Initial state maximizing the nonexponentially decaying survival probability for unstable multilevel systems

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(Received 5 June 2004; published 24 September 2004)

The long-time behavior of the survival probability for unstable multilevel systems that follows the power-decay law is studied based on the N-level Friedrichs model, and is shown to depend on the initial population in unstable states. A special initial state maximizing the asymptote of the survival probability at long times is found and examined by considering the spontaneous-emission process for the hydrogen atom interacting with the electromagnetic field.

DOI: 10.1103/PhysRevA.70.032108 PACS number(s): 03.65.Db, 32.80.−t, 42.50.Md, 42.50.Vk

One of the crucial characters of unstable systems is the famous exponential-decay law. Observations of the law were made for many quantum systems, and its theoretical description also proved to be attributed to the poles on the second Riemann sheet of the complex energy plane [1]. However, the deviation from the exponential-decay law was also predicted both for short times and for long times [2]. Indeed, despite an apparent difficulty [3], a nonexponential decay law at short times was successfully observed [4]. On the other hand, the long-time deviation has still not been detected, even though it is expected for all systems coupled with the continuum of the lower-bounded energy spectrum. The main reasons behind this could be ascribed to a too small survival probability, that is, the component of the initial state remaining in the state at long times.

The unstable systems are described by the Friedrichs model [5,6], which enables us to investigate the time evolution involving such processes as the spontaneous emission of photons from the atoms [7,8] and the photodetachment of electrons from the negative ions [8–11]. In the former, often only the first excited level is counted, while other higher ones are neglected, and in the latter the negative ion is assumed to have only one electron bound state. These single-level approximations (SLA) could be verified as long as the lowest level is quite separate from the higher ones. However, the multilevel treatment of the model gives us another advantage, namely the choice of coherently superposed initial states extending over various levels. In fact, it can yield a variety of temporal behavior that is never found in the SLA [12–15]. Such multilevel effects on temporal behavior are still not well studied except for Refs. [12–16], and much less examined with respect to nonexponential decay at long times.

In the present article, we consider the long-time behavior of the survival probability $S(t)$ by examining the N-level Friedrichs model. In particular, restricting ourselves to the weak-coupling case, we clarify how the asymptote of $S(t)$ depends on the initial states. By choosing the initial state localized at the lowest level, we look at the SLA from a multilevel treatment. Then, the result in the N-level model turns out to agree with that in the SLA in the weak-coupling regime. Furthermore, among the various initial states, we can find a special one that maximizes the asymptote of $S(t)$ at long times. Initial states that eliminate the first term of the asymptotic expansion of $S(t)$ are also obtained. For clarity of discussion, we assume all form factors to vanish at zero energy. However, the existence of such special initial states is proved to be quite general and independent of other details of the form factors.

The N-level Friedrichs model describes the couplings between the discrete spectrum and the continuous spectrum. The Hamiltonian of the model is defined by

$$H = H_0 + \lambda V, \tag{1}$$

where H_0 denotes the free Hamiltonian,

$$H_0 = \sum_{n=1}^{N} \omega_n |n\rangle\langle n| + \int_0^\infty d\omega\omega|\omega\rangle\langle\omega|, \tag{2}$$

and λV the interaction Hamiltonian,

$$V = \sum_{n=1}^{N} \int_0^\infty d\omega[v_n^*(\omega)|\omega\rangle\langle n| + v_n(\omega)|n\rangle\langle\omega|], \tag{3}$$

with the coupling constant λ. The eigenvalues ω_n of H_0 were supposed to be nondegenerate, i.e., $\omega_n < \omega_{n'}$ for $n<n'$. Both $|n\rangle$ and $|\omega\rangle$ are the bound and scattering eigenstates of H_0, respectively, and satisfy the orthonormality condition $\langle n|n'\rangle = \delta_{nn'}$, $\langle\omega|\omega'\rangle = \delta(\omega-\omega')$, and $\langle n|\omega\rangle = 0$, where $\delta_{nn'}$ is Kronecker's delta and $\delta(\omega-\omega')$ is Dirac's delta function. They also compose the complete orthonormal system with the resolution of identity. In Eq. (3), $v_n(\omega)$ denotes the form factor characterizing the transition between $|n\rangle$ and $|\omega\rangle$. In the latter discussion, we will simplify the model with the assumption that the form factor $v_n(\omega)$ is an analytic function in a complex domain including the cut $(0, \infty)$, and behaves like

$$v_n(\omega) = \begin{cases} q_n\omega^{p_n} & (\omega \to +0) \\ s_n\omega^{-r_n} & (\omega \to \infty), \end{cases} \tag{4}$$

where p_n and r_n are the positive constants, while q_n and s_n are appropriate ones. The small-energy condition ensures

