Classical limit of the interaction of a quantum system with the electromagnetic field

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The interaction of nonrelativistic matter with the quantized electromagnetic field is investigated in the classical limit of large photon numbers. Quantization of both matter, say an atom, and the field results in a time-independent Schrödinger equation (TISE). However, for very strong fields (quantum mechanically, large photon numbers) this is impractical to solve. The standard approach then is simply to replace the quantized field by a classical field to give a time-dependent Schrödinger equation (TDSE) for the atom alone. Here we show how this TDSE can be derived from the TISE for atom plus field, illustrating at each stage the approximations that are necessary to treat the field classically. An important difficulty at the semiclassical stage is a breakdown of the approximation at classical turning points. We show how the use of coherent field states can circumvent this problem. In the limit that the field can be treated classically, time emerges from the Maxwell equations and a TDSE for the atom alone results.

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I. INTRODUCTION

The development of lasers has reached the stage where the field strengths and the pulse lengths (attoseconds) are approaching those attainable with heavy-ion beams. Indeed the similarity between particle and laser beams in their action on matter is being emphasized increasingly. In the first year of quantum mechanics it was not known how to quantize a particle beam or the electromagnetic field. Hence in Schrödinger's 1926 paper [1] on the interaction of an atom with light, the electromagnetic field was treated as an external classical field coupled to the atom, i.e., in this first application of the time-dependent Schrödinger equation (TDSE) it was recognized that the time-dependence arises from classical equations (the Maxwell equations) and in this sense the TDSE is a mixed classical-quantum equation. Similarly, in the famous Born, Heisenberg, and Jordan paper [2] both an external electromagnetic field and an external α-particle beam acting on an atom are treated by classical mechanics (Maxwell's or Newton's equations, respectively), valid when the corresponding energy can be regarded as infinitely large compared with that of the atom.

In the case of a particle beam, Born's collision theory [3] showed how the beam could be quantized, in the simplest approximation as a plane wave. Already by 1931, following work by Frame, Brinkmann, and Kramers [4] and, above all, Mott [5], it was known that the collision problem described by the time-independent Schrödinger equation (TISE) with a quantized beam reduces to the problem of a TDSE with a classical, time-dependent beam, in the limit that the beam energy greatly exceeds atomic transition energies. This condition is necessary, since in the approximation of the TDSE, energy is not conserved. Rather, the beam is considered to have a fixed energy which in zeroth order is unchanged by the interaction, i.e., there is no back coupling of the atom on the beam.

Since Mott's work, the transition from an exact description of a particle beam interacting with a quantum system via the TISE to the approximate description of a classical beam via the TDSE has been studied in great detail and precise conditions have been given for the validity of each step of the reduction [6]. Quite generally, it has been shown recently how the full TISE for a quantum system interacting with a quantized environment reduces to the TDSE for the system in the presence of a classical environment, in the limit that the energy of the environment is large on a quantum scale [7]. This derivation was given in detail for the particle beam and the analogous development for a boson field was sketched out. The aim of the present paper is to give this derivation of the TDSE for laser-matter interactions in detail by enumerating the conditions that must be obtained in order to treat the electromagnetic (e.m.) field classically. The extension to quantized fields is nontrivial since plainly the treatment of the field involves oscillatory motion. Then the semiclassical approximation breaks down at turning points and the derivation given in Ref. [7] is invalid. Here we show by use of coherent states that, nevertheless, a continuous classical time variable can be defined. In the Appendix it is shown explicitly that the conditions for validity enumerated are satisfied in the limit of large photon numbers by treating the Jaynes-Cummings (JC) model [9] as an example.

In 1927 Dirac showed how the electromagnetic field for n photons can be quantized in the occupation number representation [8]. When coupled to a quantum system, say an atom, the field-atom system obeys a TISE in which energy is ex-

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changed between field and atom but total energy is conserved. For fields of few photons, e.g., in cavity QED experiments, it is practical to solve the TISE and this procedure is exact. However, clearly this is not feasible for very high-power, short-pulse lasers involving extremely large numbers of photons. Then the laser field is treated classically, the field strength is considered as fixed, and energy is not conserved in the interaction with the atom.

Below we present a derivation of the TDSE for the atom in a time-dependent classical field and emphasise the approximations necessary at each stage of the reduction. A key step along the way is the "single-channel approximation" for the full quantum state of field and atom, which amounts, in a certain sense, to an approximate disentanglement of the field state from the state of the atom. Then the limit is taken that the field energy becomes large enough that the field can be treated classically and also large enough that the classical state of the field is impervious to changes in the state of the quantum system. In fact, it was shown [7] that not the classical but only the semiclassical approximation for beam particles is necessary to derive the TDSE from the full TISE. Here the same will be shown to be true in the case of photons. However, as mentioned above, since the classical limit in the photon case is that of oscillatory motion, a more careful treatment of the semiclassical approximation is required. Hence we discuss in some detail the problem of the semiclassical limit in the vicinity of turning points. The latter are points of zero velocity $\dot{Q}(t)=0$ of the position variable $Q(t)$. An approach based on a position representation of the degree of freedom whose classical limit is to be taken breaks down at turning points. Away from turning points, the position representation can be used to derive a time-dependent Schrödinger equation since $\dot{Q}(t)>0$ at all times. There is thus a one-to-one correspondence between position and time. However, at turning points where the velocity goes to zero, the time cannot be defined in terms of the position.

