Minimum Rényi and Wehrl entropies at the output of bosonic channels

Vittorio Giovannetti,[1,*] Seth Lloyd,[1,2] Lorenzo Maccone,[1,†] Jeffrey H. Shapiro,[1] and Brent J. Yen[1]
[1]*Research Laboratory of Electronics, Massachusetts Institute of Technology, 77 Massochusetts Avenue, Cambridge, Massachusetts 02139, USA*
[2]*Department of Mechanical Engineering, Massachusetts Institute of Technology, 77 Massochusetts Avenue, Cambridge, Massachusetts 02139, USA*
(Received 6 April 2004; published 30 August 2004)

The minimum Rényi and Wehrl output entropies are found for bosonic channels in which the signal photons are either randomly displaced by a Gaussian distribution (classical-noise channel), or coupled to a thermal environment through lossy propagation (thermal-noise channel). It is shown that the Rényi output entropies of integer orders $z \geqslant 2$ and the Wehrl output entropy are minimized when the channel input is a coherent state.

DOI: 10.1103/PhysRevA.70.022328 PACS number(s): 03.67.Hk, 03.65.Db, 42.50.−p

INTRODUCTION

A principal aim of the quantum theory of information is to determine the ultimate limits on communicating classical information, i.e., the limits arising from quantum physics [1,2]. Among the various figures of merit employed in this undertaking, one of the most basic is the minimum output entropy [3]. It measures the amount of noise accumulated during the transmission and may be used to derive important properties, such as the additivity of other figures of merit, e.g., the channel capacity. Here we will focus on the Rényi and Wehrl output entropies for a class of Gaussian bosonic channels in which the input field undergoes a random displacement. The Rényi entropies $\{S_z(\rho): 0<z<\infty, z\neq 1\}$ are a family of functions that describe the purity of a state [4]. In particular, the von Neumann entropy $S(\rho)$ can be found from this family, because $S(\rho)=\lim_{z\to 1} S_z(\rho)$. So too can the linearized entropy, because it is a monotonic function of the second-order Rényi entropy [5]. On the other hand, the Wehrl entropy characterizes the phase-space localization of a bosonic state: its minimum value is realized by coherent states, whose quadratures have minimum uncertainty product and minimum uncertainty sum. In this respect, the Wehrl output entropy can be used to quantify the channel noise by measuring the phase-space "spread" of the output state (see also [6] for a previous analysis of Wehrl output entropy). For the classical-noise and thermal-noise channels that we will consider, we show that coherent-state inputs minimize the Rényi output entropies of integer orders $z \geqslant 2$, and the Wehrl output entropy. The results presented in this paper are connected with the study of the von Neumann output entropies of the classical-noise and thermal-noise channels given in [7] and with the analysis of these channels' additivity properties given in [8].

In Sec. I we introduce the classical-noise channel map. In Sec. II we analyze the Rényi entropy at the output of this channel. We first show that a coherent-state input minimizes $S_z(\rho)$ for $z \geqslant 2$ an integer, and that it minimizes $S_z(\rho)$ for all z when the input is restricted to be a Gaussian state (Sec. II A). We then provide lower bounds, for arbitrary input states, that are consistent with coherent-state inputs minimizing Rényi output entropies of all orders (Sec. II B). In Sec. III, we analyze the Wehrl output entropy, proving that it too is minimized by coherent-state inputs. Moreover, in Sec. III A, we introduce the Rényi-Wehrl entropies and show that here as well coherent-state inputs yield minimum-entropy outputs. The preceding results will all be developed for the classical-noise channel; in Sec. IV we show that they also apply to the thermal-noise channel.

