Scheme for implementing the Deutsch-Jozsa algorithm in cavity QED

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We propose a scheme for realizing the Deutsch-Jozsa algorithm in cavity QED. The scheme is based on the resonant interaction of atoms with a cavity mode. The required experimental techniques are within the scope of what can be obtained in the microwave cavity QED setup. The experimental implementation of the scheme would be an important step toward more complex quantum computation in cavity QED.

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The Deutsch-Jozsa algorithm is a simple quantum algorithm, designed to distinguish between the constant and balanced functions [1,2] on 2^n inputs. The value of the function $f(x)$ for each input is either 0 or 1. For the constant function, the function values are constant for all 2^n inputs. On the other hand, for the balanced function, the function values are equal to 1 for half of all the possible inputs, and 0 for the other half. At worst, a classical algorithm will need $2^n/2+1$ queries to unambiguously determine whether the function is constant or balanced since there may be $2^n/2$ 0's before finally a 1 appears, showing that the function is balanced.

In contrast, the Deutsch-Jozsa algorithm requires only one query. We consider here the two-qubit Deutsch-Jozsa algorithm. The input query qubit is initially prepared in the superposition state $(1/\sqrt{2})(|0\rangle_i+|1\rangle_i)$, while the auxiliary working qubit is prepared in the state $(1/\sqrt{2})(|0\rangle_j-|1\rangle_j)$. The state of the whole system is

$$\frac{1}{2}(|0\rangle_i+|1\rangle_i)(|0\rangle_j-|1\rangle_j). \tag{1}$$

Next, the function $f(x)$ is characterized by the unitary mapping transformation U_f,

$$|x,y\rangle \rightarrow |x,y\oplus f(x)\rangle, \tag{2}$$

where $\oplus$ indicates addition modulo 2. This unitary mapping function is specific to each of the functions $f(x)$, resulting in

$$\frac{1}{2}[(-1)^{f(0)}|0\rangle_i+(-1)^{f(1)}|1\rangle_i](|0\rangle_j-|1\rangle_j). \tag{3}$$

There are four possible transformations: (i) $U_{f,1}$ corresponding to $f(0)=f(1)=0$; (ii) $U_{f,2}$ corresponding to $f(0)=f(1)=1$; (iii) $U_{f,3}$ corresponding to $f(0)=0$ and $f(1)=1$; (iv) $U_{f,4}$ corresponding to $f(0)=1$ and $f(1)=0$. Then a Hadamard transformation is performed on the query qubit,

$$|0\rangle_i \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_i+|1\rangle_i), \tag{4}$$

$$|1\rangle_i \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_i-|1\rangle_i).$$

After the transformation, the state of qubit i becomes $|f(0)\oplus f(1)\rangle$. If $f(x)$ is constant, the state of qubit i becomes $|0\rangle_i$. On the other hand, if $f(x)$ is balanced, the state of qubit i becomes $|1\rangle_i$. Therefore, a measurement on qubit i yields the desired information whether the function $f(x)$ is constant or balanced, while the classical algorithm requires two queries. The Deutsch-Jozsa algorithm has been experimentally realized in the nuclear magnetic resonance system [3,4], ion trap [5], and the linear optical system [6].

On the other hand, cavity QED is another qualified system for realizing a quantum processor. A two-qubit phase gate has been experimentally realized with resonant interaction of a two-level atom with a cavity mode [7]. Schemes have been proposed for realizing a quantum discrete Fourier transform [8] and a Grover search algorithm in cavity QED [9]. However, none of these schemes have been experimentally realized. In this paper, we propose a scheme for implementing the Deutsch-Jozsa algorithm in cavity QED. As far as we know, this is the first scheme for the implementation of this algorithm in cavity QED. Our scheme only involves two atoms sequentially interacting with a resonant cavity mode. The resonant interaction of two or three atoms with a cavity mode has been demonstrated in recent experiments [7,10]. Therefore, our scheme might be experimentally realizable with presently available techniques. The experimental implementation of the scheme would be an important step toward a more complex quantum algorithm with cavity QED techniques, serving to illustrate the power of cavity QED for quantum-information processing. The aim of the following section is to show that the four operations $U_{f,1}$, $U_{f,2}$, $U_{f,3}$, and $U_{f,4}$ of Eq. (2) can be realized with a cavity QED system.

