Dark evolution in time-varying Zeno subspace

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We investigate the evolution of a quantum system under the influence of sequential measurements. The measurement scheme distinguishes whether or not the system is in a specified state $|f_n\rangle$ at the nth step, where $|f_n\rangle$ varies with n. Dark evolution corresponds to the situation when all measurements give negative results. We show that dark evolution is unitary in the continuous measurement limit. We derive the effective Hamiltonian, and indicate how $|f_n\rangle$ controls quantum state transport.

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I. INTRODUCTION

The influence of measurements on quantum systems has been an important subject since the discovery of quantum mechanics. One of the most intriguing measurement-induced phenomena is the quantum Zeno effect (QZE), in which the time evolution of a system may be frozen under very frequent observations of the initial state [1–3]. The QZE is understood as a consequence of the projection postulate and the quadratic behavior of the survival probability at short times. Experimental observations of the QZE in atomic systems have been reported [4,5]. Recently, Facchi *et al.* analyzed the Zeno problem with a more general approach [6,7]. They indicated that quantum evolution can occur in a restricted Hilbert space (Zeno subspace) defined by measurement projection operators. Such a Zeno subspace serves as a basis of useful applications, such as quantum state engineering [8] and decoherence control [9,10].

The QZE has been discussed mainly in situations where the same state or observable is frequently monitored. Since the corresponding measurement projection operators are constant in time, the underlying Zeno subspace is stationary. A natural extension is the inclusion of time-varying observations [7,11]. This involves sequential measurements such that different states are monitored at different times. Such a time-dependent problem has been studied in a two-level system [11]. Although it should be expected that time-varying projections would lead to interesting behavior [12], the detailed dynamics has not been fully explored.

In this paper we investigate this problem in an N-level system ($N \geqslant 3$ in general). The system is subjected to a prescribed sequence of measurements, such that the nth measurement detects whether the system is in the state $|f_n\rangle$ or not. $|f_n\rangle$ changes with n, and so the Zeno subspace is time dependent. The measuring apparatus is designed such that it can interact with $|f_n\rangle$ only at the nth step. Each measurement simply gives a "yes" or "no" answer, and it does not provide any further information about the system. An interesting question is how the system evolves if all measurements give *negative* results, i.e., "no" for all n. This corresponds to what we will call dark evolution in this paper. Such an evolution is driven by measurements, and it occurs even if the Hamiltonian of the measured system is zero.

Early examples of negative result experiments were discussed by Renninger [13] and Dicke [14], who indicated possible modifications of the measured system if the detector does not detect anything. Since the state of the detector is not affected by the measured system, negative result experiments are sometimes known as interaction-free measurements [15]. In this regard, the measurement scheme that we will examine is a form of interaction-free measurement generalized to time-dependent situations. In order to determine the quantum dynamics, we will present a Hamiltonian formalism of the problem. In particular, we will show that dark evolution is unitary and it is governed by an effective Schrödinger equation in the frequent measurement limit. Some of the main features of quantum state transport will be discussed.

II. DARK EVOLUTION

Let $|\Psi_n\rangle$ be the state of the system immediately after the nth measurement. The initial state $|\Psi_0\rangle$ is prepared such that it is orthogonal to $|f_1\rangle$. If any step in the measurement yields a "yes" answer, we have to reset the system to the initial condition and restart the experiment. This ensures dark evolution in a single run, and hence the system state remains pure, assuming that decoherence effects are negligible.

Dark evolution is described by the relation ($\hbar = 1$)

$$|\Psi_n\rangle = (1 - |f_n\rangle\langle f_n|)e^{-iH\tau}|\Psi_{n-1}\rangle, \tag{1}$$

where H is the Hamiltonian (assumed time independent) of the unmeasured system, and τ is the time interval between measurements. After M measurements, the system state is given by

$$|\Psi_M\rangle = P_M e^{-iH\tau} P_{M-1} e^{-iH\tau} \cdot \cdot \cdot P_2 e^{-iH\tau} P_1 e^{-iH\tau}|\Psi_0\rangle \tag{2}$$

where $P_n = 1 - |f_n\rangle\langle f_n|$ is the projection operator. Note that $|\Psi_M\rangle$ in Eq. (2) has not been normalized. It is understood that $|\langle\Psi_M|\Psi_M\rangle|^2$ corresponds to the probability of realizing a run of the experiment involving M measurements with negative results.

