Reply to "Comment on 'Detuning effects in the one-photon mazer' "

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We refute in this Reply the criticisms made by Abdel-Aty. We show that none of them are founded and we demonstrate very explicitly what is wrong with the arguments developed by this author.

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In a recent paper [1] we presented the quantum theory of the mazer in the nonresonant case, pointing out various effects compared to the resonant case. In this system, the interaction of a two-level atom with a single mode of a cavity is investigated. The atom is supposed to move unidirectionally on the way to the cavity and the interaction occurs when the atom passes through it (see Fig. 1). The effects of this interaction are then studied after the atom has left the cavity region. Compared to the conventional micromaser, the atomic motion is described quantum mechanically (see our paper [1] and references therein for the detailed description of the system and the model considered). In the nonresonant case, a detuning between the cavity mode and the atomic transition frequencies is present. In our paper, we showed that this detuning adds a potential step effect not present at resonance, resulting in a well-defined acceleration or deceleration (according to the sign of the detuning) of the atoms that emit a photon inside the cavity if they are initially excited. We also demonstrated that this photon emission inside the cavity may be completely blocked by use of a positive detuning. Finally, we characterized the properties of the induced emission probability in various regimes and demonstrated notably that the well-known Rabi formula is well recovered by the general quantum theory in the hot atom regime (where the quantization of the center-of-mass motion is not necessary). Various criticisms about our paper have been raised by Abdel-Aty and summarized in a Comment to which this Reply is intended. We give here an answer to all of these criticisms and demonstrate that none of them are founded.

First, and contrary to what is claimed in Abdel-Aty's Comment, it is obvious that all the physical effects just described and the way they have been derived in Ref. [1] do not appear in any form in any previous paper dedicated to this subject by this author, namely, Refs. [2] and [3]. This will be further demonstrated in the rest of this Reply.

Secondly, it is stated by Abdel-Aty that the evaluation of the coupled equations (5a) and (5b) of our paper is not satisfactory. In particular, it is declared that, in the first line of Eq. (5a) [(5b)], $\cos^2\theta$ ($\sin^2\theta$) should be replaced by $\cos 2\theta$ ($\sin 2\theta$). We wonder about this criticism as it is in no way argued and is actually wrong. To be very explicit, we give the details of the calculations which lead to Eqs. (5a) and (5b) and we demonstrate that they are perfectly correct.

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The Hamiltonian used to describe the mazer is given by [4]

$$H = \hbar\omega_0\sigma^\dagger\sigma + \hbar\omega a^\dagger a + \frac{p^2}{2m} + \hbar g u(z)(a^\dagger\sigma + a\sigma^\dagger) \tag{1}$$

with usual notations. In particular, ω and ω_0 are the frequencies of the cavity mode and the atomic transition, respectively. The atomic motion is defined along the z direction and $u(z)$ describes the spatial variation of the atom-cavity coupling (the so-called cavity mode function). The two atomic internal states are denoted by $|a\rangle$ (excited state) and $|b\rangle$ (ground state). The global Hilbert space of the system is given by

$$\mathcal{H} = \mathcal{E}_z \otimes \mathcal{E}_A \otimes \mathcal{E}_R \tag{2}$$

with $\mathcal{E}_z$ the space of the wave functions describing the one-dimensional atomic center-of-mass motion, $\mathcal{E}_A$ the space describing the atomic internal degree of freedom, and $\mathcal{E}_R$ the space of the cavity single mode radiation.

We introduce in the space $\mathcal{E}_A \otimes \mathcal{E}_R$ the orthonormal basis

$$|\Gamma_{-1}\rangle = |b,0\rangle,$$

$$|\Gamma_n^+(\theta)\rangle = \cos\theta|a,n\rangle + \sin\theta|b,n+1\rangle,$$

$$|\Gamma_n^-(\theta)\rangle = -\sin\theta|a,n\rangle + \cos\theta|b,n+1\rangle, \tag{3}$$

with θ an arbitrary parameter and $n \geqslant 0$. Combined with the z representation $\{|z\rangle\}$ in $\mathcal{E}_z$, the set of vectors

$$\{|z,\Gamma_{-1}\rangle, |z,\Gamma_n^\pm(\theta)\rangle\} \tag{4}$$

defines an orthonormal basis over the whole Hilbert space $\mathcal{H}$.