Here we show explicitly that the problem of turning points emerging in the harmonic motion of field quadratures may be overcome by replacing the position (alias quadrature) representation by a coherent state representation. We are thus naturally led to introduce coherent states, known to be the field states describing the classical limit of the quantized e.m. field. This new derivation is also useful since it is not based on field quadratures (similar to the particle beam case) but is couched in the language of second quantization using creation and annihilation operators, obviously a language more familiar in quantum optics. We will also discuss the similarities and differences, with respect to their interaction with matter, between the transition of a quantized particle beam to classical status and the same transition of a quantized field.

II. INTERACTION OF A QUANTUM SYSTEM WITH A BOSON FIELD

In the following we will first enumerate, step by step, the conditions that must be obtained if a TDSE is to be a good approximation to the TISE. The proof that these approximations do hold in the limit of large photon numbers, corresponding to field energies much greater than atomic transition energies, is given in the text and in more detail for the simple JC model in the Appendix. We begin by considering the time-independent problem of a quantum system, with a Hamiltonian $H_S(p,x)$, interacting with a boson field with Hamiltonian $H_F(P,Q)$ via a coupling $H_I(x,Q)$, to give a total Hamiltonian

$$H = H_S(p,x) + H_F(P,Q) + H_I(x,Q). \tag{1}$$

Here x, Q are position operators and p, P the corresponding momentum operators. The boson field will be taken as a sum over field modes with Hamiltonian

$$H_F(P,Q) = \sum_k \tfrac{1}{2}(P_k^2 + \omega_k^2 Q_k^2) \tag{2}$$

and

$$P_k = -i\hbar \frac{\partial}{\partial Q_k}. \tag{3}$$

It is more usual in quantum optics [10] to write the Hamiltonian in terms of annihilation and creation operators,

$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_k \epsilon_k \left(a_k^\dagger a_k + \frac{1}{2} \right) + \hbar \sum_{ijk} g_{ij}^k c_i^\dagger c_j (a_k^\dagger + a_k), \tag{4}$$

where the coupling constants g_{ij}^k are proportional to the dipole matrix elements $\langle \phi_i | \vec{\pi}_k \cdot \vec{p} | \phi_j \rangle$. Here the $|\phi_i\rangle$ are eigenvectors of H_S with eigenenergies ϵ_i, and we denote the polarization vector of the corresponding field mode with $\vec{\pi}_k$. We have assumed the usual dipole approximation for the interaction, often valid in both atom-photon and electron-phonon interactions. However, this specification is made for the purposes of illustration only and the general form (1) could be retained, as is shown explicitly [Eq. (30) below] in an approach using field quadratures.

In Eq. (4) the operators $a_k^\dagger$, a_k create and annihilate photons of energy $\epsilon_k = \hbar\omega_k$ and the atomic operators are $c_i^\dagger = |\phi_i\rangle\langle\phi_0|$, so that $c_i^\dagger c_j = |\phi_i\rangle\langle\phi_j|$. A simplification studied in a large number of papers in diverse fields is that where the quantum system has only two levels, corresponding to a spin half system and the field has one mode only. Then Eq. (4) can also be written in the form

$$H = \tfrac{1}{2}(\epsilon_+ - \epsilon_-)\sigma_z + \epsilon \left(a^\dagger a + \frac{1}{2} \right) + \hbar g (a^\dagger + a)\sigma_x, \tag{5}$$

and σ_x, σ_z are Pauli 2×2 spin matrices, related in the usual way to the atomic operators c_i, see Ref. [10] and the Appendix. In what follows the limit will be taken that the energy in the field, corresponding to H_F, is much greater than the energy changes in the quantum system, or in the coupling H_I. Then it is convenient to emphasize the respective energies by defining

$$b = \frac{a}{\sqrt{\langle n \rangle}}, \; b^\dagger = \frac{a^\dagger}{\sqrt{\langle n \rangle}} \tag{6}$$

where $\langle n \rangle$ is the mean photon number. In Eq. (5), this gives

$$H = \tfrac{1}{2}\hbar\omega_0\sigma_z + \langle n\rangle\epsilon\left(b^\dagger b + \frac{1}{2\langle n\rangle}\right) + \hbar\sqrt{\langle n\rangle}g(b^\dagger + b)\sigma_x. \tag{7}$$