At time $t = n\tau$, we write $|\Psi(t)\rangle = |\Psi_n\rangle$ and $|f(t)\rangle = |f_n\rangle$. Assuming $|f(t)\rangle$ and its derivatives are continuous, Eq. (1) gives: $|\Psi(t+\tau)\rangle - |\Psi(t)\rangle = [P(t+\tau)e^{-iH\tau} - 1]|\Psi(t)\rangle = [P(t) + \dot{P}(t)\tau - i\tau P(t)H - 1]|\Psi(t)\rangle + O(\tau^2)$, where $\dot{P}(t) = dP(t)/dt$ is the time derivative of the projection operator. In the frequent measurement limit $\tau \to 0$, we have

$$i\frac{d}{dt}|\Psi(t)\rangle = [P(t)HP(t) + i\dot{P}(t)]|\Psi(t)\rangle, \tag{3}$$

where the identity $P(t)|\Psi(t)\rangle = |\Psi(t)\rangle$ has been employed. We remark that there are subtle relations between pulsed observations and continuous observations in realistic systems [11]. Here the $\tau \to 0$ limit is taken for idealized situations. However, we will show that dark evolution exists in more general situations (Sec. III), and projection measurements are not crucial.

Eqation (3) describes the system evolution under the initial condition $\langle f(0)|\Psi(0)\rangle = 0$. It is easy to show that $d\langle f|\Psi\rangle/dt = -\langle f|\dot{f}\rangle\langle f|\Psi\rangle$, and so $\langle f(t)|\Psi(t)\rangle = 0$ because of the initial condition. Therefore $|\Psi(t)\rangle$ remains orthogonal to $|f(t)\rangle$ at any later time. With this result, Eq. (3) further gives $\langle\Psi(t)|\dot{\Psi}(t)\rangle + \langle\dot{\Psi}(t)|\Psi(t)\rangle = 0$. This shows that the norm

$$\langle\Psi(t)|\Psi(t)\rangle = 1 \tag{4}$$

is preserved, i.e., dark evolution is unitary in the frequent measurement limit.

A. Effective Hamiltonians

To learn how the system evolves for a given $|f(t)\rangle$, it is useful to cast Eq. (3) in the form of a Schrödinger equation,

$$i\frac{d|\Psi\rangle}{dt} = H_D|\Psi\rangle, \tag{5}$$

where H_D (D refers to dark evolution) is an effective Hamiltonian. We point out that Eq. (3) is *not* a Schrödinger equation because the $i\dot{P}(t)$ term is not Hermitian. This problem can be overcome by making use of the fact that $\langle f(t)|\Psi(t)\rangle = 0$ shown above. We can add any term $|X\rangle\langle f(t)|$ (where $|X\rangle$ is arbitrary) inside the brackets on the right side of Eq. (3) without changing the evolution of $|\Psi\rangle$. By choosing $|X\rangle = 2i|\dot{f}(t)\rangle$, we obtain the effective Hamiltonian

$$H_D(t) = P(t)HP(t) + i[|\dot{f}(t)\rangle\langle f(t)| - |f(t)\rangle\langle\dot{f}(t)|], \tag{6}$$

which is controlled by $|f(t)\rangle$.

The specification of $|f(t)\rangle$ can be made from the unitary operator that generates the motion of $|f(t)\rangle$. We assume that $|f(t)\rangle = e^{-iKt}|f(0)\rangle$, where K is a time-independent Hermitian operator. To see the effects of K, we go to a comoving frame in which $|f(t)\rangle$ is stationary. This corresponds to a unitary transformation $|\tilde{\Psi}(t)\rangle = e^{iKt}|\Psi(t)\rangle$. The corresponding Schrödinger equation reads $i|\dot{\tilde{\Psi}}(t)\rangle = \tilde{H}_D(t)|\tilde{\Psi}(t)\rangle$, where the transformed effective Hamiltonian $\tilde{H}_D$ is given by

$$\tilde{H}_D(t) = P(0)(e^{iKt}He^{-iKt} - K)P(0). \tag{7}$$

This relation indicates the explicit role of K in the effective Hamiltonian. In deriving Eq. (7), we have made use of the relation $\langle f(0)|\tilde{\Psi}(t)\rangle = 0$.

B. Formal solutions

The formal solution of $|\Psi(t)\rangle$ is given by

$$|\Psi(t)\rangle = e^{-iKt}\mathcal{T}\exp\left[-i\int_0^t dt'\tilde{H}_D(t')\right]|\Psi(0)\rangle, \tag{8}$$

where $\mathcal{T}$ is the time ordering operator. Further simplification can be made if K and H commute. In this case $\tilde{H}_D = P(0)\times(H-K)P(0)$ is time independent, Eq. (8) becomes

$$|\Psi(t)\rangle = e^{-iKt}e^{-i\tilde{H}_Dt}|\Psi(0)\rangle. \tag{9}$$

Note that K and $\tilde{H}_D$ do not commute with each other in general, we may need to solve the eigenvalue problem

$$\tilde{H}_D|u_k\rangle = \omega_k|u_k\rangle \tag{10}$$

in order to obtain the explicit form of $|\Psi(t)\rangle$. The general solution can then be expressed in the expansion

$$|\Psi(t)\rangle = \sum_{k=1}^{N-1} c_k e^{-i\omega_k t}|v_k(t)\rangle, \tag{11}$$

where $|v_k(t)\rangle \equiv e^{-iKt}|u_k\rangle$ and the coefficients c_k are determined by the initial state.

