Damping of Bogoliubov excitations in optical lattices

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Extending recent work to finite temperatures, we calculate the Landau damping of a Bogoliubov excitation in an optical lattice, due to the coupling to a thermal cloud of such excitations. For simplicity, we consider a one-dimensional Bose-Hubbard model and restrict ourselves to the first energy band. For energy conservation to be satisfied, the excitations in the collision processes must exhibit "anomalous dispersion," analogous to phonons in superfluid ^{4}He. This leads to the disappearance of all damping processes when $Un^{c0} \geqslant 6J$, where U is the on-site interaction, J is the hopping matrix element, and $n^{c0}(T)$ is the number of condensate atoms at a lattice site. This phenomenon also occurs in two-dimensional and three-dimensional optical lattices. The disappearance of Beliaev damping above a threshold wave vector is noted.

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Recently, several experimental papers have reported results on the damping of collective modes in Bose condensates in a one-dimensional (1D) periodic optical lattice potential [1]. Most theoretical studies [2–7] have concentrated on the case of a pure condensate at $T=0$ and have ignored the effect of a thermal cloud, which can cause the damping of condensate excitations at finite T. We discuss a Bose-Hubbard tight-binding model using the Gross-Pitaevskii (GP) approximation, but generalized to include a static thermal cloud of non-condensate atoms. For illustration, we consider a 1D model but emphasize the same features that appear in two-dimensional (2D) and three-dimensional (3D) optical lattices [8]. We limit our analysis to relatively strong optical lattices such that the lowest energy band is fairly narrow. We calculate the temperature dependence of the number of condensate atoms $n^{c0}(T)$ in each lattice well, as well as the Landau damping of condensate modes due to coupling to thermal excitations. We find that for damping processes to occur, the dispersion relation E_q of the Bogoliubov Bloch excitations must initially bend upward as the quasimomentum q increases. This "anomalous dispersion" is also the source of three-phonon damping in superfluid ^{4}He [9,10]. This condition leads to the disappearance of all damping in an 1D optical lattice when the normalized interaction strength $\alpha \equiv Un^{c0}/J > 6$ (where U is the on-site interaction and J is the hopping matrix element). More generally, all damping processes involving three excitations vanish in a D-dimensional simple cubic tight-binding optical lattice when $\alpha > 6D$ [8]. This ability to control the excitation damping in optical lattices may be very important in applications.

Since our major interest is in the thermal gas of excitations in an optical lattice, it is important to keep in mind the distinction between the Bloch excitations associated with linearized fluctuations of an equilibrium Bose condensate and the stationary states of the time-independent GP equation for the Bose order parameter. In a continuum model, the latter states [2,3] can be described by the eigenfunctions $\Phi_k^0(x) = e^{ikx}u_k(x)$, where the Bloch condensate function satisfies the usual periodicity condition $u_k(x)=u_k(x+ld)$, where d is the optical lattice spacing and l is an integer. Physically, $\Phi_k^0(x)$ corresponds to a solution of the static GP equation with a superfluid flow in the periodic potential, with the condensate quasimomentum being given by $\hbar k$. The fluctuations of these states are also described by a quasimomentum q in the first Brillouin zone (BZ) and will be referred to as Bogoliubov Bloch excitations of the optical lattice. The thermal cloud at finite temperatures is an incoherent gas of these Bogoliubov excitations in the first band. In our tight-binding model, the Bloch condensate function is $\Phi_k^0(l)=e^{ikld}\sqrt{n^{c0}}$ and we only consider a Bose condensate in the $k=0$ Bloch state, i.e., $\Phi_{k=0}^0(l)=\sqrt{n^{c0}}$.

