Retrodiction for optical attenuators, amplifiers, and detectors

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The transformation that an attenuator makes on the state of an optical field is the time reverse of that of an amplifier. Thus predicting the output state for an amplifier is equivalent to retrodicting the input state of an attenuator. We explore the consequences of this equivalence for simple optical quantum communication channels. One counterintuitive consequence is that the mean number of photons sent into an amplifier as retrodicted from a measurement of the number of output photons does not include the contribution of the amplifier noise.

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I. INTRODUCTION

A description of the output state of a quantum communication channel, based on knowledge of the input state and the initial state of the environment interacting with the channel, requires the use of predictive quantum mechanics. The combined initial state evolves forward in time unitarily under the total Hamiltonian for the system and environment, and the environmental trace is taken to give an evolved state of the system alone. In this way we can calculate predictive conditional probabilities of future measurement events based on knowledge of states prepared in the present (preparation events).

This is *not* the most natural way to analyze communications systems, however. In general, the communication problem is to determine the message sent from the message received. In quantum communications this translates into finding the probabilities that particular states were sent in the past, given the results of measurements in the present. The best way to do this is to use retrodictive quantum mechanics [1–7]. In this more unusual formulation the state of the system at any time between preparation and measurement is the *measured* state evolved backwards in time. The retrodictive conditional probability that a particular state was prepared is found by projecting this back-evolved retrodictive state onto the possible prepared states.

It is possible to use predictive quantum mechanics combined with Bayes's theorem, which relates predictive and retrodictive conditional probabilities [8], to analyze the communication problem. We can calculate retrodictive conditional probabilities by calculating the evolved predictive state for each possible prepared state and projecting these onto the measurement result to obtain a complete set of predictive conditional probabilities. This is hardly an efficient strategy, however, as there could be many possible prepared states. Classical computers are inefficient at modeling quantum systems. In open systems particularly this approach will be calculationally intensive. It is much more efficient to perform one retrodictive calculation to find the retrodictive state and project this onto the possible prepared states.

All quantum communication channels are currently optical and hence the study of quantum optical devices forms an important part of quantum communication theory. Attenuating and amplifying devices are essential constituents of optical communication channels. Lossy phenomena are always present in such systems and are caused by signal scattering in real fibers, transmission in nontransparent materials, or detection of the signal by an imperfect apparatus with quantum efficiency smaller than unity. Optical amplifiers, on the other hand, are used in communications to increase the amplitude of the transmitted optical signal when needed [9]. Although classical amplification and attenuation are reverse processes, in quantum mechanics following an attenuator by an amplifier does not restore the original signal [10]. However, it has been shown that retrodiction through an attenuating (amplifying) channel corresponds to prediction through an amplifying (attenuating) channel [11]. This correspondence allows us to apply the more familiar predictive quantum theory to a retrodictive system. Here we demonstrate the equivalence between the two systems by deriving the predictive and retrodictive matrix elements for the density operators describing the single mode field for an attenuating and an amplifying device, respectively. We use these to explore the consequences of this equivalence, concentrating particularly on realistic measurement schemes and possible prepared states. The quality of an attenuator or amplifier is characterized by the characteristic temperature of the environment with which the signal field mode interacts. We concentrate mainly on the ideal case where the environment is at zero temperature ($T=0$ K), before briefly dealing with the case where $T\neq 0$ K.

In Sec. II we provide a brief introduction to retrodictive quantum mechanics. The most general expressions for the predictive and retrodictive density-matrix elements for amplifiers and attenuators are derived in Sec. III, and the retrodictive elements are obtained using a simple method previously used to calculate retrodictive atomic states [12]. In Sec. IV we apply the results to particular input photon probability distributions and to photocounting experiments. In Sec. V we calculate predictive and retrodictive density-matrix elements for the field for imperfect photodetection,

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caused by a nonunit detector quantum efficiency and/or the presence of dark counts. In Sec. VI, we show by means of a particular example and by using the field density-matrix elements that the predictive-retrodictive equivalence between attenuators and amplifiers is also valid for nonideal devices with environments at a nonzero temperature; we further illustrate this property by using a master equation approach. The conclusions are given in Sec. VII.

II. PREDICTIVE AND RETRODICTIVE QUANTUM THEORY IN OPEN SYSTEMS

In this section we summarize predictive and retrodictive quantum theory in open systems [6,7]. Discussion of predictive and retrodictive master equations [7,13] is left until Sec. VI.

An open quantum system is one in which the system of interest is coupled to a large unmeasured set of systems called the environment. Almost every real quantum system is open, but systems coupled to amplifying or attenuating channels are explicitly so. Suppose that states are prepared at time t_p, undergo evolution for a time τ, and are then measured at time $t_m = t_p + \tau$. The conditional probability that the measurement result j is obtained given that the system was prepared in state $\hat{\rho}_i(t_p)$ is the trace over the system states of this density operator projected onto the element $\hat{\Pi}_j$ of the probability operator measure (POM) describing the measurement [6,7,14]

$$P(j|i) = \mathrm{Tr}_{\mathrm{S}}[\hat{\rho}_i^{pred}(t_m)\hat{\Pi}_j(t_m)]. \tag{1}$$

The state of the system at the measurement time t_m is given by the predictive density operator, i.e., the prepared state coupled to the environment, evolved forward in time, and conditioned on the result of a possible measurement of the environment,

$$\hat{\rho}_i^{pred}(t_m) = \mathrm{Tr}_{\mathrm{E}}[\hat{\Pi}_{\mathrm{E}}(t_m)\hat{U}(\tau)\hat{\rho}_i(t_p) \otimes \hat{\rho}_{\mathrm{E}}(t_p)\hat{U}^{\dagger}(\tau)]. \tag{2}$$

Here $\hat{\rho}_E(t_p)$ is the state of the environment at time t_p and $\hat{U}(\tau)$ is the unitary operator which determines the evolution of the coupled system and environment. $\hat{\Pi}_j(t_m)$ is the POM element corresponding to measurement result j. A POM is a mathematical representation of a measuring device, a set of elements $\hat{\Pi}_j$, each of which is a positive semidefinite operator associated with a different possible result j of the measurement, and which satisfies

$$\sum_j \hat{\Pi}_j = \hat{1}_S, \tag{3}$$

the unit operator in the system space. This condition ensures that the probabilities of all the possible measurement results sum to unity. For a simple von Neumann measurement [15], the POM elements are projectors onto the eigenstates of the measured observable operator, and therefore their trace is also normalized. The environment is usually unmeasurable and so in Eq. (2) $\hat{\Pi}_{\mathrm{E}}(t_m) = \hat{1}_{\mathrm{E}}$, the unit operator in the environment space.

In a quantum communication system, the sender does not send the same state every time, as this conveys no information. One state from a set of possible states is prepared and the particular state sent must be determined from the measurement result. The best way to do this is to use retrodictive quantum theory. The POM element of the system at any time prior to t_m is the measured POM element, coupled to the environment state and evolved backwards in time, conditioned on the state of and traced over the environment. The POM at the preparation time is

$$\hat{\Pi}_j(t_p) = \mathrm{Tr}_{\mathrm{E}}[\hat{\rho}_{\mathrm{E}}(t_p)\hat{U}^{\dagger}(\tau)\hat{\Pi}_j(t_m) \otimes \hat{\Pi}_{\mathrm{E}}(t_m)\hat{U}(\tau)], \tag{4}$$

and we can calculate the predictive conditional probability as

$$P(j|i) = \mathrm{Tr}_{\mathrm{S}}[\hat{\rho}_i^{pred}(t_p)\hat{\Pi}_j(t_p)]. \tag{5}$$

In fact this equation is valid for density operators and POM elements evaluated at any intermediate time t between preparation and measurement, with $\hat{\rho}_i^{pred}(t_p)$ evolved forward in time from t_p to t, and $\hat{\Pi}_j(t_m)$ evolved backwards in time from t_m to t. This amounts to an invariance of the conditional probability on the time of collapse of the system. In other words we may include the evolution as part of the system preparation, as part of the measurement, or it may be divided between the two processes. If required the retrodictive state can be found from the POM element using

$$\hat{\rho}_j^{retr}(t) = \hat{\Pi}_j(t)/\mathrm{Tr}_{\mathrm{S}}[\hat{\Pi}_j(t)]. \tag{6}$$

Suppose that we wish to calculate retrodictive conditional probabilities. These must take account of the *a priori* probabilities $P(i)$ that particular states are prepared. These probabilities are encoded in the *a priori* prepared state

$$\hat{\Lambda}(t_p) = \sum_i \hat{\Lambda}_i(t_p) = \sum_i P(i)\hat{\rho}_i^{pred}(t_p). \tag{7}$$

The retrodictive conditional probability that i was prepared given that j was measured is then

$$P(i|j) = \frac{\mathrm{Tr}[\hat{\Lambda}_i\hat{\Pi}_j]}{\mathrm{Tr}[\hat{\Lambda}\hat{\Pi}_j]} = \frac{\mathrm{Tr}[\hat{\Lambda}_i\rho_j^{retr}]}{\mathrm{Tr}[\hat{\Lambda}\rho_j^{retr}]}, \tag{8}$$

where again the operators can be evaluated at any time between preparation and measurement. In this way the receiver can construct the message with the minimum of calculation. If the preparation is *unbiased*, in other words each of the states spanning the system is equally likely to be prepared, the *a priori* prepared state, Eq. (7), becomes the unit operator for the state space of the system, $\hat{\Lambda} = \hat{1}/D$, where D is the dimension of the state space of the system. The retrodictive conditional probability, Eq. (8), can now be written

$$P(i|j) = \mathrm{Tr}[\hat{\Lambda}_i\hat{\rho}_j^{retr}], \tag{9}$$

thus displaying a symmetric form to that of the predictive conditional probability. Unbiased preparations are not the norm in physics, but they do occur in quantum communications, for example in the BB84 protocol for quantum key distribution [16].

III. PREDICTIVE AND RETRODICTIVE DENSITY-MATRIX ELEMENTS FOR IDEAL ATTENUATORS AND AMPLIFIERS

The effect of attenuation or amplification of an optical signal is reflected in the expression for the density operator of the field after transmission through the attenuating or amplifying device. Information about the signal transformation is given by the input-output relations for the matrix elements of the field density operator. We derive these input-output relations in the photon-number state basis, by using in both cases beam-splitter models [17,18]. Although these models use a single mode description of the environment field, the resulting expressions have general validity. Once the predictive reduced density operators for these open systems are known, the retrodictive density-matrix elements for the signal field are easily derived by means of a simple calculational tool [12].

A. Attenuator

Light traveling through an optical fiber exhibits a power that decreases exponentially with distance, as a result of the absorption in the fiber's material (usually fused silica glass) and of the Rayleigh scattering due to the random inhomogeneities of the material refractive index [9]. The attenuation process limits the magnitude of the optical power transmitted and the transmission factor K of the device is smaller than unity.