Here we use a ladder-type three-level atom, whose states are denoted by $|i\rangle$, $|g\rangle$, and $|e\rangle$. The transition frequency between the states $|g\rangle$ and $|e\rangle$ is resonant with the cavity frequency, while the transition frequency between the states $|g\rangle$ and $|i\rangle$ is highly detuned from the cavity frequency and thus the state $|i\rangle$ is not affected during the atom-cavity interaction. The quantum information is stored in the states $|g\rangle$ and $|i\rangle$. Assume that the atom acts as the query qubit and the cavity acts as the auxiliary working qubit.

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We should first prepare the cavity in the required state. In order to do so, we send a preparing atom through the single-mode cavity. Assume the atom is initially in the excited state $|e\rangle_a$ and the cavity mode is in the vacuum state $|0\rangle_c$. In the interaction picture, the atom-cavity interaction is described by the Jaynes-Cummings Hamiltonian,

$$H_I = ig(a^\dagger S^- - aS^+), \tag{5}$$

where $a^\dagger$ and a are the creation and annihilation operators for the cavity field, S^+ and S^- are the raising and lowering operators for the atom, and g is the atom-field coupling constant. After an interaction time t, the state of the system is given by

$$|\psi\rangle = \cos(gt)|e\rangle_a|0\rangle_c + \sin(gt)|g\rangle_a|1\rangle_c, \tag{6}$$

where t is the interaction time. With the choice $t=\pi/2g$, the cavity is prepared in the one-photon state $|1\rangle_c$ with the atom left in the state $|g\rangle_a$.

Then we send a second atom (the query qubit) through the cavity. This atom is initially in the state

$$|\phi\rangle_b = \frac{1}{\sqrt{2}}(|g\rangle_b + |i\rangle_b). \tag{7}$$

The system combined by the second atom and the cavity is in the state

$$|\Phi\rangle_1 = \frac{1}{2}(|g\rangle_b + |i\rangle_b)|1\rangle_c. \tag{8}$$

We can rewrite this state as

$$|\Phi\rangle_1 = \frac{1}{2}(|0\rangle_i + |1\rangle_i)(|0\rangle_{j,r} - |1\rangle_{j,r}), \tag{9}$$

where

$$\begin{aligned} |0\rangle_i &= |i\rangle_b, \\ |1\rangle_i &= |g\rangle_b, \\ |0\rangle_{j,r} &= \frac{1}{\sqrt{2}}(|0\rangle_c + |1\rangle_c), \\ |1\rangle_{j,r} &= \frac{1}{\sqrt{2}}(|0\rangle_c - |1\rangle_c). \end{aligned} \tag{10}$$

Therefore, with respect to the rotated basis states $|0\rangle_{j,r}$ and $|1\rangle_{j,r}$, the state $|\Phi\rangle_1$ has the same form as Eq. (1).

For the case $f(0)=f(1)=0$, the atom is tuned far off-resonant with the cavity mode through the Stark effect and thus the atom-cavity evolution is freezing. Then the system remains in the state $|\Phi\rangle_1$.

For the case $f(0)=f(1)=1$, the atom interacts with the cavity mode for a time t', leading to

$$\begin{aligned} |i\rangle_b|1\rangle_c &\rightarrow |i\rangle_b|1\rangle_c, \\ |g\rangle_b|1\rangle_c &\rightarrow \cos gt'|g\rangle_b|1\rangle_c - \sin gt'|e\rangle_b|0\rangle_c. \end{aligned} \tag{11}$$

With the choice $t'=\pi/g$, we have

$$\begin{aligned} |i\rangle_b|1\rangle_c &\rightarrow |i\rangle_b|1\rangle_c, \\ |g\rangle_b|1\rangle_c &\rightarrow -|g\rangle_b|1\rangle_c. \end{aligned} \tag{12}$$