The 1D optical lattice model (along the z axis) has been discussed in many recent papers at $T=0$ [2–5]. The optical lattice is described by $V_{\rm op}(z)=sE_R\cos^2(kz)$, where $E_R = \hbar^2k^2/2m$ is the photon recoil energy. We assume that the band-gap energy between the first and the second excitation band is large compared to the temperature $(2k_BT/E_R \ll s)$, and thus only the first band is thermally occupied. The radial trapping frequency is assumed to be very large, so that motion in this direction is effectively frozen. The depth of the optical lattice wells is assumed large enough to make the atomic wave functions well localized on the individual lattice sites, described by a tight-binding approximation. This Hamiltonian is the Bose-Hubbard model,

$$H = -J\sum_l (a_{l+1}^\dagger a_l + a_l^\dagger a_{l+1}) + \frac{1}{2}U\sum_l a_l^\dagger a_l^\dagger a_l a_l, \qquad (1)$$

where a_l and $a_l^\dagger$ are the creation and destruction operators of atoms in the radial ground state on the lth lattice site. The hopping matrix element between adjacent sites is $J = -\int dz w_l^*(z)\left[-(\hbar^2/2m)(d^2/dz^2)+V_{\rm op}(z)\right]w_{l+1}(z)$, where $w_l(z)$ is a function localized on the lth lattice site. The on-site interatom interaction is $U=g\int d\mathbf{r}_\perp|\phi_0(\mathbf{r}_\perp)|^4\int dz|w_l(z)|^4$, where $g=4\pi\hbar^2a/m$ and a is the s-wave scattering length. $\phi_0(\mathbf{r}_\perp)$ is the wave function in the radial direction.

Expanding around the minima of the optical lattice potential wells in a harmonic approximation, the well trap frequency is $\omega_s \equiv s^{1/2}(\hbar k^2/m)$, which is assumed to be much larger than the harmonic trapping frequency along the z axis. Approximating the localized atomic wave function as a Gaussian at the potential minima of lth site, $w_l(z) = (m\omega_s/\pi\hbar)^{1/4}\exp[-(m\omega_s/2\hbar)(z-z_l)^2]$, one obtains

$$\frac{J}{E_R} \sim \left[\left(\frac{\pi^2 s}{4} - \frac{s^{1/2}}{2}\right) - \frac{1}{2}s(1+e^{-s^{-1/2}})\right]e^{-\pi^2 s^{1/2}/4}. \quad (2)$$

Taking the wave function in the radial direction to be the ground-state wave function of a harmonic oscillator, one obtains ($d=\pi/k$ is the lattice period)

$$\frac{U}{E_R} \sim \frac{g}{(2\pi)^{3/2}a_\perp^2 a_s} = \frac{2^{3/2}ad}{\pi^{3/2}a_\perp^2}s^{1/4}, \quad (3)$$

where $a_s=\sqrt{\hbar/m\omega_s}$, $a_\perp=\sqrt{\hbar/m\omega_\perp}$.

We restrict ourselves to the superfluid solutions of Eq. (1), when the phase coherence between wells is well defined in the optical lattice. The superfluid-Mott transition [6,7,11] only arises when the interaction energy is much larger than we consider here. Due to Bose condensation, the destruction operator of an atom at site l can be written as $a_l=\Phi_l+\tilde{\psi}_l$, where $\Phi_l=\langle a_l\rangle$ is the condensate wave function and $\tilde{\psi}_l$ is the noncondensate field operator. This leads to the generalized discrete Gross-Pitaevskii equation [12] (we set $\hbar=1$ from now on),

$$i\frac{\partial}{\partial t}\Phi_l = -J(\Phi_{l+1}+\Phi_{l-1}) + U(n_l^{c}+2\tilde{n}_l)\Phi_l, \quad (4)$$

where $n_l^c(t)$ is the number of condensate atoms on the lth lattice site. This includes the time-dependent Hartree-Fock mean field $2U\tilde{n}_l(t)$ arising from the noncondensate atoms on the lth site. Equation (4) reduces to the usual discrete tight-binding GP equation [4,5] if all the atoms are assumed to be in the condensate (i.e., $\tilde{n}_l=0$). Introducing phase and amplitude variables, $\Phi_l=\sqrt{n_l^c}e^{i\theta_l}$, Eq. (4) reduces to