Then we note that $\langle n\rangle\epsilon$ is just the mean total energy of the field, whereas $\hbar\omega_0 = (\epsilon_+ - \epsilon_-)$ is the transition energy of the two-level system. In the optical case $\hbar\sqrt{\langle n\rangle}g$ is proportional to the Rabi-broadening energy. In the following we will consider the case where

$$\langle n\rangle\epsilon \gg \hbar\omega_0 \text{ and } \langle n\rangle\epsilon \gg \hbar\sqrt{\langle n\rangle}g. \tag{8}$$

In the general equation (4), the renormalization of the field operators according to Eq. (6) gives

$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_k \langle n_k\rangle\epsilon_k\left(b_k^\dagger b_k + \frac{1}{2\langle n_k\rangle}\right) + \hbar\sum_{ijk}\sqrt{\langle n_k\rangle}g_{ij}^k c_i^\dagger c_j(b_k^\dagger + b_k). \tag{9}$$

Note that when $\langle n_k\rangle$ becomes large the zero-point energy can be neglected as in the classical limit. Further,

$$[b_k, b_k^\dagger] = \frac{1}{\langle n_k\rangle} \tag{10}$$

so that $\langle n_k\rangle \to \infty$ gives the classical limit, corresponding to $\hbar \to 0$ for the commutator

$$[Q_k, P_k] = i\hbar \tag{11}$$

of the field operators P_k, Q_k. Hence we will consider in the following the separate but connected limits (i) that $\langle n_k\rangle \to \infty$, allowing the field to be treated classically, and (ii) the limits (8) under which the total field energy far exceeds any energy changes in the quantum system.

III. SEMICLASSICAL LIMIT

Without loss of generality the solution of the Schrödinger equation

$$(H - E)\Psi = 0 \tag{12}$$

with H in the form (1) can be written

$$\Psi(x,Q) = \sum_i \chi_i(Q)\phi_i(x,Q). \tag{13}$$

This form of expansion is exact if the ϕ_i are a complete orthonormal set of functions in the x space. The particular form of the right-hand side is chosen since it is the quantum Q variable whose classical limit will be taken.

Substitution of Eq. (13) in Eq. (12) leads to the set of coupled equations

$$\sum_i \chi_i(Q)\left[H_S + H_I - \left(E - \sum_k \frac{1}{2}\omega_k^2 Q_k^2 + \frac{1}{\chi_i}\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2}\chi_i\right) + \sum_k\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} - \frac{1}{\chi_i}\hbar^2\frac{\partial}{\partial Q_k}\chi_i\frac{\partial}{\partial Q_k}\right)\right]\phi_i(x,Q) = 0. \tag{14}$$

Equation (14) is now projected onto a state ϕ_j, i.e.,

$$\sum_k\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right)\chi_j + \sum_i\langle\phi_j|H_S + H_I|\phi_i\rangle\chi_i - \sum_{i,k}\left(\langle\phi_j|\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2}|\phi_i\rangle + \langle\phi_j|\frac{\partial}{\partial Q_k}|\phi_i\rangle\hbar^2\frac{\partial}{\partial Q_k}\right)\chi_i = E\chi_j. \tag{15}$$

These are "close-coupled" equations for the χ_j. The off-diagonal terms cause changes in the state of the e.m. field due to changes in the state of the quantum system. Neglecting all of these coupling terms gives a single-channel equation for the state χ_j of the field when the quantum system is in the state ϕ_j, i.e.,

$$\left[\sum_k\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) + E_j(Q) - E\right]\chi_j = \hbar^2\sum_k\langle\phi_j|\frac{\partial}{\partial Q_k}|\phi_j\rangle\frac{\partial}{\partial Q_k}\chi_j \tag{16}$$

with

$$E_j(Q) = \langle\phi_j|H_S + H_I - \sum_k\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2}|\phi_j\rangle. \tag{17}$$

The diagonal $\langle\phi_j|\partial/\partial Q_k|\phi_j\rangle$ terms on the right-hand side of Eq. (16) are zero for real ϕ_j and otherwise can be eliminated by a (Berry) phase transformation of χ_j. In addition, since first-order derivatives with respect to Q_k are neglected, it is consistent to neglect the second-order derivatives in the expression for E_j. Hence Eq. (16) reduces to

$$\left[\sum_k\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) + E_j(Q) - E\right]\chi_j(Q) = 0, \tag{18}$$

the defining equation for the state of the field when the quantum system is in the state described by ϕ_j.