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that the integral $\int_0^\infty d\omega v_n(\omega)v_{n'}^*(\omega)/\omega$ is definite. The large-energy condition ensures that $\int_0^\infty d\omega v_n(\omega)v_{n'}^*(\omega)/(z-\omega)$ is definite for all complex numbers $z\notin[0,\infty)$. Both of the conditions are satisfied by several systems involving the spontaneous-emission process of photons [7,8] and the photodetachment process of electrons [8–11]. Note that this small-energy condition excludes the photoionization processes associated with the Coulomb interaction [17]; however, the formulation developed below could be applied to those cases.

The initial state $|\psi\rangle$ of our interest is an arbitrary superposition of the unstable states $|n\rangle$,

$$|\psi\rangle = \sum_{n=1}^{N} c_n|n\rangle, \tag{5}$$

where c_n's are complex numbers satisfying the normalization condition $\sum_{n=1}^N|c_n|^2=1$. Then, the survival probability $S(t)$ of the initial state $|\psi\rangle$, that is, the probability of finding the initial state in the state at a later time t, is defined by $S(t)=|A(t)|^2$. The $A(t)$ denotes the survival amplitude of $|\psi\rangle$, i.e., $A(t)=\langle\psi|e^{-itH}|\psi\rangle$. In general, the Hamiltonian (1) has the possibility of possessing not only the scattering eigenstates $|\psi_\omega^{(\pm)}\rangle$, but also the bound eigenstates [18]. We shall restrict ourselves here to studying the decaying part of $A(t)$, and merely call it the survival amplitude with the same symbol as

$$A(t) = \int_0^\infty d\omega e^{-it\omega}|\langle\psi_\omega^{(\pm)}|\psi\rangle|^2. \tag{6}$$

In order to estimate the long-time behavior of $A(t)$, let us evaluate the scattering eigenstates $|\psi_\omega^{(\pm)}\rangle$ by solving the Lippmann-Schwinger equation, i.e., $|\psi_\omega^{(\pm)}\rangle=|\omega\rangle+(\omega\pm i0-H_0)^{-1}\lambda V|\psi_\omega^{(\pm)}\rangle$. In the case of our Hamiltonian, this equation can be solved in the form

$$|\psi_\omega^{(\pm)}\rangle = |\omega\rangle + \sum_{n=1}^{N} F_n^{(\pm)}(\omega)\left[|n\rangle + \int_0^\infty d\omega' \frac{\lambda v_n^*(\omega')}{\omega-\omega'\pm i0}|\omega'\rangle\right],$$

from which the integrand of $A(t)$ reads

$$\langle\psi_\omega^{(\pm)}|\psi\rangle = \sum_{n=1}^{N} F_n^{(\pm)*}(\omega)c_n. \tag{7}$$

The $F_n^{(\pm)}(\omega)$ is determined by an algebraic equation

$$\sum_{n'=1}^{N} G_{nn'}^{-1}(\omega\pm i0)F_{n'}^{(\pm)}(\omega) = -\lambda v_n(\omega), \tag{8}$$

where

$$G_{nn'}^{-1}(z) \equiv (\omega_n - z)\delta_{nn'} + \lambda^2 s_{nn'}(z), \tag{9}$$

which is the (n,n')th component of the $N\times N$ matrix $G^{-1}(z)$, and $s_{nn'}(z)$ is defined by

$$s_{nn'}(z) \equiv \int_0^\infty d\omega' \frac{v_n(\omega')v_{n'}^*(\omega')}{z-\omega'}, \tag{10}$$

for all $z=re^{i\varphi}$ $(r>0, 0<\varphi<2\pi)$. Under the large-energy condition of Eq. (4), $s_{nn'}(z)$ is guaranteed to be analytic in the whole complex plane except the cut [0, ∞). For later convenience, $G^{-1}(z)$ is defined as an inverse of $G(z)$, where $G(z)$ is assumed to be regular. Note that $G(z)$ is nothing more than the reduced (or partial) resolvent $G_{nn'}(z)=\langle n|(H-z)^{-1}|n'\rangle$. One can confirm this fact by following the discussion in Sec. 3.2 of Ref. [6]. Since the behavior of $A(t)$ at long times is characterized by that of $F_n^{(\pm)}(\omega)$ in Eq. (7) at small energies, we need to estimate the small-energy behavior of $G(z)$. Note that under the condition (4) we have