$$\frac{\partial n_l^c}{\partial t} = -2J\sqrt{n_l^c n_{l+1}^c}\sin(\theta_{l+1}-\theta_l) + 2J\sqrt{n_l^c n_{l-1}^c}\sin(\theta_l-\theta_{l-1}), \quad (5)$$

$$\frac{\partial\theta_l}{\partial t} = J\left[\sqrt{\frac{n_{l+1}^c}{n_l^c}}\cos(\theta_{l+1}-\theta_l) + \sqrt{\frac{n_{l-1}^c}{n_l^c}}\cos(\theta_l-\theta_{l-1})\right] - (Un_l^c+2U\tilde{n}_l) \equiv -\varepsilon_l^c. \quad (6)$$

In Eq. (6), ε_l^c is the energy of a condensate atom, which reduces to the chemical potential of the condensate μ_{c0} in static thermal equilibrium. The equilibrium solution (θ_l and n_l^c are independent of l) is $\theta_l(t)=-\mu_{c0}t$.

The $T=0$ Bogoliubov excitation spectrum [5,7,13] for a uniform optical lattice is easily obtained by ignoring the noncondensate atom term ($\tilde{n}_l=0$) and considering small fluctuations from equilibrium, $n_l^c=n^{c0}+\delta n_l^c$. The normal mode solutions of Eqs. (5) and (6) are

$$\delta\theta_l = \delta\theta(q)e^{i[qld-E_qt]}, \quad \delta n_l^c = \delta n^c(q)e^{i[qld-E_qt]}, \quad (7)$$

with the Bloch Bogoliubov excitation energy,

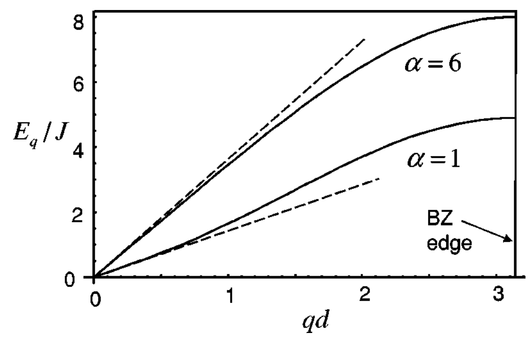

FIG. 1. The Bogoliubov excitation energy E_q (normalized to J) in the first Brillouin zone, for $\alpha=1$ and 6, as a function of the quasimomentum q. The dashed lines give $E_q=cq$.

$$E_q = \sqrt{\epsilon_q^0(\epsilon_q^0+2Un^{c0})}. \quad (8)$$

Here $\epsilon_q^0 \equiv 4J\sin^2(qd/2)$ is the kinetic energy associated with tunneling. This $T=0$ excitation spectrum is shown in Fig. 1 for two values of the interaction ratio $\alpha\equiv Un^{c0}/J$. For small q, the spectrum is phononlike $E_q\simeq cq$, with the phonon speed $c=\sqrt{2Jd^2Un^{c0}}=\sqrt{Un^{c0}/m^*}$, where $m^*\equiv 1/2Jd^2$ is an effective mass of atoms in the optical lattice. We call attention to an important feature of Eq. (8). For $\alpha\leqslant 6$, E_q bends up before bending over, as q approaches the BZ boundary. This "anomalous dispersion" also occurs in superfluid ^{4}He [9,10]. For $\alpha>6$, the spectrum simply bends over as q increases.

As we noted earlier, the energy of a condensate atom ε_l^c on site l is given by the right-hand side of Eq. (6). In static thermal equilibrium, when all sites are identical, the condensate chemical potential is given by $\mu_{c0}=\varepsilon^{c0}=-2J+U[n^{c0}(T)+2\tilde{n}^0(T)]$. We calculate $n^{c0}(T)$ self-consistently by calculating the number of noncondensate atoms $\tilde{n}^0(T)$ at a site using the Bogoliubov excitations given by the static Popov approximation. That is, we ignore the dynamics of the noncondensate [i.e., we set $\delta\tilde{n}_l(t)=0$ in Eq. (6)]. This gives the Bogoliubov-Popov excitation spectrum E_q, which is identical to Eq. (8), except that now $n^{c0}(T)$ is the temperature-dependent number of condensate atoms at any lattice site.