An optical attenuator can be modeled by an ensemble of two-level atoms. If there are N_{g} atoms in the lower (ground) atomic level and N_{e} atoms in the excited level, and the atoms are at temperature T the population ratio is given by a Boltzmann factor

$$\frac{N_{\mathrm{g}}}{N_{\mathrm{e}}} = e^{\hbar\omega/k_B T}, \quad \text{with} \quad T \geqslant 0. \tag{10}$$

The atoms absorb some of the field energy, but also add noise photons with distribution given by the thermal excitation function [18,19]

$$N_{att}(T) = \frac{N_{\mathrm{e}}}{N_{\mathrm{e}} - N_{\mathrm{g}}} = \frac{1}{1 - e^{\hbar\omega/k_B T}} \geqslant 0. \tag{11}$$

For a zero temperature attenuator the atomic population is completely uninverted with $N_{att}=0$. In fact the theoretical model of the damped harmonic oscillator gives the simplest description for the attenuation process [20]. This can be generally constructed by coupling one oscillator to a bath of oscillators, similar to the case of a single field mode resonating in a microcavity where the presence of the damping reservoir is responsible for the finite quality factor of the cavity [21].

Our model for the attenuating device is schematized in Fig. 1. We denote by $\hat{a}$ and $\hat{a}^\dagger$ the annihilation and creation operators describing the signal field mode, and by $\hat{b}$ and $\hat{b}^\dagger$ the operators associated with the environment, here assumed to be single mode and in the vacuum state $(T=0)$. The results derived take the same form for a multimode environment. If K is the attenuation factor of the system, we can define the

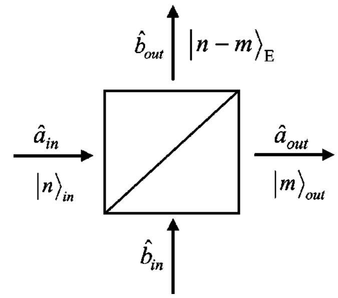

FIG. 1. Attenuating beam-splitter model.

transmission and the reflection coefficients of the beam splitter ($\mathcal{T}$ and $\mathcal{R}$, respectively) in terms of K as $\mathcal{T}=\sqrt{K}$ and $\mathcal{R}=i\sqrt{1-K}$. These coefficients satisfy the usual conditions for the validity of the field-operator commutation relations:

$$|\mathcal{T}|^2 + |\mathcal{R}|^2 = 1, \qquad TR^* + \mathcal{T}^*\mathcal{R} = 0. \tag{12}$$

The relation between the input signal operator and the output operators for a lossless attenuating beam splitter [18] is

$$\hat{a}_{in}^\dagger = \sqrt{K}\hat{a}_{out}^\dagger + i\sqrt{1-K}\hat{b}_{out}^\dagger. \tag{13}$$

As illustrated in Fig. 1, we denote by m (and m') the quantum numbers associated with the output of the attenuating channel, and by n (and n') the quantum numbers associated with the input. We denote by $\hat{\rho}_{att}$ the density operator describing the signal field sent into the attenuator at time $t=0$, and can write

$$ {}_{out}\langle m|\hat{\rho}_{att}|m'\rangle_{out} = \mathrm{Tr}_{\mathrm{E}} \sum_{n,n'} {}_{out}\langle m|n\rangle_{in\,in}\langle n|\hat{\rho}_{att}|n'\rangle_{in\,in}\langle n'|m'\rangle_{out}, \tag{14}$$

where the property of completeness of the photon-number states has been used twice. No measurement is performed on the environment output mode, and so we trace over the environment states. The input-output relation (14) for the signal field can be evaluated once the terms ${}_{out}\langle m|n\rangle_{in}$ and ${}_{in}\langle n'|m'\rangle_{out}$ are explicitly calculated. This can be done by writing

$$|n\rangle_{in} = \frac{1}{\sqrt{n!}}(\hat{a}_{in}^\dagger)^n|0\rangle = \sum_{l=0}^{n} i^{n-l}\sqrt{\frac{n!}{l\,!\,(n-l!)}}K^{l/2} \times (1-K)^{(n-l)/2}|l\rangle_{out}|n-l\rangle_{\mathrm{E}}, \tag{15}$$

where Eq. (13) has been used, and where the subscript *out* is associated with the signal output and the subscript E with the unmeasured environment channel. We then easily find, inserting this result in Eq. (14),

$$
{}_{out}\langle m|\hat{\rho}_{att}|m'\rangle_{out} \equiv \langle m|\hat{\rho}^{pred}_{att}|m'\rangle = \sum_{\substack{n=m\\ n'=m'}}^{\infty}\left(\frac{n\,!\,n'\,!}{m\,!\,m'\,!\,(n-m)\,!\,(n'-m')!}\right)^{1/2} \times K^{(m+m')/2}(1-K)^{(n-m+n'-m')/2} \times \delta_{n-m-n'+m'}\,{}_{in}\langle n|\hat{\rho}_{att}|n'\rangle_{in}. \quad (16)
$$

The above expression is the general predictive input-output relation for the density-matrix elements describing an attenuated signal field. Here K can be written in terms of an attenuation coefficient α as $K=e^{-\alpha z}$ (z being the propagation distance) if α is in km^{-1} units, or as $K=e^{-0.23\alpha z}$ if α is in dB/km units [9], or as $K=e^{-2\gamma t}$ if the attenuation is characterized by a dissipation rate 2γ of a lossy medium [21]. We will use the temporal dependence here.

From Eq. (16) both the diagonal and off-diagonal elements of the reduced density operator $\hat{\rho}_{att}$ can be found. If we set $n=n'$, and therefore $m=m'$ in Eq. (16), we obtain the output state amplitude, but not phase, for a general superposition of photon-number input states. We have in this case

$$
{}_{out}\langle m|\hat{\rho}_{att}|m\rangle_{out} = \sum_{n=m}^{\infty}\binom{n}{m}K^{m}(1-K)^{n-m}\,{}_{in}\langle n|\hat{\rho}_{att}|n\rangle_{in}, \quad (17)
$$

which shows that the effect of the attenuator is equivalent to Bernoulli sampling [21].

The general predictive density operator describing the output field after the attenuation process is given by

$$
\hat{\rho}^{pred}_{att} = \sum_{m,m'=0}^{\infty}\sum_{\substack{n=m\\ n'=m'}}^{\infty}\rho_{nn'}(0)\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2} \times (1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}|m\rangle\langle m'|, \quad (18)
$$

where we have defined $\rho_{nn'}(0)={}_{in}\langle n|\hat{\rho}_{att}|n'\rangle_{in}$, the matrix element prior to any evolution ($t=0$) describing the field before transmission through the device. Note that by rearranging the summations in Eq. (18), the latter can be re-expressed as

$$
\hat{\rho}^{pred}_{att} = \sum_{n,n'=0}^{\infty}\sum_{m=0}^{n}\sum_{m'=0}^{n'}\rho_{nn'}(0)\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2} \times (1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}|m\rangle\langle m'|, \quad (19)
$$

so that the values of n and n' are unrestricted. The restriction appears in the output quantum numbers m and m'. Since the coefficient K is implicitly a function of time, $\hat{\rho}^{pred}_{att} \equiv \hat{\rho}^{pred}_{att}(t)$ is a time-dependent operator.

We can use retrodictive quantum mechanics [6,7] to obtain information about the state of the field at a time preceding the measurement time, on the basis of the measurement result. In order to do this we utilize a simple calculational tool previously used to find retrodictive atomic states [12], in order to derive the retrodictive density-matrix elements for this system directly, given knowledge of its predictive evolution. Note that Eq. (19) is also formally equivalent to Eq. (2), which can be recast as

$$
\hat{\rho}^{pred}_{att} = \mathrm{Tr}_{\mathrm{E}}[\hat{U}_{att}\hat{\rho}_{att} \otimes \hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{att}], \quad (20)
$$

where $\hat{U}_{att}$ is the unitary operator describing the evolution of the system associated with the beam-splitter transformation [11]. The probability of getting an outcome j from the measurement performed at the output channel of the attenuator is, with the use of Eq. (19),

$$
\mathrm{Tr}_{\mathrm{ES}}[\hat{\Pi}_j\hat{U}_{att}\hat{\rho}_{att} \otimes \hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{att}] = \sum_{n,n'=0}^{\infty}\sum_{m=0}^{n}\sum_{m'=0}^{n'}\rho_{nn'}(0)\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2} \times (1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}\langle m'|\hat{\Pi}_j|m\rangle. \quad (21)
$$

The retrodictive matrix elements (and consequently the retrodictive density operator) can now be directly derived using the method found in Ref. [12],

$$
\langle n'|\hat{\rho}^{retr}_{att}|n\rangle = \frac{1}{\mathcal{N}_{att}}\frac{\partial}{\partial\rho_{nn'}}\mathrm{Tr}_{\mathrm{ES}}[\hat{\Pi}_j\hat{U}_{att}\hat{\rho}_{att} \otimes \hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{att}], \quad (22)
$$

where $\mathcal{N}_{att}$ is the normalization constant. The trace factor being simply the expectation value of the product of the POM element and the evolved predictive density operator, in general this equation allows us to obtain retrodictive density operators for a particular system from the solution of the predictive master equation describing its evolution. This straightforward connection between the predictive and the retrodictive evolutions of a system is in perfect analogy with Bayes' theorem [6,8], which in probability theory relates the probabilities of later events given probabilities of earlier events (predictive conditional probabilities), to the probabilities of earlier events given probabilities of later events (retrodictive conditional probabilities). With the summations expressed as in Eqs. (19) and (21), the evaluation in Eq. (22) of the derivatives with respect to $\rho_{nn'}$ is straightforward and we find

$$
\langle n'|\hat{\rho}^{retr}_{att}|n\rangle = \frac{1}{\mathcal{N}_{att}}\sum_{m=0}^{n}\sum_{m'=0}^{n'}\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2} \times (1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}\langle m'|\hat{\Pi}_j|m\rangle. \quad (23)
$$

The normalization constant is easily calculated,

$$
\mathcal{N}_{att} = \sum_{m=0}^{\infty}\langle m|\hat{\Pi}_j|m\rangle\sum_{n=m}^{\infty}\binom{n}{m}(1-K)^{n-m}K^{m} = \mathrm{Tr}\hat{\Pi}_j/K. \quad (24)
$$

We can substitute this in Eq. (23), and use the definition of the retrodictive density matrix (6) to find

$$\langle n'|\hat{\rho}_{att}^{retr}|n\rangle = K\sum_{m=0}^{n}\sum_{m'=0}^{n'}\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2} \times(1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}\langle m'|\hat{\rho}_j^{retr}|m\rangle. \quad (25)$$

The diagonal matrix element obtained by setting $n=n'$ (and thus $m=m'$) in Eq. (25) represents the retrodictive photon-number probability distribution, giving information on the field amplitude before the measurement and before the attenuating process. If K is time dependent, the retrodictive probability at a time t before the measurement time is given by the diagonal part of Eq. (25).