$$\begin{aligned} G_{nn'}^{-1}(\omega\pm i0) &= (\omega_n-\omega)\delta_{nn'} + \lambda^2[I_{nn'}(\omega) \mp i\pi v_n(\omega)v_{n'}^*(\omega)] \\ &= \omega_n\delta_{nn'} + \lambda^2 I_{nn'}(0) + o(1) \end{aligned} \tag{11}$$

as $\omega\to+0$, where $s_{nn'}(\omega\pm i0)=I_{nn'}(\omega)\mp i\pi v_n(\omega)v_{n'}^*(\omega)$ and

$$I_{nn'}(\omega) \equiv \mathrm{P}\int_0^\infty d\omega' \frac{\varphi_n(\omega')\varphi_{n'}^*(\omega')}{\omega-\omega'},$$

where P denotes the principal value of the integral. The existence of $I_{nn'}(0)$ may be just guaranteed by the small-energy condition of Eq. (4). Supposing that $G_{nn'}$ is of the form

$$G_{nn'}(\omega\pm i0) = g_{nn'} + o(1) \tag{12}$$

as $\omega\to+0$, one obtains that

$$\delta_{nn'} = \sum_{m=1}^{N} G_{nm}G_{mn'}^{-1} = \sum_{m=1}^{N} g_{nm}[\omega_m\delta_{mn'} + \lambda^2 I_{mn'}(0)] + o(1), \tag{13}$$

which leads to

$$g_{nn'} = \frac{1}{\omega_{n'}}\left[\delta_{nn'} - \lambda^2\sum_{m=1}^{N} g_{nm}I_{mn'}(0)\right]. \tag{14}$$

We solve this equation by assuming that $g_{nn'}$ can be expanded for small λ as

$$g_{nn'} = \sum_{j=0}^{\infty} g_{nn'}^{(j)}\lambda^{2j}. \tag{15}$$

By substituting Eq. (15) into Eq. (14), it follows that

$$g_{nn'}^{(0)} = \delta_{nn'}/\omega_{n'},\ g_{nn'}^{(1)} = -I_{nn'}(0)/\omega_n\omega_{n'}, \tag{16}$$

and for $j\geq 1$

$$g_{nn'}^{(j)} = -\frac{1}{\omega_{n'}}\sum_{m=1}^{N} g_{nm}^{(j-1)}I_{mn'}(0), \tag{17}$$

where we have assumed that all ω_n does not vanish. Note that $g_{nn'}^{(0)}$ and $g_{nn'}^{(1)}$ derived here accord with at least those for solvable cases, where $G(z)$ is explicitly obtained [14,16]. We can then obtain

$$F_n^{(\pm)}(\omega) = -\lambda f_n \omega^p + o(\omega^p), \tag{18}$$

with

$$f_n \equiv \frac{\tilde{q}_n}{\omega_n} - \lambda^2 \sum_{n'=1}^{N} \frac{I_{nn'}(0)\tilde{q}_{n'}}{\omega_n \omega_{n'}} + O(\lambda^4), \tag{19}$$

where

$$\tilde{q}_n = \begin{cases} q_n & (p_n = p) \\ 0 & (p_n \neq p), \end{cases} \tag{20}$$

where $p = \min\{p_n\}$. With use of the $\tilde{q}_n$ instead of q_n, we extracted only the dominant part of $F_n^{(\pm)}(\omega)$ at small ω.

The long-time behavior of $A(t)$ can be simply obtained by applying to Eq. (6) the asymptotic method for the Fourier integral [19]. As mentioned before, the long-time behavior is determined by the small-energy behavior of its integrand. By inserting Eq. (18) into Eq. (7), the integrand of $A(t)$ turns out to behave asymptotically,

$$|\langle \psi_\omega^{(\pm)} | \psi \rangle|^2 = \lambda^2 \left| \sum_{n=1}^{N} f_n^* c_n \right|^2 \omega^{2p} + o(\omega^{2p}) \tag{21}$$

as $\omega \to +0$. Applying the asymptotic formula for Fourier integrals, we obtain from Eq. (21) the asymptotic behavior of Eq. (6) reading

$$A(t) = \lambda^2 \frac{\Gamma(2p+1)}{(it)^{2p+1}} \left| \sum_{n=1}^{N} f_n^* c_n \right|^2 + o(t^{-2p-1}) \tag{22}$$

as $t \to \infty$, where $i^{2p+1} = e^{i(2p+1)\pi/2}$ and $\Gamma(z+1) = \int_0^\infty dx x^z e^{-x}$. We can clearly perceive $A(t) \sim t^{-2p-1}$, the power-decay law.