I. CLASSICAL-NOISE CHANNEL

The classical-noise channel is a unital Gaussian map, i.e., it transforms Gaussian input states into Gaussian output states while leaving the identity operator unaffected. It is given by the completely positive (CP) map

$$\mathcal{N}_n(\rho) = \int d^2\mu P_n(\mu) D(\mu) \rho D^\dagger(\mu), \tag{1}$$

where

$$P_n(\mu) = \frac{e^{-|\mu|^2/n}}{\pi n}, \tag{2}$$

and $D(\mu) \equiv \exp(\mu a^\dagger - \mu^* a)$ is the displacement operator of the electromagnetic mode a used for the communication. This map describes a bosonic field that picks up noise through random displacement by a Gaussian probability distribution $P_n(\mu)$. It is useful, among other things, to study the fidelity obtainable in continuous-variable teleportation with finite two-mode squeezing [9]. Moreover, this simple one-parameter map can be used to derive properties of more complicated channels, such as the thermal-noise CP map of Sec. IV. When $\mathcal{N}_n$ acts on a vacuum-state input it produces the thermal-state output

*Present address: NEST-INFM and Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126, Pisa, Italy.
†Present address: QUIT (Quantum Information Theory Group), Dipartimento di Fisica "A. Volta," Università di Pavia, via A. Bassi 6, I-27100, Pavia, Italy

$$\rho_0' \equiv \mathcal{N}_n(|0\rangle\langle 0|) = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^{a^\dagger a}. \tag{3}$$

The covariance property of $\mathcal{N}_n$ under displacement implies that a coherent-state input $|\alpha\rangle$ produces the output state $\rho_\alpha' = D(\alpha)\rho_0' D^\dagger(\alpha)$. See [7,8,10] for a more detailed description of the classical-noise map.

II. RÉNYI ENTROPIES

The quantum Rényi entropy $S_z(\rho)$ is defined as follows [5]:

$$S_z(\rho) \equiv -\frac{\ln \mathrm{Tr}[\rho^z]}{z-1} \quad \text{for} \quad 0 < z < \infty, \quad z \neq 1. \tag{4}$$

It is a monotonic function of the "z purity" $\mathrm{Tr}[\rho^z]$, and it reduces to the von Neumann entropy in the limit $z \to 1$, viz.,

$$\lim_{z\to 1} S_z(\rho) = S(\rho) \equiv -\mathrm{Tr}[\rho \ln \rho]. \tag{5}$$

For $z=2$, the Rényi entropy is a monotonic function of the linearized entropy $S_{\mathrm{lin}}(\rho) \equiv 1 - \mathrm{Tr}[\rho^2]$.

We are interested in the minimum value that $S_z(\rho)$ achieves at the output of the classical-noise channel, i.e.,

$$\mathbb{S}_z(\mathcal{N}_n) \equiv \min_{\rho \in H} S_z(\mathcal{N}_n(\rho)), \tag{6}$$

where the minimization is performed over all states in the Hilbert space $\mathcal{H}$ associated with the channel's input. The concavity of S_z implies that the minimum in Eq. (6) is achieved by a pure-state input $\rho = |\psi\rangle\langle\psi|$. Our working hypothesis is that $S_z(\mathcal{N}_n(\rho))$ achieves its minimum value when the input is a coherent state $|\alpha\rangle$, in which case we find that

$$S_z(\mathcal{N}_n(|\alpha\rangle\langle\alpha|)) = \frac{\ln[(n+1)^z - n^z]}{z-1}. \tag{7}$$

(Note that this quantity does not depend on α, thanks to the invariance of the Rényi entropy under unitary transformations.) Clearly, Eq. (7) provides an upper bound on $\mathbb{S}_z(\mathcal{N}_n)$. We conjecture that it is also a lower bound, whence

$$\mathbb{S}_z(\mathcal{N}_n) = \frac{\ln[(n+1)^z - n^z]}{z-1}. \tag{8}$$

The monotonicity of $S_z(\rho)$ with respect to the z purity permits restating the conjecture (8) as follows:

$$\mathrm{Tr}\{[\mathcal{N}_n(\rho)]^z\} \leqslant \frac{1}{(n+1)^z - n^z}, \tag{9}$$

where the right-hand side of the inequality is the z purity at the output of the classical-noise channel when its input is a coherent state. In Sec. II A. A we will show that this relation is true for integer $z \geqslant 2$, thus proving the conjecture (8) in this case [11]. There we also show that Eq. (9) holds for all $0 < z < \infty$, $z \neq 1$ when the input is restricted to be a Gaussian state. In Sec. II B we will present some lower bounds on the Rényi output entropy of arbitrary order.

