and $\mathrm{tr}[W_\rho E(\mu)] \geq 0$ [since $E(\mu)$ are projectors corresponding to completely separable pure states], the above condition reads $\Sigma_\mu \epsilon_\mu \mathrm{tr}[W_\rho E(\mu)] < |\mathrm{tr}(W_\rho \rho)|$, which can always be achieved for values of the parameters ϵ_μ close enough to zero. The perturbed state is then detected by some witness operator, therefore it is entangled. ∎

From the preceding two theorems we infer that there exists a partial neighborhood of nonzero measure of those n qubit states that are entangled but separable with respect to some bipartition that shares also the same properties, so that

these states are robust. In particular, this result holds for those states of three qubits studied in [15], and specifically to the (bound) entangled three-qubit states that are entangled yet completely biseparable, which were presented in [10] and further generalized in [16]. Let us recall at this point their definition in terms of an unextendible product basis: an unextendible product basis (UPB) [10,11] for a multipartite quantum system is an incomplete orthogonal product basis whose complementary subspace contains no product state. It has the remarkable property that if $\{|\psi_i\rangle\}_{i=1}^m$ are the (product) vectors of the UPB, then the maximally mixed state in the subspace orthogonal to the UPB, $\rho = [1/(D-m)](I - \Sigma_{i=1}^m |\psi_i\rangle\langle\psi_i|)$ (D being the dimensionality of the Hilbert space), is bound entangled (UPB states). For three-qubit systems, it was proven in [16] that the most general UPB is given by the set of vectors

$$|\psi_1\rangle = |0\rangle|0\rangle|0\rangle,$$

$$|\psi_2\rangle = |1\rangle|B\rangle|C\rangle,$$

$$|\psi_3\rangle = |A\rangle|1\rangle|\bar{C}\rangle, \tag{3}$$

$$|\psi_4\rangle = |\bar{A}\rangle|\bar{B}\rangle|1\rangle,$$

with $\langle A|\bar{A}\rangle = 0$ (similar for $|B\rangle$ and $|C\rangle$), and $|A\rangle, |B\rangle$, and $|C\rangle$ depending on only one real parameter each one (which can always be achieved by a local change of basis). In [16] it was also noted that any UPB state of three qubits is bound entangled yet completely biseparable. Our theorems imply that, apart from these UPB states, there are also mixed density matrices of this kind which cannot be associated with any UPB of three qubits, since the states $\rho(\epsilon_\mu)$ obtained by perturbing one of the UPB states are not in general related to any UPB. Neither can these states always be written as a convex combination of states associated to some UPB, since a convex combination of two different UPB states for three qubits can easily be seen to have at least rank 6, while it is possible to achieve states of rank 5 simply by mixing a UPB state with one of the pure product states of the corresponding UPB.

We now wish to analyze the distillability properties of the neighborhood of this kind of three-qubit UPB states in a more detailed way. Theorem 1 already implies a certain kind of (obvious) robustness for the nondistillability of these states, restricted to the $\epsilon_\mu \geq 0$ region $\forall_\mu$. We shall see that, indeed, this robustness is stronger due to some peculiarities of UPB states, as it can also be extended to some cases with small negative values of the parameters ϵ_μ. Our analysis focuses only on general properties of UPBs, no matter what the particular system is, therefore its range of application is not restricted only to the three-qubit case. First we present a previous lemma which we will use to prove our third theorem.

Lemma. If ρ is a UPB state in $\mathcal{H}_a \otimes \mathcal{H}_b$, then the kernel (null eigenspace) of ρ^{T_a} is spanned by product vectors.

Proof. Let $\{|a_i\rangle|b_i\rangle\}_{i=1}^m$ be a UPB with m product vectors in a Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ of dimension D. The UPB state is

$\rho=[1/(D-m)](I-\Sigma_{i=1}^{m}|a_i\rangle\langle a_i|\otimes|b_i\rangle\langle b_i|)$; its kernel is spanned by the set of vectors of the UPB. Taking the partial transposition with respect to party a we get $\rho^{T_a}=[1/(D-m)][I-\Sigma_{i=1}^{m}(|a_i\rangle\langle a_i|)^{T_a}\otimes|b_i\rangle\langle b_i|]$. Since $|a_i\rangle\langle a_i|$ is a Hermitian operator, it holds that $(|a_i\rangle\langle a_i|)^{T_a}=(|a_i\rangle\langle a_i|)^{*}=|a_i'\rangle\langle a_i'|$, being $|a_i'\rangle=(|a_i\rangle)^{*}$ the complex-conjugated vector of $|a_i\rangle$. The kernel of ρ^{T_a} is therefore spanned by the product vectors $\{|a_i'\rangle|b_i\rangle\}_{i=1}^{m}$. ∎

At this point we are in conditions of presenting our third theorem.