Using the above result, let us first consider the higher-level effects on the long-time behavior that starts from the localized initial state at the lowest level. This study is directed to an examination of the SLA. For such an initial state, i.e., $c_n = \delta_{n1}$, Eq. (22) becomes

$$A(t) = \lambda^2 \frac{\Gamma(2p+1)}{(it)^{2p+1}} \frac{|q_1|^2}{\omega_1^2} [1 + O(\lambda^2)] + o(t^{-2p-1}), \tag{23}$$

where we supposed that $\tilde{q}_1 \neq 0$. Since there are no factors related to the higher levels in Eq. (23), it implies that the long-time asymptotic behavior of $A(t)$ could agree with that in the SLA for a sufficiently small λ.

On the other hand, we can find a special superposition of unstable states $|n\rangle$ that maximizes the asymptote of $A(t)$ at long times. It is worth noting that its dependence on the initial states only appears in Eq. (22) through the factor $\Sigma_{n=1}^N f_n^* c_n$, which can be rewritten by an inner product as

$$\sum_{n=1}^{N} f_n^* c_n = \langle \chi | \psi \rangle, \tag{24}$$

where we have introduced an auxiliary vector defined by

$$|\chi\rangle \equiv \sum_{n=1}^{N} f_n |n\rangle. \tag{25}$$

Thus, resorting to the Schwarz inequality, we see that the maximum of the factor (24) is just attained if and only if $|\psi\rangle \propto |\chi\rangle$, i.e.,

$$c_n = c f_n / \|\chi\|, \tag{26}$$

where c is an arbitrary complex number with $|c| = 1$. Therefore, preparing the initial state $|\psi\rangle$ according to the above weights (26), we can maximize the asymptote of $A(t)$ at long times. Substituting Eq. (26) into Eq. (22), one obtains that

$$A(t) = \lambda^2 \frac{\Gamma(2p+1)}{(it)^{2p+1}} \|\chi\|^2 + o(t^{-2p-1}) \tag{27}$$

$$\simeq \lambda^2 \frac{\Gamma(2p+1)}{(it)^{2p+1}} \sum_{n=1}^{N} \left| \frac{\tilde{q}_n}{\omega_n} \right|^2. \tag{28}$$

It should be remarked that the initial state extended over unstable states $|n\rangle$ has the possibility of increasing the intensity of $A(t)$ more than a localized one would. This possibility may be interpreted as follows. Let us consider the spontaneous-emission process for an atom interacting with the electromagnetic field, where $|n\rangle$ is identified with the $(n+1)$th excited state of the atom with the vacuum state of the field and $|\omega\rangle$ is the ground state with the one-photon state. In this process, an initially excited atom makes a transition to the ground state with emitting a photon, while the atom that fell into the ground state can be reexcited by absorbing a photon. In the latter process, there are various candidates for the excited state. Repopulation of each excited level can make the intensity of $A(t)$ grow, providing that the initial state possesses those excited levels. However, if the initial state only consists of a specific excited state, the other excited states composing the state at a later time t are discarded without any contribution to $A(t)$ [20]. This is the reason why the decay of the $A(t)$ for extended states can be relaxed more than that for localized states.

Note that the above argument also suggests the possibility of finding other kinds of initial states that are coherently superposed to eliminate the factor (24). This is indeed achieved by the initial states that are orthogonal to $|\chi\rangle$,

$$\langle \chi | \psi \rangle = 0. \tag{29}$$

In this case, the first term on the right-hand side of Eq. (22) becomes zero. This fact means that $A(t)$ for such an orthogonal state asymptotically decays faster than t^{-2p-1}.

The maximizing initial state seems to be desirable for an experimental verification of the power-decay law. Let us now discuss the value of $|A(t)|^2$ for such an initial state at long times. In particular, we shall evaluate this value at the time t_{ep} of the transition from the exponential to the power-decay law. We have to know the exact values of both p_n and q_n for many n's; however, this requirement is satisfied by considering the spontaneous-emission process for the hydrogen atom interacting with the electromagnetic field (see also Refs. [7,8]). This time, $|n\rangle$ is interpreted as the $(n+1)p$ state of the

TABLE I. The level-number dependence of $\Sigma_{n=1}^{N}|q_n/\omega_n|^2$, the decay time t_N of the $(N+1)p$ state, and the transition time t_{ep} from the exponential to the power-decay law.