B. Amplifier

Laser amplifiers are important components of fiber-optic communication systems. They are generally constituted by Erbium-doped silica fibers for light propagation at $\lambda = 1.55\ \mu$m and offer high-gain amplification with low noise. An Erbium-doped fiber amplifier may be used as an optical-power amplifier placed directly at the output of the source laser, or as an optical preamplifier at the photodetector input. More generally amplifiers can also serve as all-optical repeaters, replacing the electronic repeaters that provide reshaping, retiming, and regeneration of the signal bits [9].

The simplest theoretical configuration for an amplifier consists of an assembly of two-level atoms with an inverted population. This is the usual model of the laser, and its linear operation regime, well below threshold, describes an amplifier. For the amplification process to occur, the population factor characteristic of standard theory satisfies [18,19,22–24]

$$N_{amp}(T) = \frac{1}{1-e^{\hbar\omega/k_B T}} \geqslant 1, \quad \text{but with } T \leqslant 0. \quad (26)$$

It can be related to the thermal factor [12] for an attenuator by

$$N_{amp}(-T) = N_{amp}(|T|) = \frac{e^{\hbar\omega/k_B T}}{e^{\hbar\omega/k_B T}-1} = 1 - N_{att}(T). \quad (27)$$

In this section we consider a linear optical amplifying channel modeled by a beam splitter, in which the signal entering the device from one input port is coupled to an inverted harmonic oscillator, the latter representing the source of noise entering the system from the other input [20,23,24]. Again we concentrate on the ideal case where the environment is at zero temperature (strictly $T\to -0$, also corresponding to a fully inverted atomic population with $N_{amp}=1$). Thus the state of the environment is different from that of the zero temperature attenuator. Here the inverted harmonic oscillator has no ground state, but it has a state of maximum energy associated with the quantum number 0. The scheme of the model we adopt is shown in Fig. 2.

If we denote the gain of the amplifier by G, the transmission and reflection coefficients of the beam splitter are

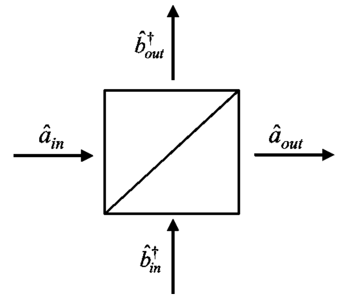

FIG. 2. Amplifying beam-splitter model.

$$\mathcal{T} = \sqrt{G} \quad \text{and} \quad \mathcal{R} = \sqrt{G-1}. \quad (28)$$

In this case, the validity of the Bosonic commutation relations for the field creation and destruction operators leads to [17,18],

$$|\mathcal{T}|^2 - |\mathcal{R}|^2 = 1, \quad (29)$$

where $|T|^2 = G \geqslant 1$, as the beam splitter is amplifying. No special requirements on the phases of the reflection and transmission coefficients are needed, and the relation between the input signal operator and the output operators is [18]

$$\hat{a}_{in}^\dagger = \sqrt{G}\hat{a}_{out}^\dagger - \sqrt{G-1}\hat{b}_{out}. \quad (30)$$

The derivation of the input-output relations for the density matrix elements describing the field sent into an amplifier can be made by a procedure similar to that outlined in Sec. II. The forms of the photon-number states characterizing the field at the output of the beam-splitter arms are now less straightforward. We denote here by $\hat{\rho}_{amp}$ the density operator describing the signal field initially sent into the amplifier, and we can write, similarly to Eq. (14),

$$_{out}\langle m|\hat{\rho}_{att}|m'\rangle_{out} = \mathrm{Tr}_{\mathrm{E}}\sum_{n,n'} {}_{out}\langle m|n\rangle_{in}\ {}_{in}\langle n|\hat{\rho}_{amp}|n'\rangle_{in}\ {}_{in}\langle n'|m'\rangle. \quad (31)$$

The photon-number input state can be found from Eq. (30),

$$|n\rangle_{in} = \frac{1}{\sqrt{n!}}(\hat{a}_{in}^\dagger)^n|0\rangle_{in} = \frac{1}{\sqrt{n!}}[\sqrt{G}\hat{a}_{out}^\dagger - \sqrt{G-1}\hat{b}_{out}]^n|0\rangle_{in}, \quad (32)$$

but where, in contrast to the attenuator, the input vacuum state $|0\rangle_{in}$ cannot be set equal to the output vacuum state. Due to the noise introduced by the amplifier, the vacuum state in the input of the device can be written only as a superposition of all the diagonal states of the output. Here these states are the result of the modeling the amplifier as a harmonic oscillator coupled to an inverted harmonic oscillator with negative energy levels, but they must also be present for other amplifier models. From this and using the principle of energy conservation, given for the amplifier by

$$\hat{a}_{out}^\dagger\hat{a}_{out} - \hat{b}_{out}^\dagger\hat{b}_{out} = \hat{a}_{in}^\dagger\hat{a}_{in} - \hat{b}_{in}^\dagger\hat{b}_{in}, \quad (33)$$

it can be shown that

$$|0\rangle_{in}=\frac{1}{\sqrt{G}}\sum_{q=0}^{\infty}\left(\frac{\sqrt{G-1}}{\sqrt{G}}\right)^{q}|q\rangle_{out}|q\rangle_{\mathrm{E}}, \tag{34}$$

where the subscript *out* refers to the observed output channel, while the subscript E refers to the environmental noise channel on which no measurements are assumed to be performed. The resulting output probability distribution for a $T=-0$ amplifier with no input signal is

$$P_{amp}(q)=\frac{(G-1)^{q}}{G^{q+1}}\equiv\frac{\overline{q}^{q}}{(1+\overline{q})^{1+q}}, \tag{35}$$

showing that the emission is chaotic [25], and the mean number of output photons is

$$\overline{q}=G-1. \tag{36}$$

The insertion of Eq. (34) into Eq. (32) and the use of the properties of the field annihilation and creation operators acting on photon-number states [25] gives

$$|n\rangle_{in}=\sum_{l=0}^{n}\sum_{s=n}^{\infty}\frac{\sqrt{n!}\sqrt{s!}}{l\,!\,(n-l)\,!\,\sqrt{(s-n)!}}\left(\frac{G}{G-1}\right)^{l}\times(-1)^{n-l}\frac{(G-1)^{(s+n)/2}}{G^{(s+1)/2}}|s\rangle_{out}|s-n\rangle_{\mathrm{E}}. \tag{37}$$

By means of Eq. (36) the quantity ${}_{out}\langle m|n\rangle_{in}$ can now be calculated and used in Eq. (31), leading to

$$\begin{aligned}{}_{out}\langle m|\hat{\rho}_{amp}|m'\rangle_{out}&\equiv\langle m|\hat{\rho}^{pred}_{amp}|m'\rangle\\&=\frac{1}{G^{(m+m'+2)/2}}\sum_{n=0}^{m}\sum_{n'=0}^{m'}\binom{m}{n}^{1/2}\binom{m'}{n'}^{1/2}\\&\quad\times(G-1)^{(m-n+m'-n')/2}\\&\quad\times\,\delta_{m-n-m'+n'}\,{}_{in}\langle n|\hat{\rho}_{amp}|n'\rangle_{in}.\end{aligned} \tag{38}$$

This is the general input-output relation for the predictive density-matrix elements of the field sent through a fully inverted amplifier. The diagonal contributions are

$${}_{out}\langle m|\hat{\rho}_{amp}|m\rangle_{out}=\frac{1}{G^{m+1}}\sum_{n=0}^{m}\binom{m}{n}(G-1)^{m-n}\,{}_{in}\langle n|\hat{\rho}_{amp}|n\rangle_{in}, \tag{39}$$

in agreement with previous results [19]. This expression describes the output photon distribution of the amplifying channel in terms of a general superposition of photon-number input states.

The predictive density operator describing the output field after the amplification process is

$$\hat{\rho}^{pred}_{amp}=\sum_{n,n'=0}^{\infty}\sum_{\substack{m=n\\m'=n'}}^{\infty}\rho_{nn'}(0)\binom{m}{n}^{1/2}\binom{m'}{n'}^{1/2}\times\frac{(G-1)^{(m-n+m'-n')/2}}{G^{(m+ni'+2)/2}}\delta_{m-n-m'+n'}|m\rangle\langle m'|, \tag{40}$$

where we have defined, $\rho_{nn'}(0)={}_{in}\langle n|\hat{\rho}_{amp}|n'\rangle_{in}$. As for the attenuator, Eq. (40) is a time-dependent density operator $\hat{\rho}^{pred}_{amp}\equiv\hat{\rho}^{pred}_{amp}(t)$, since $G=e^{2\gamma' t}$, with $2\gamma'$ being the gain rate of the device.

We can now evaluate the retrodictive density-matrix elements for this system. Here Eq. (40) represents the quantity

$$\hat{\rho}^{pred}_{amp}=\mathrm{Tr}_{\mathrm{E}}[\hat{U}_{amp}\hat{\rho}_{amp}\otimes\hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{amp}], \tag{41}$$

where $\hat{U}_{amp}$ is the unitary operator describing the amplifying process of the device [11]. If $\hat{\Pi}_j$ is the POM associated with measurement at the output of the amplifying channel, the probability that the outcome is j becomes, with the use of Eqs. (40) and (41),

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{ES}}[\hat{\Pi}_j\hat{U}_{amp}\hat{\rho}_{amp}\otimes\hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{amp}]\\&\quad=\sum_{n,n'=0}^{\infty}\sum_{\substack{m=n\\m'=n'}}^{\infty}\rho_{nn'}(0)\binom{m}{n}^{1/2}\binom{m'}{n'}^{1/2}\frac{(G-1)^{(m-n+m'-n')/2}}{G^{(m+m'+2)/2}}\\&\quad\times\,\delta_{m\text{-}n\text{-}m'+n'}\langle m'|\hat{\Pi}_j|m\rangle.\end{aligned} \tag{42}$$

Therefore, similarly to Eq. (22),

$$\langle n'|\hat{\rho}^{retr}_{amp}|n\rangle=\frac{1}{\mathcal{N}_{amp}}\frac{\partial}{\partial\rho_{nn'}}\mathrm{Tr}_{\mathrm{ES}}[\hat{\Pi}_j\hat{U}_{amp}\hat{\rho}_{amp}\otimes\hat{\rho}_{\mathrm{E}}(0)\hat{U}^{\dagger}_{amp}], \tag{43}$$

and using the same procedure as for the attenuator we find

$$\langle n'|\hat{\rho}^{retr}_{amp}|n\rangle=\sum_{\substack{m=n\\m'=n'}}^{\infty}\binom{m}{n}^{1/2}\binom{m'}{n'}^{1/2}\frac{(G-1)^{(m-n+m'-n')/2}}{G^{(m+m')/2}}\times\delta_{m\text{-}n\text{-}m'+n},\langle m'|\hat{\rho}^{retr}_{j}|m\rangle \tag{44}$$

The retrodictive conditional probability at time t is obtained by substituting G by $e^{2\gamma' t}$.