For the complete independence of the field (specified external field) from the quantum system it is clearly necessary that $E_j(Q)$ be replaced by some fixed average potential $\overline{E}(Q)$ and correspondingly $\chi_j(Q)$ by some "mean"-field state $\chi(Q)$. This is achieved by writing

$$\chi_j(Q) = a_j(Q)\chi(Q), \tag{19}$$

where the a_j are slowly varying functions of Q. Then Eq. (13) becomes

$$\Psi(x,Q) = \chi(Q)\sum_i a_i(Q)\phi_i(x,Q),$$

$$\Psi(x,Q) = \chi(Q)\psi(x,Q), \tag{20}$$

reminiscent of a single-channel Born-Oppenheimer approximation. Correspondingly, Eq. (18) is approximated by the single-channel equation,

$$\left[\sum_k\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2}+\frac{1}{2}\omega_k^2 Q_k^2\right)+\bar{E}(Q)-E\right]\chi(Q)=0. \quad (21)$$

We have now outlined the approximations necessary to allow the exact wave function (13) to be written in the factorized form (20). Note that this step is absolutely crucial to the derivation: it reflects the assumption that the field is influenced only marginally by the atom, while the atom is influenced strongly by the field. Now we are in a position to consider the effective Schrödinger equation for the quantum system wave function ψ.

Although it is not necessary, for simplicity we will now restrict our discussion to a single field mode. The following derivation starts from a slightly different angle than both the previous approach of Ref. [7], and the one to be followed for the Jaynes-Cummings Hamiltonian in the Appendix. We give this alternative derivation here in order to pave the way for the new considerations of Sec. IV, which is based on coherent states.

In fact, we may view the last line in Eq. (20) as a general ansatz and find from the TISE

$$\begin{aligned}0=(H-E)\Psi=\chi(Q)\Bigg[H_S+H_I(x,Q)-\frac{\hbar^2}{2}\Bigg(\frac{\partial^2}{\partial Q^2}\\+2\frac{\chi'(Q)}{\chi(Q)}\frac{\partial}{\partial Q}\Bigg)\Bigg]\psi(x,Q)+\psi(x,Q)\Bigg[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2}+\frac{\omega^2}{2}Q^2\\-E\Bigg]\chi(Q). \quad (22)\end{aligned}$$

Motivated by the earlier discussion, here we split the action of the total Hamiltonian on the field and quantum system degree of freedom such that the wave function $\chi(Q)$ describes the field with an energy close to the total energy E, while the remaining part of the equation describes the dynamics of the quantum degree of freedom, involving negligible energy in comparison. This complete neglecting of the back coupling of the quantum system on the field is equivalent to neglecting the potentials $E_j(Q)$ or $\bar{E}(Q)$ and allows one to choose the wave function of the field which is to become classical, to be an energy eigenstate of the fixed field Schrödinger equation, i.e.,

$$[H_F-E]\chi=\left[\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2}+\frac{\omega^2}{2}Q^2\right)-E\right]\chi(Q)=0. \quad (23)$$

Next, as explained earlier, for these large (classical) energies, and as long as we are far away from classical turning points, we may replace the true wave function $\chi(Q)$ by its WKB expression [11]

$$\chi(Q)\simeq\exp\left\{\frac{i}{\hbar}\int^Q dQ' P(Q')\right\} \quad (24)$$

and find in leading classical order the usual wave vector $\chi'(Q)/\chi(Q)=(i/\hbar)P(Q)$ with the classical momentum at position Q determined from energy conservation,

$$P(Q)=\sqrt{2(E-\omega^2Q^2/2)}. \quad (25)$$

Thus, using Eqs. (22)–(24), we get

$$\left[H_S+H_I(x,Q)-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2}-i\hbar P(Q)\frac{\partial}{\partial Q}\right]\psi(x,Q)=0 \quad (26)$$

for the remaining part of the wave function, describing the dynamics of the quantum system. We may simplify by replacing Q by a new parameter t, defined through a trajectory $Q(t)$, and determined by

$$P(Q)\frac{\partial}{\partial Q}\equiv\frac{\partial}{\partial t}. \quad (27)$$

Clearly, $P(Q)$ from Eq. (25) is nothing but the velocity $\dot{Q}$, as determined from the classical equations of motion:

$$\dot{Q}=P \text{ and } \dot{P}=-\omega^2 Q, \quad (28)$$

with solution $Q(t)=Q_0\cos(\omega t)$, so that the parameter t is just *classical time*.