Projecting the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{5}$$

onto the basis (4) yields for every $n \geqslant 0$

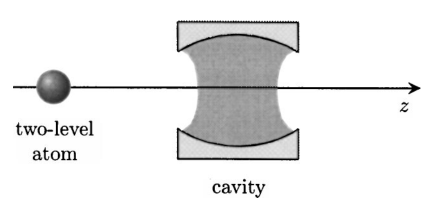

FIG. 1. General scheme of the mazer.

$$i\hbar\frac{\partial}{\partial t}\psi^{\pm}_{n,\theta}(z,t)=\langle z,\Gamma^{\pm}_n(\theta)|H|\psi(t)\rangle \tag{6}$$

with

$$\psi^{\pm}_{n,\theta}(z,t)\equiv\langle z,\Gamma^{\pm}_n(\theta)|\psi(t)\rangle. \tag{7}$$

Using the completeness relation

$$1=\int dz\left[\sum_{n=0}^{\infty}(|z,\Gamma^{+}_n(\theta)\rangle\langle z,\Gamma^{+}_n(\theta)|+|z,\Gamma^{-}_n(\theta)\rangle\langle z,\Gamma^{-}_n(\theta)|)+|z,\Gamma_{-1}\rangle\langle z,\Gamma_{-1}|\right], \tag{8}$$

the right-hand side of Eq. (6) reads

$$\langle z,\Gamma^{\pm}_n(\theta)|H|\psi(t)\rangle=\int dz'\left[\sum_{n'=0}^{\infty}[\langle z,\Gamma^{\pm}_n(\theta)|H|z',\Gamma^{+}_{n'}(\theta)\rangle\psi^{+}_{n',\theta}(z',t)+\langle z,\Gamma^{\pm}_n(\theta)|H|z',\Gamma^{-}_{n'}(\theta)\rangle\psi^{-}_{n',\theta}(z',t)]+\langle z,\Gamma^{\pm}_n(\theta)|H|z',\Gamma_{-1}\rangle\langle z',\Gamma_{-1}|\psi(t)\rangle\right]. \tag{9}$$

Straightforward calculations yield

$$\langle\Gamma^{+}_n(\theta)|\sigma^{\dagger}\sigma|\Gamma^{+}_{n'}(\theta)\rangle=\cos^2\theta\delta_{nn'}, \tag{10}$$

$$\langle\Gamma^{+}_n(\theta)|\sigma^{\dagger}\sigma|\Gamma^{-}_{n'}(\theta)\rangle=-\frac{1}{2}\sin 2\theta\delta_{nn'}, \tag{11}$$

$$\langle\Gamma^{+}_n(\theta)|a^{\dagger}a|\Gamma^{+}_{n'}(\theta)\rangle=(n+\sin^2\theta)\delta_{nn'}, \tag{12}$$

$$\langle\Gamma^{+}_n(\theta)|a^{\dagger}a|\Gamma^{-}_{n'}(\theta)\rangle=\frac{1}{2}\sin 2\theta\delta_{nn'}, \tag{13}$$

$$\langle\Gamma^{+}_n(\theta)|a^{\dagger}\sigma+a\sigma^{\dagger}|\Gamma^{+}_{n'}(\theta)\rangle=\sqrt{n+1}\sin 2\theta\ \delta_{nn'}, \tag{14}$$

$$\langle\Gamma^{+}_n(\theta)|a^{\dagger}\sigma+a\sigma^{\dagger}|\Gamma^{-}_{n'}(\theta)\rangle=\sqrt{n+1}\cos 2\theta\ \delta_{nn'} \tag{15}$$

and

$$\langle z,\Gamma^{+}_n(\theta)|H|z',\Gamma_{-1}\rangle=0. \tag{16}$$

We thus have from Eq. (9)

$$\langle z,\Gamma^{+}_n(\theta)|H|\psi(t)\rangle=\int dz'\left[\sum_{n'=0}^{\infty}\delta_{nn'}\delta(z-z')\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z'^2}+\hbar\omega_0\cos^2\theta+\hbar\omega(n+\sin^2\theta)+\hbar g\sqrt{n+1}u(z')\sin 2\theta\right)\psi^{+}_{n',\theta}(z',t)+\delta_{nn'}\delta(z-z')\left(-\frac{\hbar\omega_0}{2}\sin 2\theta+\frac{\hbar\omega}{2}\sin 2\theta+\hbar g\sqrt{n+1}u(z')\cos 2\theta\right)\psi^{-}_{n',\theta}(z',t)\right]. \tag{17}$$