C. Linear attenuation and amplification: Predictive-retrodictive inverses

A direct correspondence between *predicting* the signal output state from a quantum optical attenuator or amplifier, and *retrodicting* the signal input state for an amplifier or attenuator, respectively, has been previously established by deriving a relation between the unitary operators associated with amplification and attenuation [11]. Here, using the results obtained in Sec. II, we confirm the validity of this correspondence in the situation where the environment is initially at zero temperature.

We recall that the signal field retrodictive matrix elements for the attenuator are given by Eq. (25). We now denote by $\rho_{m'm}(0)$ the *input* density matrix elements associated with the signal state sent into the amplifier. If we write the gain of the amplifier as the inverse of an attenuation factor, $G=1/K$, Eq. (38) becomes

$$\langle n'|\hat{\rho}_{amp}^{pred}|n\rangle = K\sum_{m=0}^{n}\sum_{m'=0}^{n'}\binom{n}{m}^{1/2}\binom{n'}{m'}^{1/2}K^{(m+m')/2}\times(1-K)^{(n-m+n'-m')/2}\delta_{n-m-n'+m'}\rho_{m'm}(0), \tag{45}$$

where the quantum numbers m,m' and n,n' are exchanged. Finally if we assume that the elements $\rho_{m'm}(0)$ associated with the amplifier input signal, are proportional to the measured quantities $\langle m'|\hat{\rho}_j^{retr}|m\rangle$ at the output of the attenuator, i.e., if

$$\rho_{m'm}(0) = \langle m'|\hat{\rho}_j^{retr}|m\rangle, \tag{46}$$

we find, by comparing Eq. (25) with Eq. (45), that

$$\langle n'|\hat{\rho}_{att}^{retr}|n\rangle = \langle n'|\hat{\rho}_{amp}^{pred}|n\rangle. \tag{47}$$

In other words, given the result of a measurement, the retrodictive state that we calculate at the input of an attenuator with loss K is the same as the calculated predictive state at the output of an amplifier with gain $G=1/K$ when its input state is the measured output state of the attenuator; this is shown here for the environment for the two devices initially in vacuum states. We should bear in mind the different nature of this state in the two cases: strictly speaking it is a state of minimum energy for the attenuator and a state of maximum energy for the amplifier. The opposite equivalence can be shown by following an identical procedure to the one adopted above and we have

$$\langle n'|\hat{\rho}_{amp}^{retr}|n\rangle = \langle n'|\hat{\rho}_{att}^{pred}|n\rangle. \tag{48}$$

In this case the retrodictive state for an amplifier with gain G is the same as the predictive state for an attenuator with loss $K=1/G$ given that the measured output state of the amplifier is the input state of the attenuator.

Note that, although we have proved the equivalence for the particular case of zero-temperature environment, it is valid for any temperature, as is shown in Sec. VI.

IV. APPLICATION TO PHOTOCOUNTING EXPERIMENTS

The aims of this section are to evaluate the retrodictive matrix elements of the attenuator field for different photon distributions and to compare the expressions with the predictive forms for the amplifier. Two types of experimental procedure are considered, and explicit forms for the measurement POM element $\hat{\Pi}_j$ are given. In order to confirm the validity of the beam-splitter method used in Sec. II for the derivation of the density-matrix input-output relations, some calculations for the predictive density operator are based here on the solution of the predictive master equation describing the system under consideration. For a cavity mode coupled to a zero-temperature environment, the evolution of the field is described by [20]

$$\dot{\hat{\rho}}(t) = \gamma[2\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger\hat{a}], \tag{49}$$

where 2γ is the damping rate. The effect of the coupling to the environment is equivalent to the attenuation process.

In this section we confine the discussion to perfect photodetection. A nonunit detection efficiency can easily be accommodated by giving the whole system an overall loss factor. The effect of detector dark counts can also be taken into account. Both of these detection scenarios are discussed in Sec. V.

A. Initial superposition of photon-number states and single measurements

In photocounting experiments, where usually the statistics of a radiation field have to be determined, the diagonal part of the density operator is sufficient, as this gives information on the photon-number probability distribution, and we concentrate on these diagonal terms only. Other detection schemes such as homodyne detection are sensitive to the phase of the field, and we will consider these elsewhere [26]. In this subsection we further limit the discussion to single measurements. In optical communications or quantum cryptography, where the receiver must analyze single bits of information sent through a communication channel, the detection can only be made with a single measurement on each signal bit. We also restrict the discussion in this section to *unbiased* sources of radiation. The state space of the system is spanned by the complete set of photon number states, but usually the message sender and receiver restrict this artificially by agreeing to send one of a limited number of states within a space spanned by the first few number states. In either case, by an unbiased source we mean one for which the *a priori* probability that it produces any particular number state is the reciprocal of the dimension of the state space. We assume that both the sender and the receiver know the state space of the system and that the detector used for the measurement is an ideal photocounting detector with unit quantum efficiency.

The initial state of any diagonal field can be expressed in the general form $\hat{\rho}(0)=\hat{\rho}_{att}=\Sigma_{n=0}^{\infty}\rho_{nn}(0)|n\rangle\langle n|$, so the solution of Eq. (49) is given by [20]

$$\hat{\rho}_{att}(t) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\rho_{nn}(0)\binom{n}{m}e^{-2\gamma tm}(1-e^{-2\gamma t})^{(n-m)}|m\rangle\langle m|, \tag{50}$$

which describes a mixed state with a binomial photon-number probability distribution. With $e^{-2\gamma t}=K$, we clearly see that Eq. (50) agrees with the diagonal part of Eq. (18) derived using the beam-splitter model. We first assume that exactly m photons are counted by the detector in a *single* measurement. This is a simple von Neumann measurement [15] and the POM element can be written as

$$\hat{\Pi}_j \equiv \hat{\Pi}_m = |m\rangle\langle m|. \tag{51}$$

If the state space is the complete set of number states the retrodictive probability that the input contained n photons given m detected output photons is given by Eqs. (23) and (51) as

$$P_{att}^{retr}(n|m) = \langle n|\hat{\rho}_{att}^{retr}|n\rangle = \binom{n}{m} K^{m+1}(1-K)^{n-m}$$
$$= \binom{n}{m} \frac{(G-1)^{n-m}}{G^{1+n}} \quad \text{with } n \geqslant m, \tag{52}$$

where we have set $K=1/G$. If we have no information about the prepared input state other than its random selection the result of such an experiment does not tell us much about the initial photon distribution of the prepared signal field. In fact Eq. (52) represents the best available description of the initial photon distribution that we can obtain from a single measurement, given our lack of knowledge of the particular state that was sent. By comparing the above expression with Eq. (39), we can see that Eq. (52) also describes the probability of counting n photons at the output channel of an amplifier with gain G, given a field input state $|m\rangle$ and the environment initially at zero temperature. Therefore in this particular case we reobtain the diagonal part of Eq. (47).

Similarly we can illustrate for single measurements the inverse case, where the predictive density-matrix element for the amplifier is equivalent to the retrodictive matrix element for the attenuator. Without going into details, we find that the retrodictive probability for n input photons having detected m photons at the output of the amplifier is

$$P_{amp}^{retr}(n|m) = \langle n|\hat{\rho}_{amp}^{retr}|n\rangle = \binom{m}{n} \frac{(G-1)^{m-n}}{G^m} \quad \text{with} \quad m \geqslant n. \tag{53}$$

Finally setting $G=1/K$, we again obtain the diagonal part of Eq. (47).

One particularly striking consequence of the equivalence between attenuators and amplifiers is that ideal attenuators add noise photons backwards in time. Similarly, ideal amplifiers, which must add noise photons to an input signal, do not appear to do so backwards in time. An example of this can be given if we suppose that a communication channel contains an ideal amplifier with a particular gain (say $G=2$). If we send a one-photon pulse of light through the channel the mean number of photocounts at the output is $\langle n_{out}\rangle = G\langle n_{in}\rangle + G - 1 = 3$. If, however, we do not know how many photons were sent, and wish to determine this from the results of the measurement we obtain a counterintuitive result. If we measure three photons then the mean number of input photons is determined by considering the retrodictive amplifier as an ideal predictive attenuator with loss $K=1/G$. Thus $\langle n_{in}\rangle = K\langle n_{out}\rangle = 1.5$, which is not in line with the intuitive result, 1, given by subtracting a photon and dividing by the gain. This result is also obtainable directly by using the retrodictive density operator for the amplifier obtained from the diagonal part of Eq. (44), and computing the mean number of input photons using $\langle n_{in}\rangle = \mathrm{Tr}(\hat{\rho}_{amp}^{retr}\hat{n})$. While the result is surprising it is what one might intuitively expect for a channel that contains a classical noise-free amplifier, and is thus in line with the correspondence principle [27]. In classical physics the beam intensity retrodicted at the input of an amplifier would be the detected intensity divided by the gain. This is the classical limit of the retrodictive result, but it is not the exact classical limit of the result based on predictive amplifier theory.

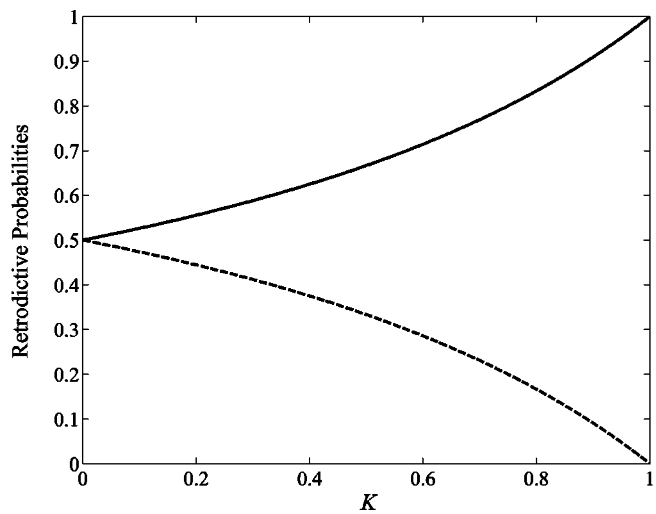

FIG. 3. Retrodictive conditional probabilities $P_{att}^{retr}(0|0)$ (full curve) and $P_{att}^{retr}(1|0)$ (dashed curve), in the case where only zero or one photon could have been sent through the attenuator.