Expressing $\tilde{n}^0(T)$ in terms of these Bogoliubov-Popov excitations, we have [5,7,12]

$$n = n^{c0} + \frac{1}{I}\sum_{|q|\geqslant q_c}[(u_q^2+v_q^2)f^0(E_q)+v_q^2], \quad (9)$$

with the standard Bogoliubov transformation functions $u_q^2 = 1/2(\tilde{E}_q/E_q+1)$, $v_q^2=1/2(\tilde{E}_q/E_q-1)$, and $f^0(E_q) = [\exp(\beta E_q)-1]^{-1}$. The Hartree-Fock (HF) excitation spectrum is $\tilde{E}_q\equiv[\epsilon_q^0+2U(n^{c0}+\tilde{n}^0)]-\mu_{c0}=4J\sin^2(qd/2)+Un^{c0}$. Apart from the limiting case of $\alpha\ll 1$, we note that E_q is quite different from the HF spectrum $\tilde{E}_q$. Strictly speaking, in our 1D model, there are no solutions of Eq. (9) for an infinite system because the summation diverges due the contribution from small momentum, in accordance with the Mermin-Wagner-Hohenberg theorem. However, we consider finite systems which introduce a lower momentum cutoff q_c

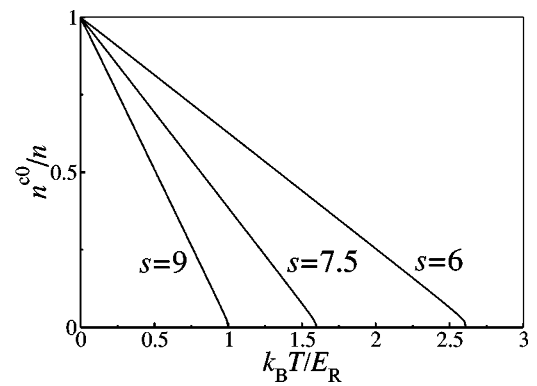

FIG. 2. The site condensate fraction n^{c0}/n as a function of temperature, where s is the strength of the optical lattice potential. Following Ref. [1], we take $n=1200$.

$\equiv 2\pi/Id$. (As a typical value [1], we take the total number of lattice sites $I=250$.) In this case, Eq. (9) has a solution and the condensate can exist. We are only considering a 1D optical lattice for illustration and emphasize that our main conclusions also hold for 2D and 3D optical lattices [8].

The condensate number $n^{c0}(T)$ is found by solving Eq. (9) self-consistently for a fixed value of n. In Fig. 2, we plot $n^{c0}(T)/n$ as a function of T. Using these results, we plot the dimensionless interaction $\alpha \equiv Un^{c0}(T)/J$ as a function of temperature in Fig. 3. The ratio α and the results in Fig. 3 will play a crucial role in our subsequent analysis. Since we limit our discussion to the first energy band of the optical lattice, our results only apply when $s \gg 2k_B T/E_R$. Higher bands would be thermally populated if we considered lower values of s.