Equation (11) reveals the basic structure of the solution when $[K,H]=0$. Initially, $\{|v_k(0)\rangle\}$ corresponds to the set of eigenvectors of $\tilde{H}_D$. As time increases, each $|v_k(t)\rangle$ evolves unitarily according to e^{-iKt}, the same operator that generates the evolution of $|f(t)\rangle$. Therefore, all $|v_k(t)\rangle$ remain orthogonal to $|f(t)\rangle$. These eigenvectors are treated as a natural set of (time-evolving) basis vectors of the system. A remarkable feature is the emergence of "new" eigen-frequencies ω_k associated with these time-varying basis vectors. These frequencies are eigenvalues of neither H nor K.

We illustrate the intricate coupling between ω_k and $|f(t)\rangle$ in a three-level system with $H=0$. Let $|f(t)\rangle$ be a coherent superposition: $|f(t)\rangle = a_1e^{-i\Omega_1t}|k_1\rangle + a_2e^{-i\Omega_2t}|k_2\rangle + a_3e^{-i\Omega_3t}|k_3\rangle$, where Ω_j and $|k_j\rangle$ $(j=1,2,3)$ are the eigenvalues and eigenvectors of K. In this case, $\tilde{H}_D = P(0)KP(0)$ is a 2×2 matrix; the calculation of its two eigenvalues $\omega_\pm$ gives $\omega_\pm = (\xi \pm \sqrt{\xi^2 - 4\eta})/2$, where $\xi = \Omega_1 + \Omega_2 + \Omega_3 - |a_1|^2\Omega_1 - |a_2|^2\Omega_2 - |a_3|^2\Omega_3$ and $\eta = |a_1|^2\Omega_2\Omega_3 + |a_2|^2\Omega_1\Omega_3 + |a_3|^2\Omega_1\Omega_2$. We see that $\omega_\pm$ depend on a_j and Ω_j nontrivially. The situations can become more complicated for higher-dimensional systems.

As a general remark, we note that if K has commensurate eigenvalues then $|f(t)\rangle$ is cyclic with a certain period T. This means $e^{-iKT} = 1$ and hence

$$|\Psi(T)\rangle = \sum_{k=1}^{N-1} c_k e^{-i\omega_k T}|v_k(0)\rangle \tag{12}$$

according to Eq. (11). Since K and $\tilde{H}_D$ generally do not share the same spectrum, we have $e^{-i\omega_k T} \neq 1$. Therefore the system in general does not return to the initial state for a cyclic $|f(t)\rangle$.

C. Quantum state transport

For the purpose of quantum state transport, a relevant problem is to find an $|f(t)\rangle$ such that the system evolves in a

prescribed function of time. Such an inverse problem has a simple solution. It follows from Eq. (3) that $H|\Psi(t)\rangle - i|\dot{\Psi}(t)\rangle$ and $|f(t)\rangle$ must be parallel. This implies $|f(t)\rangle$ in the form

$$|f(t)\rangle = \mathcal{N}_f[H|\Psi(t)\rangle - i|\dot{\Psi}(t)\rangle], \tag{13}$$

where $\mathcal{N}_f$ is a factor that can be time dependent. This factor is determined by the normalization condition $\langle f(t)|f(t)\rangle = 1$, which gives

$$\mathcal{N}_f^{-2} = \langle\dot{\Psi}(t)|\dot{\Psi}(t)\rangle + \langle\Psi(t)|H^2|\Psi(t)\rangle. \tag{14}$$

Equation (13) indicates how the measurement state $|f(t)\rangle$ is designed in order to steer the system state to evolve in a specified way. However, it is important to remark that $|\Psi(t)\rangle$ cannot be arbitrary because $\langle\Psi(t)|f(t)\rangle = 0$ must be satisfied. A direct calculation of the inner product $\langle\Psi(t)|f(t)\rangle$ in Eq. (13) leads to the condition

$$i\langle\Psi(t)|\dot{\Psi}(t)\rangle = \langle\Psi(t)|H|\Psi(t)\rangle. \tag{15}$$

This is a fundamental restriction that all $|\Psi(t)\rangle$ must obey in dark evolution.