In the above examples the detection of a number of photons tells us little about the input because the state space of the system is large. If the space is artificially restricted by prior agreement between the sender and receiver, then the amount of information gained from a single measurement can be greatly increased. Suppose that an ideal attenuating optical fiber is used to send signals in the form of photon number states. Consider first the case where no photons are detected at the output. Then the retrodictive density matrix for the input to the fiber is, from Eq. (25),

$$\hat{\rho}_{att}^{retr} = \sum_{m=0}^{\infty} K(1-K)^m |m\rangle\langle m|, \tag{54}$$

where K is the transmission of the fiber. If it is known that only either zero or one photon was sent with equal prior probability (an unbiased source) this restricts the state space in Eq. (54). Therefore the sum stops at $m=1$, and the density operator is renormalized,

$$\hat{\rho}_{att}^{retr} = \frac{1}{2-K}|0\rangle\langle 0| + \frac{1-K}{2-K}|1\rangle\langle 1|. \tag{55}$$

Figure 3 shows the retrodictive conditional probabilities that 0 or 1 photon was sent, given that none were found at the detector. These probabilities both tend to 1/2 as the attenuation increases ($K \to 0$), and so the measurement gives no information about the input state if the fiber is highly attenuating. This retrodictive decay to the no information state has

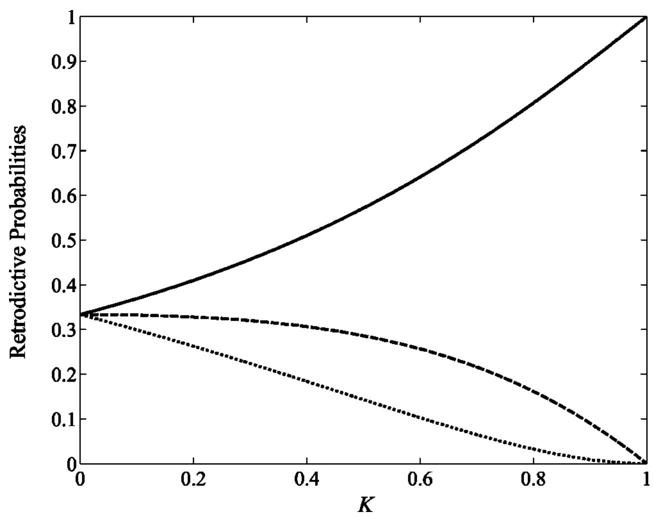

FIG. 4. Retrodictive conditional probabilities $P_{att}^{retr}(0|0)$ (full curve), $P_{att}^{retr}(1|0)$ (dashed curve), and $P_{att}^{retr}(2|0)$ (dotted curve), in the case where only zero, one, or two photons could have been sent through the attenuator.

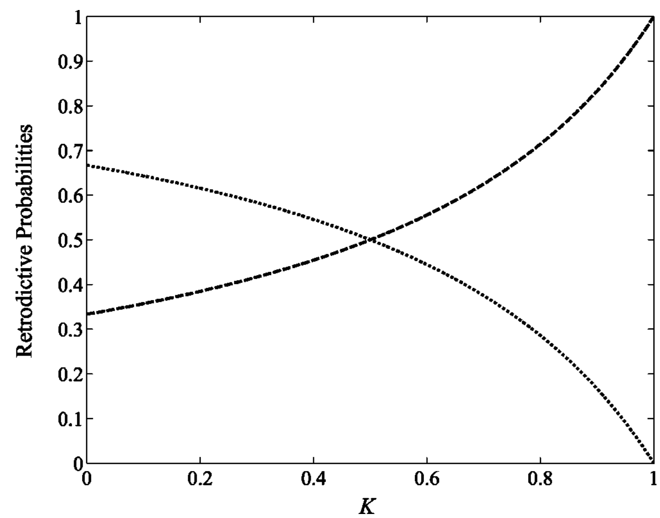

FIG. 5. Retrodictive conditional probabilities $P_{att}^{retr}(1|1)$ (dashed curve) and $P_{att}^{retr}(2|1)$ (dotted curve), in the case where only zero, one, or two photons could have been sent through the attenuator.

already been highlighted for two-level atoms interacting with a single-mode electromagnetic field [12].

If the state space is extended so that zero, one or two photons could have been sent with equal prior probabilities (again unbiased), then for no detected photons the retrodictive density operator becomes

$$\hat{\rho}_{att}^{retr} = \sum_{m=0}^{2} \frac{(1-K)^m}{3-3K+K^2}|m\rangle\langle m|. \tag{56}$$

In this case the retrodictive conditional probabilities all decay to 1/3 as the attenuation increases, as shown in Fig. 4. This is the no-information state for the three-state basis. These retrodictive results tally with the results for the output of a predictive amplifier with a vacuum input. In the high gain limit all low photon numbers are equally likely at the output, so if the output space is artificially restricted to a finite set of low number states then the output density operator will tend to the no information state.

The situation is different if the detector registers a photon. For the two-state system a count at the output of the ideal attenuator means that a photon must have been sent. There is no other mechanism for obtaining a count when the attenuator temperature is zero. The situation becomes a little more interesting for the three-state system. The retrodictive density operator obtained from one count is

$$\hat{\rho}_{att}^{retr} = \sum_{m=1}^{\infty} mK(1-K)^m|m\rangle\langle m|, \tag{57}$$

which, if not more than two photons were sent, becomes

$$\hat{\rho}_{att}^{retr} = \frac{1}{3-2K}|1\rangle\langle 1| + \frac{2(1-K)}{3-2K}|2\rangle\langle 2|. \tag{58}$$

The probabilities that one and two photons were sent are plotted as a function of the transmission coefficient K for an unbiased source in Fig. 5. It is seen that in the high attenuation limit the one-photon probability tends to 1/3 and the two-photon probability tends to 2/3. This result is easy to understand if the system is considered to be a predictive amplifier with a one-photon input. If the output is limited to two photons then only two processes can occur. Either the photon passes through the amplifier and is detected, or on its way through it stimulates an emission of a second photon. The probability of this second process is proportional to the stimulated emission factor $(m+1)$, so it is twice as likely to occur in this case.

The situation can be different for *biased* sources for which we have *a priori* information about the state preparation. If, for example, only zero or one photon can be sent, but one photon is twice as likely to be sent as no photons, then, if no photons are detected at the output the retrodictive state is as given in Eq. (55). However, the retrodictive conditional probabilities that zero or one photon were sent are not the same as in Fig. 3. The probabilities must take account of the *a priori* distribution, which is encoded in the prepared state (7),

$$\hat{\Lambda} = \hat{\Lambda}_0 + \hat{\Lambda}_1 = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|. \tag{59}$$

The probabilities that 0 and 1 photons were sent are respectively, from Eq. (8),

$$P_{att}^{retr}(0|0) = \frac{\mathrm{Tr_S}[\Lambda_0\hat{\rho}^{retr}]}{\mathrm{Tr_S}[\Lambda\hat{\rho}^{retr}]} = \frac{1}{3-2K},$$

$$P_{att}^{retr}(1|0) = \frac{\mathrm{Tr_S}[\Lambda_1\hat{\rho}^{retr}]}{\mathrm{Tr_S}[\Lambda\hat{\rho}^{retr}]} = \frac{2-2K}{3-2K}. \tag{60}$$

In this case the retrodictive conditional probabilities at high attenuation decay to the *a priori* probabilities, which are 1/3 and 2/3, and this time there is some remaining information about the prior distribution of photons.

In real optical communications systems, signals consist of a sequence of single bits of information, denoted by 0 or 1. The essential part of the communication problem is to determine the transmitted message from the received signal and here retrodiction is the natural approach to calculate the retrodictive probabilities associated with the preparation states of the signal bits of information. A study of retrodiction applied to optical communications for different detection schemes will be presented in a subsequent paper, but here we illustrate how information about the prepared states can be accurately derived in this biased case, given the knowledge of the outcome of the single measurement.

Conventional systems operate in the classical multiphoton regime, where a received "1" may be represented by optical pulses containing for example about 10 000 photons [28]. We make here the assumption that the two optical bits "0" and "1" characterizing the binary signal can be described in first approximation by the vacuum state $|0\rangle$ and by a quantum state $|\overline{n}\rangle$ with an extremely large number of photons $\overline{n} \gg 1$. Thus the system represents another example of a biased source. The retrodictive matrix elements, and hence the normalized retrodictive conditional probabilities that, given m recorded photocounts, 0 or $\overline{n}$ photons were sent into a zero-temperature attenuator are now found to be

$$P_{att}^{retr}(0|m \neq 0) = 0, \quad P_{att}^{retr}(\overline{n}|m \neq 0) = 1, \tag{61}$$

when a number of photons $m \neq 0$ is recorded during the single measurement, while if no counts are registered

$$P_{att}^{retr}(0|0) = \frac{1}{1+(1-K)^{\overline{n}}} \quad \text{and} \quad P_{att}^{retr}(\overline{n}|0) = \frac{(1-K)^{\overline{n}}}{1+(1-K)^{\overline{n}}}, \tag{62}$$

where Eqs. (61) and (62) have been renormalized taking into account the fact that here

$$\langle n|\hat{\rho}_{att}^{retr}|n\rangle = P_{att}^{retr}(n|m) = 0 \quad \text{for} \quad n \neq 0, \overline{n}. \tag{63}$$

The above expressions depend on the transmission of the device but also on the number of photons contained in the strong optical pulse describing the "1" bit. While in this ideal situation the relations (61) give a deterministic answer to the problem of determining the nature of the signal bit sent into the communication channel, the probabilities in Eq. (62) respectively tend to 1 and 0 when $K \to 1$. Note that for a fixed value of K, the bigger $\overline{n}$ the better the discrimination between the two input pulses can be made by means of this retrodiction process. The behavior of these retrodictive conditional probabilities is shown in Fig. 6 where $P_{att}^{retr}(0|0)$ is plotted as a function of K for two different values of the input photon number $\overline{n}$.

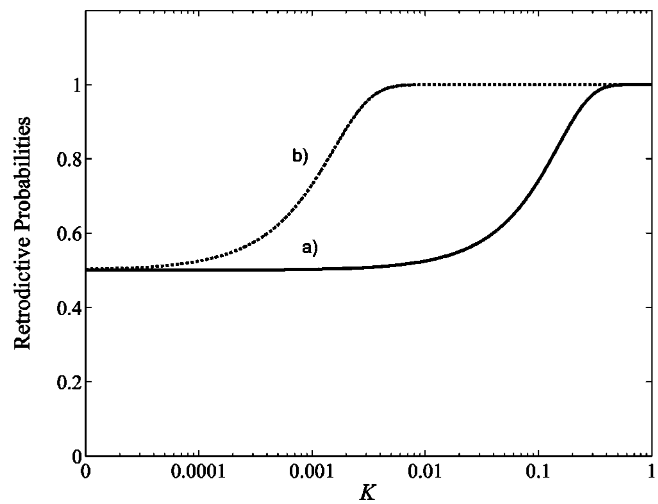

FIG. 6. Retrodictive conditional probability that no photons were sent given that none were detected, $P_{att}^{retro}(0|0)$, for the case where (a) either zero or $\overline{n}=10$ photons and (b) either zero or $\overline{n}=1000$ photons could have been sent through the attenuator.