A. Integer-z Rényi entropy

From the definition of the classical-noise channel, we see that

$$\mathrm{Tr}\{[\mathcal{N}_n(\rho)]^k\} = \int d^2\mu_1 \cdots d^2\mu_k P_n(\mu_1)\cdots P_n(\mu_k) \times \mathrm{Tr}[D(\mu_1)\rho D^\dagger(\mu_1) D(\mu_2)\rho D^\dagger(\mu_2)\cdots D^\dagger(\mu_k)], \tag{10}$$

with $k \geqslant 1$ an integer. For a pure-state input $|\psi\rangle$, the trace can be expressed as

$$\begin{aligned}\mathrm{Tr}&[D(\mu_1)\rho D^\dagger(\mu_1)\cdots D^\dagger(\mu_k)] \\ &= \langle\psi|D^\dagger(\mu_1)D(\mu_2)|\psi\rangle\langle\psi|D^\dagger(\mu_2)D(\mu_3)|\psi\rangle\cdots \\ &\quad \times\langle\psi|D^\dagger(\mu_k)D(\mu_1)|\psi\rangle = \mathrm{Tr}\{(\rho\otimes\rho\otimes\cdots\otimes\rho) \\ &\quad \times[D_1^\dagger(\mu_1)D_1(\mu_2)\otimes D_2^\dagger(\mu_2)D_2(\mu_3)\otimes\cdots \\ &\quad \otimes D_k^\dagger(\mu_k)D_k(\mu_1)]\},\end{aligned} \tag{11}$$

where the k scalar products in the input Hilbert space $\mathcal{H}$ in the second term were replaced with a single expectation value on the tensor-product Hilbert space $\mathcal{H}^{\otimes k}$ in the third term of the equation. Here $D_j(\mu)$ is a displacement operator that acts on the jth annihilation operator a_j of this enlarged Hilbert space. With this replacement, Eq. (10), which is nonlinear in ρ, can be evaluated as the linear expectation value of an operator Θ on $\mathcal{H}^{\otimes k}$, i.e.,

$$\mathrm{Tr}\{[\mathcal{N}_n(\rho)]^k\} = \mathrm{Tr}[(\rho\otimes\cdots\otimes\rho)\Theta], \tag{12}$$

with Θ being a convolution of tensor products of the displacements $\{D_j\}$, namely,

$$\Theta = \int \frac{d^2\vec{\mu}}{(\pi n)^k} e^{-\vec{\mu}\cdot C\cdot\vec{\mu}^\dagger + \vec{\mu}\cdot G^\dagger\cdot\vec{a}^\dagger - \vec{a}\cdot G\cdot\vec{\mu}^\dagger}, \tag{13}$$

where $\vec{\mu}$ is the complex vector $(\mu_1,\ldots,\mu_k)$ and $\vec{a} \equiv (a_1,\ldots,a_k)$. In Eq. (13), $C \equiv \mathbb{1}/n + A/2$ and G are $k\times k$ real matrices, with $\mathbb{1}$ being the identity and

$$A \equiv \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{14}$$

$$G \equiv \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}. \tag{15}$$

(The matrix A is null when $k=2$.) A and G are commuting circulant matrices [12]; hence they possess a common basis of orthogonal eigenvectors. This means that there exists a

unitary matrix Y such that $D \equiv YCY^\dagger$ and $E \equiv YGY^\dagger$ are diagonal. Rewriting Θ from Eq. (13) in factored form by performing the change of integration variables $\vec{\nu} \equiv \vec{\mu} \cdot Y^\dagger$, and then introducing the new annihilation operators $\vec{b} \equiv \vec{a} \cdot Y^\dagger$, we find that

$$\Theta = \bigotimes_{j=1}^{k} \Theta_j, \tag{16}$$

with

$$\Theta_j \equiv \frac{1}{n|e_j|^2} \int \frac{d^2\nu}{\pi} e^{-d_j|\nu|^2/|e_j|^2} D_{b_j}(\nu), \tag{17}$$