Let us discuss $H=0$ systems that highlight the pure influence of time-varying projective measurements. Physical examples of $H=0$ systems may be found in the degenerate Zeeman levels of an atom, and $|f(t)\rangle$ corresponds to a coherent superposition of these levels. For $H=0$, Eq. (15) implies $\langle\Psi(t)|\dot{\Psi}(t)\rangle = 0$, which corresponds to the condition of *parallel transport*. It means that $|\Psi(t+\delta t)\rangle$ and $|\Psi(t)\rangle$ share the same quantum phase to first order in δt, i.e., the local phase change $\arg[\langle\Psi(t)|\Psi(t+\delta t)\rangle] \approx 0$. However, as the system evolves, there is an overall phase accumulated by the system. Such an accumulated phase is purely geometrical under the parallel transport condition [16].

For $H=0$ systems, Eq. (13) indicates that $|f(t)\rangle = \mathcal{N}_f|\dot{\Psi}(t)\rangle$. To provide an explicit example, suppose $|\Psi(t)\rangle$ is prescribed by

$$|\Psi(t)\rangle = \sum_{j=1}^{N} \sqrt{p_j} e^{-i\nu_j t}|j\rangle, \tag{16}$$

where p_j and ν_j are real constants so that $\Sigma_{j=1}^{N} p_j\nu_j = 0$ is satisfied for the parallel transport condition. The required $|f(t)\rangle$ is given by

$$|f(t)\rangle = \mathcal{N}_f \sum_{j=1}^{N} \sqrt{p_j}\nu_j e^{-i\nu_j t}|j\rangle. \tag{17}$$

We note that the possibility of steering an $(H=0)$ system into an arbitrary state via suitably designed continuous measurements was noticed by von Neumann many years ago [1]. This is usually understood in "bright" measurement configurations, i.e., "yes" detection answers leading to a complete state reduction [17]. In contrast, our approach exploits the dark Zeno subspace from which the detector cannot extract any information (except for two-level systems in which dark and bright measurements are equivalent). Finally, we remark that our mechanism of transporting quantum states should be distinguished from adiabatic passage [18], a technique that is commonly employed for state preparation. Here dark evolution is guided by projections onto a Zeno subspace, and adiabatic changes of energy eigenstates are not necessarily required.

III. DISCUSSION AND SUMMARY

Our formulation so far is based on state projections triggered by measurements. In essence, dark evolution is due to the existence of a time-varying state $|f(t)\rangle$ that the system cannot access. As long as $|f(t)\rangle$ can be "simulated" in the system, dark evolution will occur *without* involving any measurements. One possible mechanism is to shift the energy of $|f(t)\rangle$ by a large amount relative to the energies of all other states. Because of energy constraint, a system is forbidden to reach $|f(t)\rangle$, if the initial state is orthogonal to $|f(0)\rangle$.

To elaborate the idea, let us consider a system with a model Hamiltonian

$$\mathcal{H} = E|f(t)\rangle\langle f(t)|. \tag{18}$$

By writing the system state vector $|\psi_s(t)\rangle$ as $|\psi_s(t)\rangle = |\Psi(t)\rangle + \alpha(t)|f(t)\rangle$, where $|\Psi(t)\rangle$ is orthogonal to $|f(t)\rangle$, the Schrödinger equation $H|\psi_s\rangle = i|\dot{\psi}_s\rangle$ gives

$$|\dot{\Psi}\rangle + \dot{\alpha}|f\rangle + \alpha|\dot{f}\rangle = -i\alpha E|f\rangle \tag{19}$$

and $\alpha(t)$ obeys the equation $i\dot{\alpha} = (E - i\langle\dot{f}|f\rangle)\alpha - i\langle f|\dot{\Psi}\rangle$. When E is sufficiently large such that $E \gg \langle f|\dot{\Psi}\rangle, \langle f|\dot{f}\rangle$, we have $\dot{\alpha}(t) \approx 0$ and $\alpha(t) \approx i\langle f(t)|\dot{\Psi}(t)\rangle/E$ as an adiabatic solution (where terms with fast oscillatory phases are neglected). This allows us to recover Eq. (3) ($H=0$ case) from Eq. (19) by keeping the leading terms and using $\langle f(t)|\dot{\Psi}(t)\rangle = -\langle\dot{f}(t)|\Psi(t)\rangle$. The idea of applying a large coupling term to generate Zeno dynamics was suggested in Ref. 10. The above discussion provides a generalization in time-varying situations. In particular, we indicate the required conditions on the large parameter E and the speed of $|f(t)\rangle$.

To summarize, we show how a nonconstant sequence of projections would force a measured system to evolve. In particular, we introduce the notion of dark evolution caused by negative result measurements in the context of the QZE. By varying $|f(t)\rangle$ with time, dark evolution enables quantum state transport under certain basic constraints. Since the state of the detector is unaffected, quantum coherence of the measured system is preserved in the Zeno subspace. Our study provides a Hamiltonian formalism to determine the quantum dynamics in the continuous measurement limit.

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