We introduce $\psi(x,t)=\psi(x,Q(t))$ and find from Eq. (26)

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi(x,t)=\Bigg[H_S+H_I(x,Q(t))+\frac{\hbar^2}{2}\Bigg(\frac{\ddot{Q}(t)}{\dot{Q}^3(t)}\frac{\partial}{\partial t}\\-\frac{1}{\dot{Q}^2(t)}\frac{\partial^2}{\partial t^2}\Bigg)\Bigg]\psi(x,t). \quad (29)\end{aligned}$$

The derivatives on the right-hand side emerge from the second-order derivative $\partial^2/\partial Q^2$ with respect to position, when expressed in terms of derivatives with respect to time t. In the limit of large ("classical" amount of) energy E, contained almost exclusively in the classical degree of freedom Q, these additional terms vanish, as we are going to show next. First, observe that $\hbar\partial/\partial t$ is of the order of the energy of the quantum system $E_S=\langle H_S+H_I\rangle$, i.e., small compared to E, if the the additional derivative terms on the right-hand side of Eq. (29) may be dropped. Now, by self-consistency, and furthermore using $\dot{Q}=P\approx\sqrt{E}$ for the harmonic oscillator away from the turning points and $\ddot{Q}\approx\omega\dot{Q}$, we can estimate the order of magnitudes: First,

$$\left\langle\frac{\hbar^2}{2}\frac{\ddot{Q}(t)}{\dot{Q}^3(t)}\frac{\partial}{\partial t}\right\rangle\approx E_S\frac{\hbar\omega}{E}, \text{ and } \left\langle\frac{\hbar^2}{2}\frac{1}{\dot{Q}^2(t)}\frac{\partial^2}{\partial t^2}\right\rangle\approx E_S^2/E$$

such that the additional derivative terms on the right-hand side of Eq. (29) are of the order $E_S((E_S+\hbar\omega)/E)$. Thus, compared to the remaining terms on the right-hand side, which are of the order E_S, the additional terms are smaller by a factor of the order $\hbar\omega/E\approx 1/n$, where n is the number of photons in the field mode. The latter is a tiny number for a classical field. Therefore it is safe to write

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=[H_S+H_I(x,Q(t))]\psi(x,t), \quad (30)$$

which is the usual TDSE for a quantum system interacting with a (single mode of the) classical electromagnetic field

with time dependence cos ωt. Here, however, the derivation rests on the TISE for both atom and field. Time emerges from a classical motion, i.e., as a derived classical parameter only. Note also that the dipole approximation is not necessary. However, the arguments above are valid only away from turning points where the velocity $\dot{Q}(t)$ is nonzero. Clearly the TDSE of Eq. (30) is valid for all times. The reason for this shortcoming is obvious: it is the choice of Q, the position (or rather quadrature) representation, for the field mode. While helpful in diagonalizing the coupling Hamiltonian H_I, in this representation the real field quadrature $Q(t)$ undergoes harmonic motion with periodic zeroes in its time derivative $\dot{Q}(t)$. Therefore the position representation fails to provide a global time. In Sec. IV it is shown how this shortcoming may be overcome by using a coherent state representation of the field state.

There is increasing emphasis, particularly for intense, few-cycle laser pulses, on the similarities of the interaction of light sources and particle beams with matter. Therefore it is of interest to contrast the derivation of the TDSE for the e.m. field interacting with a quantum system with that obtained from the interaction of a particle beam with a quantum system. In the latter case, the defining equations (18) for states χ_j of the particle beam as environment are the "perturbed stationary states" of Mott and Massey [12] describing ion-atom collisions and dating from the 1930s. These states of the particle ion beam entangled with a target atom describe the diffraction pattern of the deflection of the ion beam. Measurement of the ion-beam energy fixes the state of the atomic target (delayed choice measurement). In the case of the photon field χ_j gives the amplitude (i.e., occupation number) of each mode k and measurement of the field energy gives the excitation state of the atom. In the TISE energy is fixed.

In the case of the ion beam, the classical limit in which the beam is decoupled from the atom corresponds to classical motion along a fixed trajectory. This is the classical limit of Eq. (21) in which the field Hamiltonian is replaced simply by the kinetic energy of the beam which moves in the fixed potential $\bar{E}$ provided by the atom. The energy of the beam is fixed, i.e., energy transfer to or from the quantum system is ignored in deciding the ion-beam motion. The analogy in the case of the field is the classical limit of Eq. (21) in which the field is described by a classical solution of Maxwell's equations in the oscillator "potential" of H_F plus the polarization field $\bar{E}(Q)$ of the atom. Again the field solutions are at fixed energy. By contrast, in both cases the quantum system is described by the TDSE (30) in which changes in both in state and energy are induced by the field, explicitly through $H_I(t)$ and implicitly through the field motion $\partial/\partial t = \dot{Q}(\partial/\partial Q)$.