where $D_{b_j}(\nu) \equiv \exp[\nu b_j^\dagger - \nu^* b_j]$ is the displacement operator associated with b_j, while d_j and e_j are the jth diagonal elements of the matrices D and E, respectively (i.e., they are the jth eigenvalues of C and G). As discussed in Appendix A, the operator Θ_j is diagonal in the Fock basis of the mode b_j and takes the thermal-like form [9]

$$\Theta_j = \frac{2/n}{2d_j + |e_j|^2} \left(\frac{2d_j - |e_j|^2}{2d_j + |e_j|^2} \right)^{b_j^\dagger b_j}. \tag{18}$$

Because the $\{d_j\}$ have positive real parts equal to $1/n$ [see Eq. (A3)], the vacuum state of b_j is the Θ_j-eigenvector whose associated eigenvalue has the maximum absolute value $2/[n(2d_j + |e_j|^2)]$. It then follows from Eq. (16) that for any state $R \in \mathcal{H}^{\otimes k}$ we have

$$|\mathrm{Tr}[R\Theta]| \leq \prod_{j=1}^{k} \frac{2/n}{2d_j + |e_j|^2} = \frac{1/n^k}{\det[C + G^\dagger G/2]} = \frac{1}{(n+1)^k - n^k}, \tag{19}$$

where in deriving the first equality we have used the invariance of the determinant under the unitary transformation Y. Because inequality (9) now follows directly from Eq. (12), this completes the proof: for integer $k \geq 2$ the maximum k purity [or, equivalently, the minimum Rényi entropy $\mathbb{S}_k(\mathcal{N}_n)$] is provided by a coherent state input (see also Appendix. B).

Gaussian-state inputs. Suppose that the channel input is restricted to be a Gaussian state ρ_G. It is easy to show that a coherent-state input minimizes $S_z(\rho_G)$ for all $0 < z < \infty$, $z \neq 1$. A Gaussian state is completely characterized by its mean $\langle a \rangle$ and its covariance matrix,

$$\Gamma \equiv \begin{bmatrix} \langle \{\Delta a, \Delta a^\dagger\} \rangle / 2 & \langle (\Delta a)^2 \rangle \\ \langle (\Delta a)^2 \rangle & \langle \{\Delta a, \Delta a^\dagger\} \rangle / 2 \end{bmatrix}, \tag{20}$$

where $\langle \cdot \rangle \equiv \mathrm{Tr}[\cdot \rho_G]$ is the expectation with respect to ρ_G, $\Delta a \equiv a - \langle a \rangle$, and $\{\cdot,\cdot\}$ denotes the anticommutator. As shown in [7], the classical-noise channel's output state ρ_G', when its input is ρ_G, is also Gaussian. The mean $\langle a \rangle$ is unaffected by the CP map $\mathcal{N}_n$, but the covariance matrix is modified by the presence of classical noise, viz., $\Gamma \to \Gamma' = \Gamma + n\mathbb{1}$.

By concatenating two unitary transformations—a displacement to drive $\langle a \rangle$ to zero, and a squeezing operator to symmetrize the quadrature uncertainties —ρ_G' can be converted into the thermal state

$$\tau_G' = \frac{1}{n'+1} \left(\frac{n'}{n'+1} \right)^{a^\dagger a}, \tag{21}$$

where $n' = \sqrt{\det \Gamma'} - 1/2$. The state (21) has Rényi entropy

$$S_z(\tau_G') = \frac{\ln[(n'+1)^z - n'^z]}{z-1} \quad \text{for} \quad 0 < z < \infty,\ z \neq 1. \tag{22}$$

Moreover, because Rényi entropy is invariant under unitary transformations, we have $S_z(\rho_G') = S_z(\tau_G')$. Equation (22) thus shows that $S_z(\rho_G')$ is monotonically increasing with increasing $n' = \sqrt{\det \Gamma'} - 1/2$, and in [7] we showed that $\min_{\rho_G}(\sqrt{\det \Gamma'} - 1/2) = n$ is achieved by coherent-state inputs. It follows that $S_z(\mathcal{N}_n(\rho_G))$ is minimized, for all $0 < z < \infty$, $z \neq 1$, when the channel input is a coherent state. The corresponding Gaussian-state result for the von Neumann entropy at the classical-noise channel's output was derived in [7].