Defining the detuning $\delta=\omega-\omega_0$ and inserting Eq. (17) into Eq. (6) yields the Schrödinger equation for the $\psi^{+}_{n,\theta}(z,t)$ component, that is,

$$i\hbar\frac{\partial}{\partial t}\psi^{+}_{n,\theta}(z,t)=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+(n+1)\hbar\omega-\cos^2\theta\hbar\delta+\hbar g u(z)\sqrt{n+1}\sin 2\theta\right]\psi^{+}_{n,\theta}(z,t)+\left[\hbar g u(z)\sqrt{n+1}\cos 2\theta+\frac{1}{2}\sin 2\theta\hbar\delta\right]\psi^{-}_{n,\theta}(z,t), \tag{18}$$

which is exactly Eq. (5a) of our paper [1] (where, of course, $\frac{1}{2}\theta\hbar\delta\sin 2$ must be read $\frac{1}{2}\hbar\delta\sin 2\theta$; this glaring typographic error was introduced after proof corrections and is beyond the scope of this Comment). Therefore and contrary to what is claimed by Abdel-Aty, the $\cos^2\theta$ term in the first line of our Eq. (5a) in Ref. [1] *must not* be replaced by $\cos 2\theta$, confirming that this equation is perfectly correct.

We demonstrate in a similar manner that Eq. (5b) of our paper [which yields the Schrödinger equation for the $\psi^{-}_{n,\theta}(z,t)$ component] is also error free and that again the $\sin^2\theta$ term that appears in the first line of this equation *must not* be replaced by $\sin 2\theta$. We therefore refute the criticism made in Abdel-Aty's Comment that these equations would not be satisfactory. They are, and there is absolutely no need to replace them with those proposed by this author, namely, Eq. (9) in his Comment. We demonstrate even at the end of this Reply that this replacement cannot be done as it leads to inconsistencies and wrong results in some cases, especially in the mesa mode case that is presently investigated by us and Abdel-Aty.

A third criticism states that we would have overlooked the formulas for $\cos 2\theta_n$ and $\sin 2\theta_n$ where θ_n defines the dressed-state basis and is given by

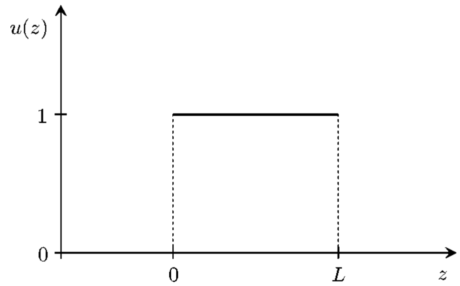

FIG. 2. The mesa mode function.

$$\cot 2\theta_n = -\frac{\delta}{\Omega_n}, \tag{19}$$

with $\Omega_n = 2g\sqrt{n+1}$. According to this criticism, we would have avoided great simplifications of our equations. Nothing could be more wrong. We did not overlook anything. It is well explained in our paper that in the case of the mesa mode function $(u(z)=1$ for $0<z<L$, 0 elsewhere [5], where L is the length of the cavity that is supposed to be located in the $[0,L]$ region$)$, the coupled equations (5a) and (5b) reduce to a much simpler decoupled form in the local dressed state basis, namely, Eqs. (7) and (17) of our paper (for outside and inside the cavity, respectively). Equation (20) of our paper does not express anymore than Eq. (10) of Abdel-Aty's Comment. Indeed, according to our notations, Eq. (20) of Ref. [1] yields explicitly

$$\cos 2\theta_n = 1 - 2\sin^2\theta_n = 1 - \frac{\Lambda_n + \delta}{\Lambda_n} = -\frac{\delta}{\Lambda_n}, \tag{20}$$

$$\sin 2\theta_n = 2\sin\theta_n\cos\theta_n = \frac{\sqrt{\Lambda_n^2 - \delta^2}}{\Lambda_n} = \frac{\Omega_n}{\Lambda_n}, \tag{21}$$

with $\Lambda_n = \sqrt{\delta^2 + \Omega_n^2}$.