When the ion-beam energy is very high, such that the kinetic energy is much greater than $\bar{E}$, it is a good approximation (much used in atomic collisions) to ignore $\bar{E}$. Then the beam motion is a straight line trajectory, completely independent of the quantum system. The corresponding approximation in the field case is to ignore $\bar{E}$ in Eq. (21). Then the classical limit is simply the free field modes $Q_k(t)$ of the cavity, without influence of the radiation field of the atom, as used in Eq. (23). Plainly, this is a good approximation when the total field energy $\Sigma_k n_k \hbar \omega_k$ greatly exceeds $\bar{E}$. As we have seen in (30), this limit corresponds, for example in dipole coupling, to putting

$$H_I(t) \propto \underline{r} \cdot \underline{E}(\underline{r}) \cos(\omega t) \tag{31}$$

(where $\underline{r}$ is the quantum system dipole operator and $\underline{E}$ the electric field amplitude), an approximation usually used in laser-atom interactions.

IV. COHERENT STATE DERIVATION

Coherent states are the classical states of field modes, corresponding to minimal uncertainty in both field quadratures Q and P. Therefore, it is not too surprising that they also play a role in the derivation of a time dependent Schrödinger equation for the quantum degree of freedom from a time independent Schrödinger equation for both the coupled quantum system and field mode. We should emphasise that the classical harmonic motion of the field amplitudes can be used as clock, provided by the classical system to monitor the quantum system with which it interacts. It resembles a real clock more naturally than the linear, unbound motion of a scattered particle, which was previously employed [7] to define time for an interacting quantum system.

We use usual annihilation and creation operators as in (5), and write the total Hamiltonian (4) (for a single field mode) in the form

$$H_F = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

$$H_I = \hbar S(a + a^\dagger) \tag{32}$$

with the atomic operator $S = \Sigma_{ij} g_{ij} c_i^\dagger c_j$ from Eq. (4).

As in the last section, we set out to solve the time-independent equation $[H-E]\Psi = 0$, here, however in coherent state representation. Coherent states $|\alpha\rangle = e^{-|\alpha^2|/2+\alpha a^\dagger}|0\rangle$ are eigenstates of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$, labeled by a complex number α. Crucial properties that are relevant for us are [10]

$$\langle\alpha|a^\dagger = \alpha^* \langle\alpha|, \tag{33}$$

$$\langle\alpha|a = \left(\frac{\partial}{\partial\alpha^*} + \frac{1}{2}\alpha\right)\langle\alpha|.$$

Coherent states form an overcomplete basis, $\mathbb{1} = \int (d^2\alpha/\pi)|\alpha\rangle\langle\alpha|$ with the overlap between two coherent states given by $\langle\alpha|\beta\rangle = \exp\{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\}$. Any state may be expanded in a coherent state representation according to $|\chi\rangle = \int (d^2\alpha/\pi)\tilde{\chi}(\alpha,\alpha^*)|\alpha\rangle$ with $\tilde{\chi}(\alpha,\alpha^*) = \langle\alpha|\chi\rangle$. Note that we can always write $\tilde{\chi}(\alpha,\alpha^*) = \exp(-\frac{1}{2}|\alpha|^2)\chi(\alpha^*)$, with a function $\chi(\alpha^*)$ depending on α^* only, highlighting the particular dependence of $\langle\alpha|\chi\rangle$ on α and its complex conjugate α^* (see Bargmann [14]).

In contrast to the position (Q) representation used in the previous sections, we employ the above properties of coher-

ent states to write the TISE in coherent state representation with respect to the classical degree of freedom. We start from

$$\langle\alpha|\Psi\rangle = \exp\left\{-\frac{1}{2}|\alpha|^2\right\}\chi(\alpha^*)|\psi(\alpha^*)\rangle, \tag{34}$$

implying as discussed after Eq. (22) that we consider $\exp\{-\frac{1}{2}|\alpha|^2\}\chi(\alpha^*)$ to describe the classical degree of freedom, while the dependence of the quantum part $|\psi(\alpha^*)\rangle$ on the classical degree of freedom is only secondary, without any back reaction from the quantum part onto the classical degree of freedom.

Before proceeding we should mention another important property of coherent states, namely, that their overlap with a photon number state is $\langle\alpha|n\rangle = \exp\{-|\alpha|^2/2\}(\alpha^*)^n/\sqrt{n!}$ In the relevant limit of large photon number n, it is easy to see (and physically obvious through energy considerations) that only those coherent states $|\alpha\rangle$ with

$$|\alpha|^2 = n \tag{35}$$

will contribute significantly to the number state $|n\rangle$.