Number of levels N	1	10	50
$\vert\omega_1/q_1\vert^2 \sum_{n=1}^{N} \vert q_n/\omega_n\vert^2$	1.00	1.28	1.29
t_N (s)	1.60×10^{-9}	3.18×10^{-7}	3.18×10^{-5}
t_{ep} (s)	2.00×10^{-7}	4.23×10^{-5}	4.59×10^{-3}

atom with the vacuum state of the field, and $|\omega\rangle$ as the 1s state with the one-photon state. It then follows that $p_n=\frac{1}{2}$ for all n [21], and $|q_n|$ is determined through the relation $\gamma_n = 2\pi\lambda^2|v_n(\omega_n)|^2+O(\lambda^2)\simeq 2\pi\lambda^2|q_n|^2|\omega_n|$, where the last estimation is confirmed in the dipole approximation. γ_n is the decay rate of the $(n+1)p$ state, which is estimated as $\gamma_n \simeq 8.0\times10^9\times2^8(n+1)n^{2n}/9(n+2)^{2n+4}$ s^{-1} [22]. From these facts, it follows that

$$\lambda^2\left|\frac{q_n}{\omega_n}\right|^2 \simeq \frac{8.0\times10^9\times6(n+1)^7n^{2n}}{\pi\Omega^3(n+2)^{2n+4}[(n+1)^2-1]^3}\mathrm{s}^2, \qquad (30)$$

where $\omega_n=\frac{4}{3}\Omega[1-(n+1)^{-2}]$ with $\Omega=1.55\times10^{16}$ s^{-1}, and we also choose $\lambda=6.43\times10^{-9}$. Thus, we see that $|q_n|^2/|\omega_n|^2 \sim n^{-3}$ for large n. In Table I, the numerical values of $\Sigma_{n=1}^{N}|q_n/\omega_n|^2(\simeq\|\chi\|^2)$, the decay time $t_N(=1/\gamma_N)$ of the $(N+1)p$ state, and the time t_{ep} are listed for the three cases of the level numbers $N=1$, 10, and 50. Here, we define t_{ep} as the maximum time which equates the square modulus of the asymptote (28) to that of the following $A(t)$ at intermediate times [13,16]:

$$A(t)\simeq\sum_{n=1}^{N}|c_n|^2e^{-it\omega_n-t\gamma_n/2}, \qquad (31)$$

where c_n is chosen as Eq. (26). It is worth noting that when $t\gg t_N$, $|A(t)|^2$ can approximate $|c_N|^4e^{-t\gamma_N}$ because the decay time t_n lengthens with n in this case. We see from Table I that t_{ep} is much longer than t_N, so that t_{ep} is roughly estimated as the root of the equation,

$$|c_N|^4e^{-t\gamma_N}=\frac{\lambda^4}{t^4}\left|\sum_{n=1}^{N}\left|\frac{q_n}{\omega_n}\right|^2\right|^2. \qquad (32)$$

On the other hand, the factor $\Sigma_{n=1}^{N}|q_n/\omega_n|^2$ is essentially unchanged with N, whereas t_{ep} rapidly increases with N (see Table I). These facts and Eq. (32) imply that $A(t_{ep})$ rather decreases as N increases [23]. Hence, we should unfortunately conclude that the maximizing initial state does not provide any help for an observation of the power-decay law for the spontaneous emission from a hydrogen atom.

In summary, we have considered the long-time behavior of the unstable multilevel systems and estimated the asymptotic behavior of the survival amplitude $A(t)$ for an arbitrary initial state in the long-time region where $A(t)$ obeys a power-decay law. We have then found two special initial states. One of them asymptotically maximizes $A(t)$ at long times, and the other eliminates the first term of the asymptotic expansion of $A(t)$. The latter fact may imply that the exponent of the power decay of $A(t)$ is determined by not only the small-energy behavior of the form factors but also the initial population in unstable states. Such relations between the initial states and the power-decay law were studied with respect to the long-time behavior of wave packets, both for the free-particle system [24] and for finite-range potential systems [25]. In the case of the experimental verification of the power-decay laws, the existence of the maximizing initial states seems preferable. This expectation is probably misplaced for the spontaneous-emission process of a hydrogen atom, however a possibility still could remain for the systems allowing the photodetachment or the photoionization process. We then should require of them the property that both γ_n and $|q_n/\omega_n|$ do not decrease as n increases. More important states for this aim are those states which maximize $A(t)$ at the transition time from the exponential to the power-decay law. The relation between such a maximizing state and the discussed one is still unclear. It will be addressed in a future issue.

The author would like to thank Professor I. Ohba and Professor H. Nakazato for useful comments, and Dr. J. Harada for helpful discussions. He would also like to thank the Yukawa Institute for Theoretical Physics at Kyoto University, where this work was initiated during the YITP-03-16, Quantum Mechanics and Chaos: From Fundamental Problems through Nanosciences. This work is partly supported by a grant for the 21st Century COE Program at Waseda University from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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