Finally, we prove now that Eq. (9) in Abdel-Aty's Comment does not express another point of view equivalent to our equations and that, contrary to the claim of this author, it cannot replace Eqs. (5a) and (5b) of our paper. Their equations read [taking into account that the first partial derivatives on the left-hand side of Eqs. (8) and (9) in Abdel-Aty's Comment must evidently be read $\partial/\partial t$ instead of $\partial/\partial z$]

$$i\frac{\partial C_n^+}{\partial t} = \left[-\frac{1}{2M}\frac{\partial^2}{\partial z^2} + V_n^+ - \left(\frac{d\theta_n}{dz}\right)^2\right]C_n^+ - \left[2\frac{\partial C_n^-}{\partial z}\left(\frac{d\theta_n}{dz}\right) + C_n^-\left(\frac{d\theta_n}{dz}\right)^2\right], \tag{22}$$

$$i\frac{\partial C_n^-}{\partial t} = -\left[-\frac{1}{2M}\frac{\partial^2}{\partial z^2} + V_n^- - \left(\frac{d\theta_n}{dz}\right)^2\right]C_n^- + \left[2\frac{\partial C_n^+}{\partial z}\left(\frac{d\theta_n}{dz}\right) + C_n^+\left(\frac{d\theta_n}{dz}\right)^2\right],$$

using the notations as defined in the Comment, where $C_n^\pm$ are the components of the wave function over the dressed states, the θ_n angle is z dependent through the relation

$$\cot 2\theta_n = -\frac{\delta}{2gu(z)\sqrt{n+1}} \tag{23}$$

and where

$$V_n^\pm = \left(n + \frac{1}{2}\right)\omega \pm \frac{1}{2}\sqrt{\delta^2 + 4g^2u^2(z)(n+1)}. \tag{24}$$

At resonance ($\delta=0$), the θ_n angle that defines the dressed state basis takes the value $\pi/4$ everywhere. Thus $d\theta_n/dz=0$ whatever the cavity mode function and Eq. (22) must reduce to the well known equations of the mazer in that case [4,6], namely,

$$i\frac{\partial C_n^\pm}{\partial t} = \left(-\frac{1}{2M}\frac{\partial^2}{\partial z^2} + V_n^\pm\right)C_n^\pm, \tag{25}$$

as was also stated by Abdel-Aty and Obada in Ref. [2]. It is important to note that Eq. (25) covers the whole z axis. It describes elementary scattering processes of the $C_n^\pm$ components over the potentials $V_n^\pm$ defined by the atom-cavity interaction. However, we observe that Eq. (22) does not reduce to Eq. (25) in the resonant case, but rather to

$$i\frac{\partial C_n^\pm}{\partial t} = \pm\left(-\frac{1}{2M}\frac{\partial^2}{\partial z^2} + V_n^\pm\right)C_n^\pm \tag{26}$$

which is a wrong result.

On the contrary, we may notice that in the resonant case, our equations (5a) and (5b) in Ref. [1] well reduces to Eq. (25) using $\theta=\pi/4$. Abdel-Aty's Eq. (22) most probably contains a sign inaccuracy. Considering that this problem should be solved on the author's own responsibility, we should conclude that both approaches are equivalent in the resonant case. However, we are dealing here with the detuning effects and must focus our attention on this nonresonant case, for which the criticisms were written. In the mesa mode case, it is claimed by Abdel-Aty that Eq. (22) is equivalent to Eq. (25) over the whole z axis, arguing that $d\theta_n/dz$ vanishes identically. This is not true and we will be very explicit in our demonstration. The mesa mode function (illustrated in Fig. 2) is constant everywhere and presents two discontinuous variations at $z=0$ and $z=L$. The θ_n angle given by Eq. (23) is thus equal to 0 or $\pi/2$ (according to the sign of the detuning) outside the cavity [where $u(z)=0$] and to θ_n^c given by $\cot 2\theta_n^c = -\delta/2g\sqrt{n+1}$ inside the cavity [where $u(z)=1$]. Therefore, θ_n is also a discontinuous function: constant everywhere, but with different values inside and outside the cavity. Consequently, it is wrong to say that $d\theta_n/dz$ vanishes identically and that this factor may be eliminated directly from Eq. (22). More precisely, $d\theta_n/dz$ vanishes everywhere *except* at the entrance and at the end of the cavity where it is infinite. This is the key point that has been overlooked by Abdel-Aty. What happens at the cavity borders is essential as this precisely defines the heart of mazer physics, namely, the study of the atom-cavity interaction for atoms passing through the cavity. The special properties predicted by Scully *et al.* [7] for the induced emission probability of a photon inside the micromazer cavity by ultracold atoms initially excited occur because the interaction between the atoms and the field is drastically different inside and outside the cavity, resulting in a strong effect on the atomic motion when the cold atoms enter the cavity [5]. A correct description of the cavity borders is therefore an essential feature in the presently investigated system and any approximation that would elude any characteristics of these points will necessarily profoundly affect the predictions about the system, resulting in possible wrong results.