B. Coherent input state and repeated measurements

In this subsection and the following we illustrate how some information on the statistics of a radiation field can be derived using retrodiction, when *repeated* measurements are performed on the signal. This is not usually the case for quantum communications but we imagine here that the nature of a radiative source has to be analyzed, and that the latter continuously produces identical field states that can be measured separately.

If the cavity field mode at time $t=0$ is in a coherent state $|\alpha\rangle$, so that $\hat{\rho}(0)=\hat{\rho}_{att}=|\alpha\rangle\langle\alpha|$ and, as before, the environment is assumed to be at zero temperature, then the solution of the master equation (49) is [21]

$$\hat{\rho}_{att}(t) = |\alpha e^{-\gamma t}\rangle\langle\alpha e^{-\gamma t}| = e^{-|\alpha|^2 e^{-2\gamma t}} \sum_{m,m'=0}^{\infty} \frac{\alpha^m \alpha^{*m'}}{\sqrt{m!\,m'!}} e^{-\gamma t(m+m')} |m\rangle\langle m'|. \tag{64}$$

By setting $e^{-2\gamma t}=K$, the diagonal contribution of $\hat{\rho}_{att}(t)$ can be written as a superposition of photon-number state operators similar to Eq. (50) and consequently the general retrodictive matrix elements are given by the diagonal part of Eq. (25),

$$\langle n|\hat{\rho}_{att}^{retr}|n\rangle = K \sum_{m=0}^{n} \binom{n}{m} K^m (1-K)^{n-m} \langle m|\hat{\rho}_j^{retr}|m\rangle. \tag{65}$$

This further underlines the fact that one *single measurement* is not enough to extract accurate information about the statistics of the input radiation field.

In a situation where *repeated* counts are recorded, we introduce the following measured density operator:

$$\hat{\rho}^{meas} = \sum_{j=0}^{\infty} P(j)|j\rangle\langle j|, \tag{66}$$

where $P(j)$ is the *experimental* probability distribution constructed from the results of the repeated measurements, and normalized. A coherent field is described by Poissonian statistics [21,25]. The fact that this input is coherent is not known to the measurer, and the statistics of this field must be determined from the measured statistics. If the input is Poissonian the output statistics are also Poissonian so the constructed experimental probability distribution (in the limit of an "infinite" number of single measurements) is

$$\langle m|\hat{\rho}^{meas}|m\rangle = P(m) = \frac{e^{-|\beta|^2}|\beta|^{2m}}{m!}, \tag{67}$$

where $|\beta|^2=K|\alpha|^2$, and in agreement with the output probability distribution characterized by Eq. (17) for a coherent input field. By inserting Eq. (65) in Eq. (64) (with $\hat{\rho}_j$ replaced by $\hat{\rho}^{meas}$) and setting $K=1/G$, we obtain the retrodictive density-matrix elements,

$$\langle n|\hat{\rho}_{att}^{retr}|n\rangle = \frac{(G-1)^n}{G^{n+1}}e^{-|\beta|^2}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{m!}\left(\frac{|\beta|^2}{G-1}\right)^m \tag{68}$$

$$= \frac{(G-1)^n}{G^{n+1}}e^{-|\alpha|^2/G}L_m\left(\frac{-|\alpha|^2}{G(G-1)}\right) = \langle n|\hat{\rho}_{amp}^{pred}|n\rangle, \tag{69}$$

where L_m is the Laguerre polynomial. This expression also describes the superposition of chaotic and coherent light at the output of an amplifier [amplified vacuum field of the environment with mean number of photons $G-1$ in accordance with Eq. (35), and coherent light of amplitude $|\alpha| = G^{1/2}|\beta|$], for an input state given by a coherent field of amplitude $|\beta|$, [19,20]. This can also be shown from Eq. (38), when substituting $\rho_{nn}(0)$ by the initial Poissonian distribution described by Eq. (67). Compared to the single measurement experiment, additional information on the field statistics can now be extracted from the retrodictive density-matrix elements. Nevertheless, it is true that, because of the noise involved in the amplification process, an uncertainty about the input field of the attenuator remains. It is not possible to say that the input state had Poissonian statistics with a mean number of photons $|\alpha|^2$ merely on the basis of the measurement statistics.

C. Chaotic input state and repeated measurements

We now consider a chaotic input field, and we assume that the radiation source emits continuously. As before the environment is assumed to be at zero temperature, and the density operator describing the radiation amplitude at time $t=0$ can be written as [21,23]

$$\hat{\rho}(0) = \hat{\rho}_{att} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{1+n}}|n\rangle\langle n|, \tag{70}$$

where $\bar{n}$ is the mean number of thermal photons of the input field. We again assume that repeated measurements are performed during the experiment, and thus that the statistics of counted photons can be built up. The retrodictive density-matrix elements for the system are again given by Eq. (65). In this case, the expectation value of the POM element $\hat{\rho}^{meas}$ (instead of $\hat{\rho}_j$) in the form Eq. (66), must be given (in the limit of an infinite number of single measurements) by the chaotic distribution

$$\langle m|\hat{\rho}^{meas}|m\rangle = \frac{\bar{m}^m}{(1+\bar{m})^{1+m}}, \tag{71}$$

where the average number of detected photons is $\bar{m}=K\bar{n}$. We obtain

$$\langle n|\hat{\rho}_{att}^{retr}|n\rangle = \frac{(G-1+G\bar{m})^n}{(G+G\bar{m})^{n+1}} = \langle n|\hat{\rho}_{amp}^{pred}|n\rangle. \tag{72}$$

The retrodictive matrix elements for the attenuator are thus equivalent to the predictive matrix elements for an amplifier where the input field is represented by chaotic light with mean photon number equal to $\bar{m}$. This can be shown from Eq. (38) when the initial probability distribution is assumed to be given by Eq. (71). The distribution in Eq. (72) represents a chaotic distribution with mean photon number equal to the sum of the chaotic contributions constituting the output of the amplifier [19]. These are the amplified vacuum field of the environment and the amplified chaotic light, characterized by mean photon numbers given by $G-1$ and $G\bar{m}$, respectively. As for the coherent input, by retrodicting the result arising from successive measurements on the output signal, we are able to derive information about the mean photon number of the input field and on the field statistics.

V. PREDICTIVE AND RETRODICTIVE DENSITY-MATRIX ELEMENTS IN THE CASE OF IMPERFECT DETECTION

In order to complete the description of retrodiction for ideal attenuators and amplifiers (at zero temperature environment) presented in Secs. II and III in the context of photocounting experiments, we give here the expressions for the predictive and the retrodictive density-matrix elements of the field, first when the detector placed at the output of the devices has a quantum efficiency smaller than unity, and second when dark counts affect the measurement. Both situations correspond more closely to real experimental conditions. Devices which produce postselected states based on measurements of exact numbers of photons are likely to be useful in quantum information processing, and form the basis of linear optical quantum computing. Therefore the results in this section have direct relevance to the fidelity of such devices when detection is imperfect.

A. Inefficient photodetection

1. Attenuation and detection

An imperfect photodetector can always be modeled by a beam splitter (or attenuator) followed by a perfect detector, since its effect on the radiation falling onto its detection area is an effective attenuation of the field. If we consider first the case where an attenuating channel characterized by a transmission factor K is followed by the imperfect detector with quantum efficiency $\eta<1$, we expect the whole system to be equivalent to an attenuator with global transmission factor $K\eta$. In the context of photocounting experiments we concentrate our attention on the field amplitude, and so we only present the diagonal contributions of the field density-matrix elements. By using the result valid for the attenuation process of a single mode signal, the photon probability distribution after the measurement by the photodetector can be written as [25]

$$\langle p|\hat{\rho}^{pred}_{att+D}|p\rangle = \sum_{m=p}^{\infty} \langle m|\hat{\rho}^{pred}_{att}|m\rangle \binom{m}{p} \eta^p (1-\eta)^{m-p}, \quad (73)$$

assuming the surrounding thermal noise to be at zero temperature. The distribution $\langle m|\hat{\rho}^{pred}_{att}|m\rangle$ is given by Eq. (17), and it can be shown by combining Eq. (73) with Eq. (17), that the predictive matrix elements are simply

$$\langle p|\hat{\rho}^{pred}_{att+D}|p\rangle = \sum_{n=p}^{\infty} \rho_{nn}(0) \binom{n}{p} (K\eta)^p (1-K\eta)^{n-p}. \quad (74)$$

The whole system can therefore be represented, as expected, by an attenuating device characterized by a transmission coefficient $K' = K\eta$.

The diagonal retrodictive matrix elements for the system attenuator + photodetector can be straightforwardly derived, and the procedure of Sec. II, we obtain

$$\langle n|\hat{\rho}^{retr}_{att+D}|n\rangle = K\eta \sum_{p=0}^{\infty} \binom{n}{p} (K\eta)^p (1-K\eta)^{n-p} \langle p|\hat{\rho}^{retr}_{j}|p\rangle, \quad (75)$$

where $\hat{\rho}^{retr}_{j}$ represents the density operator associated with the outcome j of the measurement. It is easy to show that we have here

$$\langle n'|\hat{\rho}^{retr}_{att+D}|n\rangle = \langle n'|\hat{\rho}^{pred}_{amp}|n\rangle, \quad (76)$$

with the amplifier's gain given by $G' = 1/K\eta$. The result (75) is precisely that which was derived in Ref. [4] using Bayes' theorem and the Bernoulli sampling formula.