With Eq. (34), the TISE with the Hamiltonians H_F and H_I from Eq. (32) reads

$$0 = \langle\alpha|(H-E)|\Psi\rangle = \chi(\alpha^*)\left[H_S + \hbar g\left(\alpha^* + \frac{\chi'(\alpha^*)}{\chi(\alpha^*)} + \frac{\partial}{\partial\alpha^*}\right)\sigma_x + \hbar\omega\alpha^*\frac{\partial}{\partial\alpha^*}\right]|\psi(\alpha^*)\rangle + |\psi(\alpha^*)\rangle \times\left[\hbar\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*} + \frac{1}{2}\right) - E\right]\chi(\alpha^*). \tag{36}$$

This is to be compared to the corresponding Eq. (22) of the position space approach. Just as before, we chose χ to be an eigenstate of the field Hamiltonian, i.e., to be a number state $|n\rangle$ with energy $E=\hbar\omega(n+\frac{1}{2})$. In coherent state representation,

$$\chi(\alpha^*) = (\alpha^*)^n/\sqrt{n!} \tag{37}$$

Indeed, one easily confirms that this choice ensures that the second part of Eq. (36) disappears. For the remaining equation we need the logarithmic derivative $\chi'(\alpha^*)/\chi(\alpha^*)$, which, by virtue of Eqs. (37) and (35) may be written as

$$\frac{\chi'(\alpha^*)}{\chi(\alpha^*)} = \frac{n}{\alpha^*} = \alpha. \tag{38}$$

Thus from Eqs. (36) and (38) we read off the defining equation for the quantum system,

$$\left[H_S + \hbar S\left(\alpha^* + \alpha + \frac{\partial}{\partial\alpha^*}\right) + \hbar\omega\alpha^*\frac{\partial}{\partial\alpha^*}\right]|\psi(\alpha^*)\rangle = 0, \tag{39}$$

which should be compared with Eq. (26) of the position space approach. In complete analogy we now introduce a new parameter t, replacing α^*. It is defined through a complex trajectory $\alpha^*(t)$ for the coherent state field amplitude, such that the first-order derivative term defines the time development:

$$\hbar\omega\alpha^*\frac{\partial}{\partial\alpha^*} \equiv -i\hbar\frac{\partial}{\partial t}. \tag{40}$$

Thus time is determined from the classical, here harmonic, motion of the field amplitude, with

$$\alpha(t) = \alpha_0 e^{-i\omega t}. \tag{41}$$

Crucially, while the position space expression (27) fails to provide a time near the classical turning points due to $P(Q)=0$, the coherent state expression (40) remains finite for all times. Moreover, it is remarkable to see that the coherent state equation (39) contains first-order derivatives only. Naturally, it closely resembles the first-order time-dependent Schrödinger equation, whereas the position space expression (26) is of second order.

We still have not quite achieved our goal: with $\partial/\partial\alpha^*$ replaced by $(1/\dot{\alpha}^*)\partial/\partial t$, we see that the field part of the interaction Hamiltonian becomes $[\alpha^* + \alpha - \alpha(\hbar\omega|\alpha|^2)^{-1}(i\hbar\partial/\partial t)]$. With $1/|\alpha|^2 \approx 1/n$ vanishing in the large photon number limit $n\to\infty$, the additional time derivative vanishes, which may be compared to the similar discussion after Eq. (29).

Therefore we find for the state $|\psi(t)\rangle = |\psi(\alpha^*(t))\rangle$, in the limit of large photon number, the TDSE

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = [H_S + \hbar S(\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t})]|\psi(t)\rangle, \tag{42}$$

describing the interaction of the atom with a time-dependent classical e.m. field. If a two-level approximation is valid, we have $S=g\sigma_x$. In contrast to the identical position space result (30), however, the coherent state derivation of Eq. (42) holds for all times. We emphasize again that we started from the time-independent Schrödinger equation, the time parameter in Eq. (42) being nothing but a convenient label for the classical harmonic motion of the e.m. field amplitude $\alpha(t)$.

V. CONCLUSION

It has been shown earlier [7] that the time-dependent Schrödinger equation may be regarded as a (semi)classical approximation of the time-independent Schrödinger equation in one higher dimension. In the limit that this variable becomes classical, the dynamics of the "classical" degree of freedom provides a "clock" for the remaining quantum part. In this work, the idea of a "clock" as a periodic physical system rather than a linearly extended one as in [7] has been investigated in detail. This is the case when the extra degree of freedom corresponds to a quantized electromagnetic field whose classical limit is oscillatory motion of the field amplitude. A key step in the quantum to classical transition has been shown to be the "single-channel approximation" which amounts to an approximate disentanglement of the quantum field state from that of the atom and a semiclassical treatment of the ensuing field dynamics. However, the usual semiclassical approximation breaks down at turning points. It has

been shown that an approach based on coherent states circumvents this problem. Hence the derivation is valid for all times rather than restricted to those time spans well between the classical turning points. The appearance of coherent states comes as no surprise, as they are the classical states of a harmonic oscillator. In the Appendix we illustrate our approach with the Jaynes-Cummings model, whose large photon-number dynamics, described by the time-dependent Schrödinger equation with a classical driving field, is here derived from the time-independent Schrödinger equation where both atom and field are treated quantum mechanically.