In the mesa mode case, Eq. (22) contains two singularities at the cavity borders that are removed in Eq. (25). That means that both systems of equations *are not* equivalent. They are equivalent everywhere, except at $z=0$ and $z=L$. Following Abdel-Aty's approach, the correct way to solve the Schrödinger equation would consist of directly considering Eq. (22). However, as we just showed, this approach leads to singularities in the mesa mode case. This points out the limitations of the method and questions the validity of the treatment. It is easy to understand the origins of these limitations. In Abdel-Aty's approach, the Hamiltonian (1) is diagonalized in the local Hilbert space $\mathcal{E}_A \otimes \mathcal{E}_R$ and the atomic center-of-mass motion is supposed to move according to the spatial variations of the z-dependent energy levels [as given by Eq. (4) in Abdel-Aty's Comment]. It is well known that this approach is restricted in the framework of the *adiabatic approximation*. In the presence of a detuning, its validity requires smooth varying mode functions (so that $d\theta_n/dz$ does not contain any singularity). This condition excludes longitudinal modes of closed cavities, whose electric fields exhibit discontinuities at the points where the atom enters and leaves the cavity (Haroche and Raymond [8]). The mesa mode belongs to this exclusion category (as it is obviously not a smoothly varying function) and explains why Abdel-Aty's approach cannot be followed.

On the contrary, our Eqs. (5a) and (5b) in Ref. [1] have been derived outside any restricted scheme in the global Hilbert space (2). Their validity is extremely general and they can be used for any mode function, any initial atomic wave function (including plane waves), and any detuning. They do not contain singularities and we have shown that analytical solutions can be found in the mesa mode case. We have redemonstrated in this Reply the correctness of these equations and are thus confident about all the physical results deduced from them in our paper. It is obvious that these results differ significantly from those obtained by Abdel-Aty and Obada in Refs. [2] and [3] (compare, for example, the divergent expressions for the reflection and transmission coefficients of atoms by the cavity) and that all the physical effects we have recalled at the beginning of this Reply cannot be deduced from their papers.

The reflection and transmission coefficients presented in Refs. [2] and [3] are those obtained in the framework of the well-known one-dimensional scattering problems over square potentials $V_n^{\pm}$. They are deduced from Eq. (25), considering that this equation would describe scattering processes of $C_n^{\pm}$ components representing the *same* wave function projections along the whole z axis. However, this assumption is *only* true in the resonant case where the dressed state bases inside and outside the cavity are identical ($\theta_n=\pi/4$). In the nonresonant case, these bases differ, $C_n^{\pm}$ does not represent the same wave function projections inside and outside the cavity, and the left-hand side of Eq. (25) becomes z dependent. In the presence of a detuning, the system *does not* reduce to elementary scattering processes over potentials V_n^+ and V_n^- defined by the cavity [1]. The reflection and transmission coefficients are less evident to compute. This explains why our results and their results are not in agreement, why we question them, and why we cannot follow Abdel-Aty's suggestion to replace our set of equations (5a) and (5b) in Ref. [1] by Eq. (22). We already pointed out this problem in a separate Comment [9] at the same time of the publication of our paper [1]. A Reply to our Comment has been recently addressed by these authors [10]. However, as it contains exactly the same criticisms as those presented in Abdel-Aty's Comment [and the same inaccuracies connected with Eq. (22)], we refute all of them for the same reasons detailed here and we question clearly the arguments developed therein, where the mesa mode is dramatically confused with a constant function along the whole z axis and the discontinuous variations of this mode are ignored. In this sense, the arguments developed by Abdel-Aty and Obada in Ref. [10] and in Abdel-Aty's Comment are strongly inconsistent. These authors justify the basic equations used in Refs. [2] and [3] [namely, Eq. (25)] by arguing that the mode function does not contain any variations. However, they consider in these papers scattering processes and reflection mechanisms of the atoms by the cavity, although these effects are strictly related to variations of the mode function. If the scattering potentials were constant everywhere, obviously no scattering could occur and this would describe a free particle problem. This is contradictory.

As a conclusion, we have shown in this Reply that all the criticisms raised in Abdel-Aty's Comment are not founded and cannot be considered further. We have also demonstrated precisely what is wrong with Abdel-Aty and Obada's arguments and why the validity of their results published in Refs. [2], [3], and [10] must be seriously questioned. Finally, we have justified again the correctness of our equations and their very general validity, proving that we may be confident with the results contained in Ref. [1].

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