In the rotated basis, this corresponds to a controlled-NOT gate,

$$\begin{aligned} |0\rangle_i|0\rangle_{j,r} &\rightarrow |0\rangle_i|0\rangle_{j,r}, \\ |0\rangle_i|1\rangle_{j,r} &\rightarrow |0\rangle_i|1\rangle_{j,r}, \\ |1\rangle_i|0\rangle_{j,r} &\rightarrow |1\rangle_i|1\rangle_{j,r}, \\ |1\rangle_i|1\rangle_{j,r} &\rightarrow |1\rangle_i|0\rangle_{j,r}. \end{aligned} \tag{13}$$

Next, we perform the single-qubit transformation on the atom,

$$|0\rangle_i \rightarrow |1\rangle_i, \quad |1\rangle_i \rightarrow -|0\rangle_i. \tag{14}$$

This transformation can be achieved by using a π-Ramsey pulse. Then we repeat the controlled-NOT operation and perform the transformation

$$|0\rangle_i \rightarrow -|1\rangle_i, \quad |1\rangle_i \rightarrow |0\rangle_i. \tag{15}$$

This transformation can be achieved by using another π-Ramsey pulse with a phase difference π relative to the first Ramsey pulse. This leads to

$$\begin{aligned} |\Phi\rangle_2 &= \frac{1}{2}(|0\rangle_i + |1\rangle_i)(|0\oplus 1\rangle_{j,r} - |1\oplus 1\rangle_{j,r}) \\ &= \frac{1}{2}(-|0\rangle_i - |1\rangle_i)(|0\rangle_{j,r} - |1\rangle_{j,r}). \end{aligned} \tag{16}$$

For the case $f(0)=0$ and $f(1)=1$, we perform the controlled-NOT operation of Eq. (13). Then the atom-cavity system evolves to

$$\begin{aligned} |\Phi\rangle_3 &= \frac{1}{2}[|0\rangle_i(|0\rangle_{j,r} - |1\rangle_{j,r}) + |1\rangle_i(|0\oplus 1\rangle_{j,r} - |1\oplus 1\rangle_{j,r}) \\ &= \frac{1}{2}(|0\rangle_i - |1\rangle_i)(|0\rangle_{j,r} - |1\rangle_{j,r}). \end{aligned} \tag{17}$$

For the case $f(0)=1$ and $f(1)=0$, we first perform the single-qubit transformation of Eq. (14). Then we perform the controlled-NOT operation of Eq. (13). Finally we perform the single-qubit transformation of Eq. (15). This leads to

$$\begin{aligned} |\Phi\rangle_4 &= \frac{1}{2}[|0\rangle_i(|0\oplus 1\rangle_{j,r} - |1\oplus 1\rangle_{j,r}) + |1\rangle_i(|0\rangle_{j,r} - |1\rangle_{j,r}) \\ &= \frac{1}{2}(-|0\rangle_i + |1\rangle_i)(|0\rangle_{j,r} - |1\rangle_{j,r}). \end{aligned} \tag{18}$$

In this way we obtain the unitary mapping transformation U_f of Eq. (2), which leads to

$$\frac{1}{2}[(-1)^{f(0)}|0\rangle_i + (-1)^{f(1)}|1\rangle_i)(|0\rangle_{j,r} - |1\rangle_{j,r}). \tag{19}$$

After the Hadamard transformation on the atom, if $f(x)$ is constant, the state of the atom becomes $|0\rangle_i$. If $f(x)$ is bal-

anced, the state of the atom becomes $|1\rangle_i$. Thus, a single measurement of the atomic state is sufficient to determine whether the function $f(x)$ is constant or balanced.

Finally we briefly address the experimental feasibility of the proposed scheme. For the Rydberg atoms with principal quantum numbers 50 and 51, the radiative time is $T_r = 3 \times 10^{-2}$ s, and the coupling constant is $g = 2\pi \times 25$ kHz [7,10,11]. Thus the interaction time of the atom with the cavity field is $2\pi/g = 4.0 \times 10^{-5}$ s. Then the time required to complete the procedure can be assumed to be 4.0×10^{-4} s, much shorter than T_r. In recent experiments, the decay time of the cavity was $T_c \simeq 1.0 \times 10^{-3}$ s [7,10,11], longer than the required time. Therefore, based on cavity QED techniques presently available, the proposed scheme might be realizable. The experimental implementation of the scheme may serve to be an intermediate step for more complex quantum-information processing in cavity QED.

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