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APPENDIX: JAYNES-CUMMINGS MODEL

In the field case it is of interest to develop the specific example of the Jaynes-Cummings model [9] in more detail, since it has been so widely used. We follow the general discussion of the first half of Sec. III. If the quantum system is a two-level atom coupled to a laser field consisting of one quantized mode only, the total Hamiltonian is of the form (1) and is written

$$H = H_S + H_F + H_I$$

$$\text{with } H_S = \tfrac{1}{2}\hbar\omega_0\sigma_z,$$

$$H_F = \tfrac{1}{2}(P^2 + \omega^2 Q^2),$$

$$H_I = \sqrt{2\hbar\omega}gQ\sigma_x. \tag{A1}$$

Here, the energy of a single photon of the mode is $\hbar\omega$, $\hbar\omega_0$ is the energy of the atomic transition, and σ_x, σ_z are (2×2) Pauli matrices. This Hamiltonian is written in Eq. (5) in terms of creation and annihilation operators by using the transformation on the field quadratures,

$$Q = \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger),$$

$$iP = \sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger). \tag{A2}$$

The key question in practical applications is the choice of the complete set ϕ_j of functions in which to expand the state of the quantum system. In the case of the two-level atom one requires two states $|+,Q\rangle$ and $|-,Q\rangle$ corresponding to the atom in the upper or lower state of the doublet, i.e.,

$$\Psi = \chi_+(Q)|+,Q\rangle + \chi_-(Q)|-,Q\rangle = \chi(Q)(a_+(Q)|+,Q\rangle + a_-(Q)\times|-,Q\rangle). \tag{A3}$$

Two choices of the "atomic" basis are practically useful. One is the adiabatic basis in which H_S+H_I is diagonalized. In the Jaynes-Cummings model these potentials have been studied by Graham and Höhnerbach [13]. An alternative, used frequently in laser-atom interactions, is to take the unperturbed atomic eigenstates $|+\rangle$, $|-\rangle$ as basis ϕ_j, independent of Q. Then Eq. (A3) becomes

$$\Psi = \chi(Q)(a_+(Q)|+\rangle + a_-(Q)|-\rangle). \tag{A4}$$

This wave function for both the field and the two-level system is of the form (20). We substitute Eq. (A4) into the TISE $(H-E)\Psi=0$ with the Hamiltonian H given in Eq. (A1) and project onto the wave function of the system $\psi=a_+(Q)|+\rangle + a_-(Q)|-\rangle$.

Neglecting all derivatives of the coefficients $a_\pm(Q)$ with respect to Q, we find the effective equation for the field [cf. Eq. (21)]

$$\left(\frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2 + \overline{E}(Q) - E\right)\chi(Q) = 0, \tag{A5}$$

with

$$\overline{E}(Q) = \langle\psi|H_S + H_I|\psi\rangle = |a_+|^2(Q)E_+(Q) + |a_-|^2(Q)E_-(Q) + a_+^*(Q)a_-(Q)\langle+|H_I|-\rangle + \text{c.c.} \tag{A6}$$

Here, the potentials $E_\pm(Q)$ are defined in Eq. (17) neglecting the $\partial^2/\partial Q^2$ terms. Following the discussion after Eq. (14), and using the form (A4) of the wave function allows us to neglect the off-diagonal terms $\langle+|H_I|-\rangle$ in Eq. (A6). Therefore the average potential reduces to

$$\overline{E}(Q) = |a_+|^2(Q)E_+(Q) + |a_-|^2(Q)E_-(Q) = \tfrac{1}{2}\hbar\omega_0[|a_+|^2(Q) - |a_-|^2(Q)]. \tag{A7}$$

Clearly, as $E\to\infty$, the energy $\hbar\omega_0$ and thus $\overline{E}(Q)$ may be neglected with respect to E and the field energies are those of the free mode, completely independent of any influence of the two-level atom. This is the high-energy situation where the number n of photons far exceeds unity and therefore is unaltered by emission or absorption of a single photon by the atom. In the particle beam case, as explained above, the analog is a plane wave for the particle beam, i.e., zero effective potential $\overline{E}(Q)$.

Although in the above approximation the field is unaffected by the atom, the atom is still strongly affected by the field. Substitution of Eq. (A4) into Eq. (22) and making the same approximations that led to Eq. (30) gives the coupled equations for the atom alone,

$$\left(\frac{1}{2}\hbar\omega_0\sigma_z + \sqrt{2\hbar\omega}gQ(t)\sigma_x\right)\begin{pmatrix}a_+(t)\\ a_-(t)\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}a_+(t)\\ a_-(t)\end{pmatrix}. \tag{A8}$$

This is exactly the TDSE obtained by assuming at the outset a classical field with amplitude $Q(t)$ driving the two-level atom whose state vector is expanded in the unperturbed basis, i.e.,

$$|\psi(t)\rangle = a_+(t)|+\rangle + a_-(t)|-\rangle. \tag{A9}$$

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