We next turn to the calculation of Landau damping. A collective mode in a condensate coupled to a thermal cloud of excitations in thermal equilibrium has a complex frequency, $\omega \equiv E_q - \mathrm{i}\Gamma_q$, with the Landau damping given by [14,15]

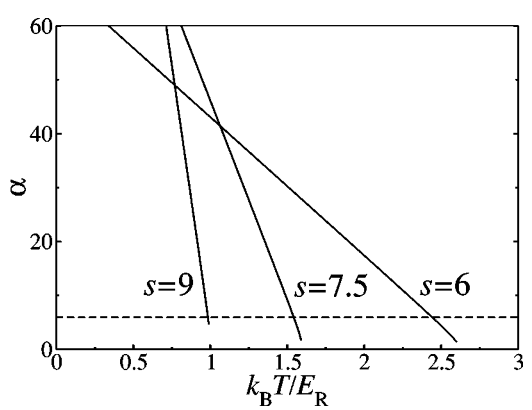

FIG. 3. Plot of the interaction parameter $\alpha = Un^{c0}(T)/J$, as a function of temperature, for several values of s. The number of condensate atoms at a lattice site $n^{c0}(T)$ is given in Fig. 2. The dashed line is $\alpha=6$.

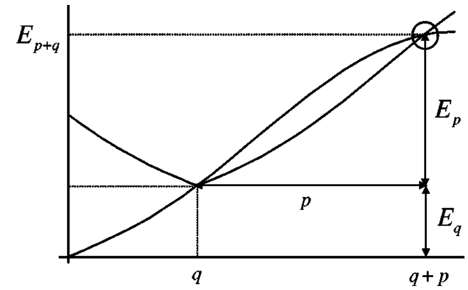

FIG. 4. The Bogoliubov excitation energy E_q for $\alpha < 6$. The intersection of the two dispersion curves at $p+q$ shows that the energy conservation condition is satisfied.

$$\Gamma_q = \pi \sum_{|p| \geq q_c} |M_{q,p;q+p}|^2 [f^0(E_p) - f^0(E_{q+p})] \times \delta(E_q + E_p - E_{q+p}). \quad (10)$$

Here E_p is the Bloch Bogoliubov excitation energy given in Eq. (8) for a uniform condensate in an optical lattice, and the momentum sum is over the first Brillouin zone. The expression in Eq. (10) describes a condensate excitation of (quasi) momentum q being absorbed by an excitation p of the optical lattice thermal gas, leading to a thermal excitation with momentum $q+p$.

The energy conservation condition $E_q + E_p = E_{q+p}$ needs to be satisfied in Eq. (10). This is illustrated in Fig. 4 [16]. Clearly, the intersection at $(q+p, E_{q+p})$ requires that the dispersion relation E_q first bends up as q increases, before bending over. This anomalous dispersion is also a feature of phonons in superfluid ^{4}He [9,10]. The values of (q,p) satisfying $E_q + E_p = E_{q+p}$ are shown in Fig. 5. For a given q, we see that as $\alpha \to 6$, the value of p decreases to zero. There is no solution for $\alpha > 6$, indicating the disappearance of Landau damping.

The matrix element in Eq. (10) is given by [14,15]

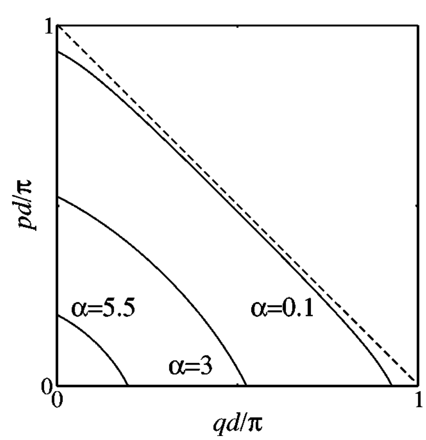

FIG. 5. The values (q,p) satisfying the energy conservation condition $E_q + E_p = E_{q+p}$, for several values of α.

$$M_{q,p_1;p_2} = 2U\sqrt{\frac{n^{c0}}{I}}\sum_{G_n}[(u_{p_1}u_{p_2}+v_{p_1}v_{p_2}-v_{p_1}u_{p_2})u_q - (u_{p_1}u_{p_2}+v_{p_1}v_{p_2}-u_{p_1}v_{p_2})v_q]\delta_{q+p_1,p_2+G_n}, \quad (11)$$

where $G_n = 2\pi n/d$ is a reciprocal lattice vector (n is an integer). In Fig. 5, the curves of the solution of the energy conservation condition never go across the dashed line $q+p = \pi/d$. Thus, Umklapp scattering processes ($G_n \neq 0$) do not contribute to Landau damping [16].