2. Amplification and detection

The system consisting of an amplifying channel followed by an imperfect detector can be represented by an amplifier of transmission G followed by an attenuator of transmission η. The amplifier environment is initially in the vacuum state, and we also assume that the thermal field surrounding the detector is at zero temperature. The photon-number probability distribution after the detection process can be written as in Eq. (73) as

$$\langle p|\hat{\rho}^{pred}_{amp+D}|p\rangle = \sum_{m=p}^{\infty} \langle m|\hat{\rho}^{pred}_{amp}|m\rangle \binom{m}{p} \eta^p (1-\eta)^{m-p}, \quad (77)$$

with

$$\langle m|\hat{\rho}^{pred}_{amp}|m\rangle = \frac{1}{G^{m+1}} \sum_{n=0}^{m} \rho_{nn}(0) \binom{m}{n} (G-1)^{m-n}. \quad (78)$$

By combining Eq. (78) with Eq. (77), we then obtain

$$\langle p|\hat{\rho}^{pred}_{amp+D}|p\rangle = \sum_{m=p}^{\infty} \sum_{n=0}^{m} \rho_{nn}(0) \binom{m}{n} \binom{m}{p} \times \eta^p (1-\eta)^{m-p} \frac{1}{G^{m+1}} (G-1)^{m-n}. \quad (79)$$

On the other hand, although we will not go into details here, it can be shown that the retrodictive matrix elements for the system constituted by an amplifier followed by an imperfect photodetector are given by

$$\langle n|\hat{\rho}^{retr}_{amp+D}|n\rangle = \sum_{m=n}^{\infty} \sum_{p=0}^{m} \binom{m}{n} \binom{m}{p} \eta^{p+1} (1-\eta)^{m-p} \frac{1}{G^m} \times (G-1)^{m-n} \langle p|\hat{\rho}^{retr}_{j}|p\rangle. \quad (80)$$

We have shown in Sec. II that the predictive matrix elements for an attenuator or amplifier are equivalent to the retrodictive matrix elements for an amplifier or attenuator, respectively. We therefore expect Eq. (80) to be identical to the photon-number probability distribution at the output of the system where an amplifier with gain $G' = 1/\eta$ is followed by an attenuator with loss $K' = 1/G$. The environment is initially at zero temperature and the noise entering the attenuating channel is given only by the contribution coming from the output of the amplifier. By evaluating the output photon distribution for this system we find

$$\langle n|\hat{\rho}'^{pred}_{amp+att}|n\rangle = \sum_{m=n}^{\infty} \sum_{p=0}^{m} \rho_{pp}(0) \binom{m}{n} \binom{m}{p} \eta^{p+1} \times (1-\eta)^{m-p} \frac{1}{G^m} (G-1)^{m-n}. \quad (81)$$

Therefore, if

$$\rho_{pp}(0) = \langle p|\hat{\rho}^{retr}_{j}|p\rangle, \quad (82)$$

we have, as expected,

$$\langle n|\hat{\rho}'^{pred}_{amp+att}|n\rangle = \langle n|\hat{\rho}^{retr}_{amp+D}|n\rangle \equiv \langle n|\hat{\rho}^{retr}_{amp+att}|n\rangle, \quad (83)$$

valid for a general initial photon probability distribution ρ_{nn} of the field. Note that the predictive-retrodictive equivalence shown in Eq. (83) is not a property of the diagonal elements only, and its validity can be extended to the off-diagonal ones. Retrodiction of the process of amplification followed by imperfect photodetection is thus equivalent to prediction for an amplifier followed by an attenuator. A similar property applies for an attenuator followed by an amplifier.

B. Measurements including dark counts

Real photodetectors exhibit extra counts not associated with the absorption of photons. These dark counts are random detection events that are independent of the incident light, and are usually described by a Poissonian probability distribution [4,9],

$$P(d) = \frac{e^{-\nu}\nu^d}{d!}, \tag{84}$$

with mean number ν. In quantum communications where low intensity signal pulses constitute the information bits, the dark count events are the main source of error. For instance, in quantum cryptography, the bit error rate of the key distribution process strongly depends on the rate of detected dark counts; typically for an effective bit rate of 100 s^{-1}, the error rate due to dark counts is of the order of 6 s^{-1} [28]. The presence of dark counts with mean number ν associated with each measurement event, with a value between 0.01 and 0.1, can have a significant effect on the measurement result when signals consist of about 0.1 photons per bit, as is often the case in quantum cryptography.

In this section we briefly illustrate how the presence of dark counts in the measurement affects the expressions for the predictive and retrodictive density-matrix elements for the attenuating or amplifying system at zero temperature. We limit our attention to the diagonal contributions only and for simplicity we consider a detection quantum efficiency η equal to unity.

1. Attenuation and detection

The detector does not discriminate between dark counts and photoelectron counts, and the probability of recording N total counts is derived by combining the two independent output probabilities, given respectively by the photon-number distribution at the output of the attenuating channel (associated with the quantum number m) and the probability of detecting $N-m$ dark counts. We have

$$\begin{aligned}\langle N|\hat{\rho}^{pred}_{att+d}|N\rangle &= \sum_{m=0}^{N} \frac{e^{-\nu}\nu^{N-m}}{(N-m)!}\langle m|\hat{\rho}^{pred}_{att}|m\rangle \\ &= \sum_{m=0}^{N} \frac{e^{-\nu}\nu^{N-m}}{(N-m)!}\sum_{n=m}^{\infty}\rho_{nn}(0)\binom{n}{m}K^m(1-K)^{n-m},\end{aligned} \tag{85}$$

where $N-m=d$.

On the other hand, for a single measurement where N counts are recorded, we find, by the same method used throughout the paper, the retrodictive matrix elements for the system given by

$$\begin{aligned}\langle n|\hat{\rho}^{retro}_{att+d}|n\rangle = K\left(\sum_{l=0}^{N}\frac{\nu^l}{l!}\right)^{-1}\sum_{m=0}^{Min[n,N]}\frac{\nu^{N-m}}{(N-m)!}\binom{n}{m} \\ \times K^m(1-K)^{n-m},\end{aligned} \tag{86}$$

in accordance with previous results [4].

2. Amplification and detection

Similar calculations can be performed for the amplifier. The probability of detecting N photons is given by

$$\begin{aligned}\langle N|\hat{\rho}^{pred}_{amp+d}|N\rangle &= \sum_{m=0}^{N}\frac{e^{-\nu}\nu^{N-m}}{(N-m)!}\langle m|\hat{\rho}^{pred}_{amp}|m\rangle \\ &= \sum_{m=0}^{N}\frac{e^{-\nu}\nu^{N-m}}{(N-m)!}\sum_{n=0}^{m}\rho_{nn}(0)\binom{m}{n}\frac{1}{G^{m+1}}(G-1)^{m-n},\end{aligned} \tag{87}$$

and inversely given a number N of recorded counts, the retrodictive density-matrix elements for the system amplifier +photodetector are now

$$\langle n|\hat{\rho}^{retro}_{amp+d}|n\rangle = \left(\sum_{l=0}^{N}\frac{\nu^l}{l!}\right)^{-1}\sum_{m=n}^{N}\frac{\nu^{N-m}}{(N-m)!}\binom{m}{n}\frac{1}{G^m}(G-1)^{m-n}. \tag{88}$$

Note that, because of the additional *independent* random counts (detector dark counts), the equivalence between the attenuator (amplifier) predictive matrix elements and the amplifier (attenuator) retrodictive matrix elements, shown in Eqs. (47) and (48), is lost.

VI. ATTENUATORS AND AMPLIFIERS AT ELEVATED TEMPERATURE

The aim of this section is to illustrate the fact that quantum linear amplification and attenuation remain predictive-retrodictive inverses for a general temperature T of the environment. We use expressions for the predictive output distributions from an attenuator and an amplifier which have been previously derived [19]. We only compare the retrodictive attenuator with the predictive amplifier in the case of single measurements. Recently a retrodictive master equation has been derived [13], which allows the evolution of a retrodictive state to be tracked backwards in time for any open system. The general equivalence between nonideal attenuators and amplifiers is relatively easily proved using this formalism. An account of this is given in Sec. VI C.

A. Attenuation and retrodiction of single measurement result

The output photon distribution from an attenuating channel at general temperature, and for arbitrary input light can be written as [19]

$$\begin{aligned}\langle s|\hat{\rho}^{pred}_{att}|s\rangle = \sum_{m=0}^{s}\sum_{n=m}^{\infty}\rho_{nn}(0)\binom{s}{m}\binom{n}{m}\frac{\overline{n}_{ch}^{s-m}}{(1+\overline{n}_{ch})^{1+s}} \\ \times\left(\frac{K}{1+\overline{n}_{ch}}\right)^m\left(1-\frac{K}{1+\overline{n}_{ch}}\right)^{n-m},\end{aligned} \tag{89}$$

where $\overline{n}_{ch}$ is the number of chaotic noise photons added by the device. As we have already pointed out, the diagonal contributions of the density matrix are sufficient for a description of the field in terms of its amplitude and they are in fact the only relevant contributions in the context of photocounting experiments. If the attenuator is modeled by an ensemble of two-level atoms, $\overline{n}_{ch}$ can be written in terms of the thermal factor (11) as [19]

$$\overline{n_{ch}} = N_{att}(1-K). \tag{90}$$

Note that for $T\to 0$, $N_{att}\to 0$. Then for an ideal attenuator (at zero temperature) we have $\overline{n_{ch}}=0$ as expected. In general, given the mean number $\overline{n}$ of photons of the input distribution, the mean number of output photons from the attenuating channel is

$$\overline{s} = \overline{n_{ch}} + K\overline{n}, \tag{91}$$

the sum of the chaotic contribution and the contribution from the attenuated signal.

The diagonal retrodictive matrix elements for the field can be calculated by the same method as in earlier sections. In the case where s photons are recorded (by an ideal detector) at the output of the device we have

$$\langle n|\hat{\rho}_{att}^{retro}|n\rangle = \frac{1}{\mathcal{N}_{att}}\frac{\partial}{\partial\rho_{nn}(0)}\langle s|\hat{\rho}_{att}^{pred}|s\rangle, \tag{92}$$

where $\mathcal{N}_{att}$ is the appropriate normalization constant. The double summation in Eq. (89) can be rewritten as

$$\sum_{n=0}^{s-1}\sum_{m=0}^{n} + \sum_{n=s}^{\infty}\sum_{m=0}^{s},$$

so from Eq. (92)

$$\langle n|\hat{\rho}_{att}^{retro}|n\rangle = K\sum_{m=0}^{Min[n,s]}\binom{s}{m}\binom{n}{m}\frac{\overline{n_{ch}}^{s-m}}{(1+\overline{n_{ch}})^{1+s}} \times\left(\frac{K}{1+\overline{n_{ch}}}\right)^{m}\left(1-\frac{K}{1+\overline{n_{ch}}}\right)^{n-m}. \tag{93}$$

B. Amplification of photon-number state input and comparison

The general output probability distribution from an amplifying channel, and for a photon-number input state $|s\rangle$, can be written as [19]

$$\langle n|\hat{\rho}_{amp}^{pred}|n\rangle = \sum_{m=0}^{Min[n,s]}\binom{n}{m}\binom{s}{m}\frac{\overline{n'_{ch}}^{n-m}}{(1+\overline{n'_{ch}})^{1+n}} \times\left(\frac{G}{1+\overline{n'_{ch}}}\right)^{m}\left(1-\frac{G}{1+\overline{n'_{ch}}}\right)^{s-m}, \tag{94}$$

and is characterized by a mean number of output photons

$$\overline{n} = \overline{n'_{ch}} + Gs, \tag{95}$$

being the sum of a pure chaotic contribution and the contribution from the amplified signal. The mean number of chaotic photons $\overline{n'_{ch}}$, deriving from the amplification of the environment noise, can be written in terms of the thermal factor (26) as [19]

$$\overline{n'_{ch}} = N_{amp}(G-1), \tag{96}$$

and it reduces to the amplified vacuum field fluctuation contribution shown in Eq. (36), when $T\to -0$ and thus $N_{amp}\to 1$.