Theorem 3. Any UPB state ρ perturbed by a small enough amount of noise ρ_1, such that $\rho_1^{T_a}>0$ in the kernel of ρ^{T_a}, remains nondistillable (nondistillability is a conditionally *robust* property).

Before we prove Theorem 3, let us recall that any physical noise can always be represented in terms of a mixture (with positive weights) in the space of density matrices, that is, as a probabilistic combination of the original (unperturbed) mixed state and a noise-induced density matrix.

Proof. The proof is based on degenerate perturbation theory. Let us consider a UPB $\{|a_i\rangle|b_i\rangle\}_{i=1}^{m}$ in $\mathcal{H}_a\otimes\mathcal{H}_b$ and call ρ the corresponding UPB state. We wish to note here that any UPB can always be written in this way, by joining the different parties into two different sets a and b. Consider also ρ_1 as any other possible quantum state in the same Hilbert space. We write a small perturbation of ρ with ρ_1 as $\rho(\epsilon)=[1/(1+\epsilon)](\rho+\epsilon\rho_1)$, $\epsilon>0$ being an infinitesimal noise parameter. Taking the partial transposition with respect to one of the parties we obtain $\rho(\epsilon)^{T_a}=[1/(1+\epsilon)](\rho^{T_a}+\epsilon\rho_1^{T_a})$. The null eigenvectors of ρ can be chosen to be the states of the UPB $\{|a_i\rangle|b_i\rangle\}_{i=1}^{m}$, while the states from the $(D-m)$-dimensional subspace orthogonal to this set have eigenvalue $1/(D-m)$. According to the previous lemma, the kernel of ρ^{T_a} is then spanned by the set of vectors $\{|a_i'\rangle|b_i\rangle\}_{i=1}^{m}$, while the vectors from its orthogonal subspace have eigenvalue $1/(D-m)$. Using degenerate perturbation theory, for ϵ small enough the lowest eigenvalues of $\rho(\epsilon)^{T_a}$ are given by $\epsilon\lambda_r+O(\epsilon^2)$, $\{\lambda_r\}_{r=1}^{m}$ being the eigenvalues of the $m\times m$ matrix A defined by $A_{ij}\equiv\langle a_i'|\langle b_i|\rho_1^{T_a}|b_j\rangle|a_j'\rangle$. If $\lambda_r<0$ for some r, this leads to a negative eigenvalue in the spectrum of $\rho(\epsilon)^{T_a}$, therefore turning $\rho(\epsilon)$ into an NPT state. The case in which $\lambda_r=0$ for some r needs a second-order analysis

in perturbation theory, which easily leads to negative eigenvalues in the spectrum of $\rho(\epsilon)^{T_a}$ as well, and therefore to similar conclusions. Consequently, in order for the perturbed operator to remain PPT, we must demand the condition $A>0$, which means that $\rho_1^{T_a}>0$ in the kernel of ρ^{T_a}. ∎

Note that, in terms of the projectors $E(\mu)$ from Theorem 1, any ρ_1 can always be decomposed as $\rho_1=\Sigma_{\mu}c_{\mu}E(\mu)$, being c_{μ} certain real parameters. Defining $\epsilon_{\mu}\equiv\epsilon c_{\mu}$, we observe that the hypothesis of Theorem 3 does not restrict ϵ_{μ} to be non-negative. It is also worth pointing out the case in which the UPB is composed of real vectors only. In this situation, the kernel of the partially transposed UPB state ρ^{T_a} coincides with the space spanned by the vectors of the UPB, and therefore the condition imposed on the noise in Theorem 3 gets simplified. This simplification applies to a very large variety of UPBs, such as all the three-qubit and many two-qutrit examples [10,11,16].

When particularizing to the three-qubit case, and bringing together the results from the previous three theorems, we conclude that mixed three-qubit states that are entangled yet completely biseparable are robust against local noise and beyond, in the sense that their entanglement, their complete biseparability, and their nondistillability are not modified by small local effects and some nonlocal noisy effects. Relevant properties of these states are then of physical significance, and not just a matter of mathematical interest. We wish to note the dependence of these states on the existence of UPBs for three qubits, which are associated with Hilbert spaces of dimension 4 with no product vectors in them, and we wonder, as a possible generalization of this concept, whether there exist subspaces of dimension 5 of three-qubit Hilbert spaces such that there are less than five independent product vectors in them (we have not succeeded in finding them). In such a case, the properties of these subspaces would probably be of interest, as are the properties of UPBs, in order to bring further insight and knowledge about entanglement and distillability properties for three-partite systems.

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