When the excitation q has a wavelength much larger than the thermal excitation p (i.e., $q \ll p,\ \pi/d$), the energy conservation condition $E_q + E_p = E_{q+p}$ in Eq. (10) means that for $q \ll p$, the Landau damping of the excitation $E_q = \mathrm{c}q$ comes from absorbing a thermal excitation E_p with a group velocity $\partial E_p/\partial p$ equal to c [14]. The unique value of p_0 such that $\partial E_{p_0}/\partial p_0 = \mathrm{c}$ is found to be given by the condition $4\sin^2(\frac{1}{2}p_0 d) = -(\alpha-2)+\sqrt{2(\alpha+2)}$. This is only valid for α smaller than 6, such that $p_0 \gg q$.

For $q \ll p$, the matrix element reduces to [the analogue of Eq. (10.76) in Ref. [14]]

$$M_{q,p;q+p} = U\left(\frac{\epsilon_p^0}{E_p} + \frac{E_p}{\tilde{E}_p}\right)\left(\frac{q}{2m^*\mathrm{c}}\right)^{1/2}\frac{N_{c0}^{1/2}}{I}, \quad (12)$$

where $N_{c0} \equiv n^{c0} I$. In deriving Eq. (12) from Eq. (11), we have used the energy conservation factor in Eq. (10). Using the delta function for energy conservation to integrate over p, the Landau damping is given by

$$\Gamma_q = \frac{\alpha}{8(\alpha+2)}\left(\frac{E_{p_0}}{\epsilon_{p_0}^0}\right)^3\left(\frac{\epsilon_{p_0}^0}{E_{p_0}} + \frac{E_{p_0}}{\tilde{E}_{p_0}}\right)^2 \frac{\beta J U q d}{\sinh^2\frac{\beta E_{p_0}}{2}}. \quad (13)$$

One finds there is no damping when $\alpha > 6$ (see Fig. 3 for α as a function of s and T). Γ_q diverges at $\alpha = 6$, but this is due to the 1D nature of our system and does not occur in 2D lattices. Figure 3 shows that, for $n = 1200$ [1], α is smaller than 6 only when T is very close to T_c, for s in the range of 6–9. For a smaller number of atoms on a lattice site, the temperature range where α is less than 6 becomes larger.

For our 1D model to apply, one would need a much tighter magnetic trap (in the radial direction) than used in Ref. [1]. We call attention to the recent technique [17] of producing a two-dimensional array (in the xy plane) of very long, tightly confined condensate tubes along the z axis. With an additional optical lattice imposed along these 1D condensate tubes, one has an ideal system to test the damping predictions of the present paper.

One also has damping from collisions that transfer atoms between the condensate and thermal cloud [18] and Beliaev damping (present even at $T=0$) involving the spontaneous decay of an excitation into two excitations. All such processes [8] also involve an energy conservation factor of the kind $\delta(E_1 - E_2 - E_3)$, which can only be satisfied if the excitations exhibit anomalous dispersion, i.e., $\alpha \leq 6D$, where D is the dimension of the optical lattice. Finally, for a fixed value of $\alpha < 6D$, there is a finite threshold momentum q^* such that the decay (Beliaev damping) of an excitation E_q is impossible when $q > q^*$. The same phenomenon has been discussed for phonons in superfluid ^{4}He [9,10] and one finds that $q^* \sim p_0$ [8,19]. This disappearance of Beliaev damping for $q > q^*$ at $T=0$ should be easy to confirm experimentally.

In conclusion, assuming that only the lowest energy excitation band is thermally occupied, we have shown that Landau damping of Bogoliubov excitations in uniform optical lattices is only possible if they exhibit anomalous dispersion. The latter can be turned on or off by adjusting the number of condensate atoms per site and the optical lattice depth.

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