The comparison between the retrodictive attenuator and the predictive amplifier is easily made by looking at Eqs. (93) and (94). The retrodictive matrix element (93) is identical to the predictive matrix element (94) provided that in the two expressions the quantities raised to the various powers 0, s, n, and m are the same. It can be easily checked that the four resulting conditions are all satisfied if $G=1/K$ and the thermal factors N_{att} and N_{amp} are related by Eq. (27). This simple example proves the equivalence between the predictive attenuator and the retrodictive amplifier in the general case where the environment is at temperature different from zero.

In real amplifiers the atomic population is never fully inverted and so the population factor N_{amp} can never be exactly equal to 1. Typically for Erbium doped fiber amplifiers $N_{amp}\approx 1.4$, corresponding to a ratio between the populations of the ground and excited levels given by $N_g/N_e\approx 0.3$ [29]. Nevertheless, a large range of population factor values can be achieved since for instance in Ar^+ lasers, where the lower level population varies between 60% and 70% of that of the upper level, N_{amp} varies between 2.5 and 3.3 [30]. On the other hand, in Raman amplifiers characterized by $\hbar\omega/k_BT \ll 1$, N_{amp} can be close to 1 [29].

C. Predictive and retrodictive master equations for attenuators and amplifiers

The evolution of an open quantum system can be tracked using unitary evolution operators which act on both system and environment variables. After the evolution time the trace over the environment gives the density operator for the system alone. A more popular approach is to derive an evolution equation for the system in which the environment variables have been already traced out of the problem. This means that the environment variables do not have to be tracked during the evolution. This equation, called a master equation, generally corresponds to nonunitary evolution [21], and is difficult to solve in general. If a state $\hat{\rho}_i^{pred}(t_p)$ is prepared at the preparation time t_p, and evolved for a time $\tau=t_m-t_p$ until the measurement time t_m, the conditional probability that the measurement result $\hat{\Pi}_j(t_m)$ will be obtained is given by Eq. (1). The evolution of the prepared state for an attenuator or an amplifier is the solution of the predictive Born-Markov master equation, which takes the form

$$\dot{\hat{\rho}}(t) = \gamma N_{att}(T)(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \gamma[N_{att}(T)+1](2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \tag{97}$$

for the attenuator, and the form

$$\dot{\hat{\rho}}(t) = \gamma[N_{att}(|T|)+1](2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \gamma N_{att}(|T|)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \tag{98}$$

for the amplifier [24], where the dot means derivative with respect to t. The parameter γ is related to the attenuation and amplification factors by equations $K=e^{-2\gamma t}$ and $G=e^{2\gamma t}$ as before, so these two equations represent an attenuator and an amplifier, which satisfy $KG=1$. In this section we will derive the retrodictive master equation for an attenuator whose predictive master equation is Eq. (97) using the general procedure followed in Ref. [13], and show that it is identical to Eq. (98).

We start by using the fact that the conditional probability does not depend on the time of collapse,

$$P(j|i) = \mathrm{Tr}[\hat{\rho}_i^{pred}(t)\hat{\Pi}_j(t)], \tag{99}$$

for all t between preparation and measurement. Thus the derivative of the probability with respect to this intermediate time must vanish, and using the rule for differentiating a product we have

$$\mathrm{Tr}[\hat{\rho}_i^{pred}(t)\dot{\hat{\Pi}}_j(t)] = -\mathrm{Tr}[\dot{\hat{\rho}}_i^{pred}(t)\hat{\Pi}_j(t)]. \tag{100}$$

We can use the master equation for the density operator to find the equation for the time derivative of the POM element, the evolution equation that it must satisfy as it evolves backwards in time to the intermediate time. We substitute Eq. (97) into Eq. (100), and use the cyclic property of the trace to give (all indices are suppressed)

$$\begin{aligned}\mathrm{Tr}[\hat{\rho}(t)\dot{\hat{\Pi}}(t)] &= -\gamma\mathrm{Tr}(\{N_{att}(T)(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ &\quad + [N_{att}(T)+1](2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})\}\hat{\Pi}(t)) \\ &= -\gamma\mathrm{Tr}\{N_{att}(T)(2\hat{\rho}\hat{a}\hat{\Pi}\hat{a}^\dagger - \hat{\rho}\hat{\Pi}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\Pi}) \\ &\quad + [N_{att}(T)+1](2\hat{\rho}\hat{a}^\dagger\hat{\Pi}\hat{a} - \hat{\rho}\hat{\Pi}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\Pi})\}.\end{aligned} \tag{101}$$

This equation is true for all density operators so the evolution equation for the POM element is

$$\begin{aligned}\dot{\hat{\Pi}}(t) &= -\gamma N_{att}(T)(2\hat{a}\hat{\Pi}\hat{a}^\dagger - \hat{\Pi}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\Pi}) \\ &\quad -\gamma[N_{att}(T)+1](2\hat{a}^\dagger\hat{\Pi}\hat{a} - \hat{\Pi}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\Pi}).\end{aligned} \tag{102}$$

The retrodictive density operator is given by $\hat{\rho}^{retr} = \hat{\Pi}/\mathrm{Tr}\hat{\Pi}$, so substituting Eq. (102) we obtain

$$\begin{aligned}\dot{\hat{\rho}}^{retr} &= \frac{\dot{\hat{\Pi}}\mathrm{Tr}\hat{\Pi} - \hat{\Pi}\mathrm{Tr}\dot{\hat{\Pi}}}{(\mathrm{Tr}\hat{\Pi})^2} = \frac{\dot{\hat{\Pi}}}{\mathrm{Tr}\hat{\Pi}} - \hat{\rho}^{retr}\frac{\mathrm{Tr}\dot{\hat{\Pi}}}{\mathrm{Tr}\hat{\Pi}} \\ &= -\gamma N_{att}(T)(2\hat{a}\hat{\rho}^{retr}\hat{a}^\dagger - \hat{\rho}^{retr}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}^{retr}) \\ &\quad -\gamma[N_{att}(T)+1](2\hat{a}^\dagger\hat{\rho}^{retr}\hat{a} - \hat{\rho}^{retr}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}^{retr}) \\ &\quad + \gamma\hat{\rho}^{retr}.\end{aligned} \tag{103}$$

where we have used the commutator $[\hat{a},\hat{a}^\dagger]=1$ and the unit trace of the density operator. A little more algebra using the commutator gives

$$\begin{aligned}\dot{\hat{\rho}}^{retr} &= -\gamma N_{att}(T)(2\hat{a}\hat{\rho}^{retr}\hat{a}^\dagger - \hat{\rho}^{retr}\hat{a}^\dagger\hat{a} \\ &\quad - \hat{a}^\dagger\hat{a}\hat{\rho}^{retr}) - \gamma[N_{att}(T)+1](2\hat{a}^\dagger\hat{\rho}^{retr}\hat{a} - \hat{\rho}^{retr}\hat{a}\hat{a}^\dagger \\ &\quad - \hat{a}\hat{a}^\dagger\hat{\rho}^{retr}),\end{aligned} \tag{104}$$

which is the evolution equation for the retrodictive density operator in the forward time direction. If we define the premeasurement time $\tilde{t}=t_m-t$ we obtain the retrodictive master equation for the attenuator,

$$\begin{aligned}\frac{d\hat{\rho}^{retr}}{d\tilde{t}} &= \gamma N_{att}(T)(2\hat{a}\hat{\rho}^{retr}\hat{a}^\dagger - \hat{\rho}^{retr}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}^{retr}) \\ &\quad + \gamma[N_{att}(T)+1](2\hat{a}^\dagger\hat{\rho}^{retr}\hat{a} - \hat{\rho}^{retr}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}^{retr}).\end{aligned} \tag{105}$$

This is the reverse time evolution equation for the retrodictive density operator for the attenuator. It has the same form as Eq. (98), provided that the attenuator and amplifier temperatures satisfy the relation $T_{att}=-T_{amp}=|T_{amp}|$. Thus the retrodictive master equation for an attenuator is identical to the predictive master equation for an amplifier provided that the gain of the amplifier is the inverse of the loss of the attenuator, and provided that the effective noise temperatures of the two devices are identical. The converse result is similarly proved.

VII. CONCLUSIONS

In this paper we have shown, by direct calculation of the input and output density-matrix elements, that optical attenuation and amplification are predictive/retrodictive inverse processes. In other words, the state transformations that they effect on the optical field are the time reverse of one another. Retrodictive quantum mechanics saves calculational effort when reconstructing messages in quantum communication. This paper shows that instead of performing, say, a retrodictive calculation on an attenuating system, it is possible to perform a predictive calculation using an amplifier, and obtain identical results.

We have applied our results to ideal devices, which add the minimum amount of noise, to the particular example of direct detection. We have derived retrodictive density matrix elements for single-shot measurements by perfect detectors, with unit quantum efficiency, imperfect detectors with quantum efficiency less than unity, and to detectors that exhibit dark counts. We have extended the method further to apply to the case where multiple measurements are made on many identical copies of the same state, allowing for the reconstruction of the density matrix of the input to the device.

In classical physics, noiseless attenuation and amplification are allowed, and these two processes are the reverse of one another. If we pre- or post-amplify an attenuated signal using an amplifier whose gain is the reciprocal of the attenuator loss we would obtain the unattenuated signal at the detector. Thus a lossy detector could easily be compensated by using a classical amplifier. Such a noise-free amplifier is disallowed in quantum physics. Both attenuators and amplifiers add noise to a quantum signal, merely by changing the probabilities that the state contains particular numbers of photons. For an ideal attenuator the mean output photon number need not contain any extra noise photons. Amplification, however, adds noise photons to the amplified mean input photon number, an ideal minimum of $G-1$ noise photons, where G is the multiplicative gain of the amplifier. This limits our attempts to reconstruct the unattenuated signal output from a noise-free attenuator exactly, even by ideal amplification, as there is always noise added to the signal. In quantum physics amplification and attenuation do not seem to have the useful classical reversing property, and one question which might be legitimately asked is whether the quantum theories describing them satisfy the correspondence principle.

This paper restores a pleasing time-reverse symmetry to the two processes in quantum theory. An interesting consequence of this is that, for ideal devices, the noise properties of amplification and attenuation are reversed in retrodictive quantum theory. If we detect an amplified signal and wish to reconstruct the signal prior to amplification we do not have to subtract any amplifier added noise photons, as the retrodictive state will be the attenuated measured state. The mean number of photons retrodicted at the input contains no noise photons for an ideal device. If, however, we detect an attenuated state, as must always occur for a detector with nonunit detector efficiency, the reconstructed state prior to the device will be affected by amplifier noise photons. These relationships between the mean numbers of input and output photons in prediction and retrodiction for both devices ensure that the theories satisfy the correspondence principle in the classical limit.

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