Comments. The most interesting cases for integer-order Rényi output entropy are $k=2$ and $k \to \infty$, where we have

$$\mathbb{S}_2 = \ln(2n+1), \tag{23}$$

$$\mathbb{S}_\infty = \ln(n+1). \tag{24}$$

Equation (23) was used in [7] to derive lower bounds for the von Neumann entropy at the output of the classical-noise channel. On the other hand, Eq. (24) establishes an upper bound on the maximum eigenvalue $\lambda_{\max}$ of any output state $\mathcal{N}_n(\rho)$ of the channel. This is so because the Rényi entropy becomes $S_\infty(\mathcal{N}_n(\rho_G)) = -\ln(\lambda_{\max})$ in the limit $k \to \infty$ [5], and Eq. (24) requires that $\lambda_{\max} \leq 1/(n+1)$.

B. Rényi entropy lower bounds

In this section we develop four lower bounds on $\mathbb{S}_z$ for arbitrary z, which support the conjecture (8).

1. Lower bound 1

The Rényi entropy $S_z(\rho)$ is a decreasing function of z [5]. So, using our knowledge of $\mathbb{S}_k(\mathcal{N}_n)$ for integers $k \geq 2$, we have that

$$\mathbb{S}_z(\mathcal{N}_n) \geq \mathbb{S}_k(\mathcal{N}_n) = \frac{\ln[(n+1)^k - n^k]}{k-1}, \tag{25}$$

for all $z \leq k$. For $z \leq 1$, we can employ the best of the von Neumann output entropy lower bounds that we established in [7] to derive a tighter lower bound on the Rényi entropy. Together with Eq. (25), this additional bound produces the staircase function 1 shown in Figs. 1 and 2.

2. Lower bound 2

The definition of the Rényi entropy leads to the following monotonicity property [5]:

$$\frac{z-1}{z} S_z(\rho) \geq \frac{z'-1}{z'} S_{z'}(\rho) \tag{26}$$

for any $z \geq z'$ and for all ρ. Allowing ρ to be an arbitrary output state from the channel $\mathcal{N}_n$, and minimizing both sides

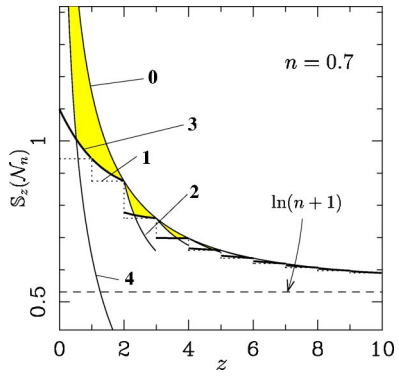

FIG. 1. Bounds on the minimum Rényi entropies $\mathbb{S}_z(\mathcal{N}_n)$ (in nats) as a function of z: $\mathbb{S}_z$ is restricted to the gray region. The upper bound 0 is the Rényi output entropy of the vacuum input, i.e., the right-hand side of Eq. (8). Lower bound 1 (dotted staircase) derives from the fact that S_z is a decreasing function of z. For $z>1$, it follows from Eq. (25) and the integer-z minimum Rényi entropy calculated in Sec. II A; for $z\leqslant 1$, it is equal to $\overline{\mathbb{S}}(\mathcal{N}_n)$, the best of the the von Neumann output entropy lower bounds from [7] (in this case $\overline{\mathbb{S}}\sim 0.95$). Lower bounds 2 and 4 are given by Eqs. (28) and (32), respectively. Lower bound 3 (thick line) is the greater of Eqs. (29) and (30). For high values of z, the maximum of lower bounds 1–4 asymptotically coincides with the upper bound 0; in the figure lower bound 2 becomes indistinguishable from the upper bound once $z\gtrsim 5$. The dashed line is $S_\infty=\ln(n+1)$.

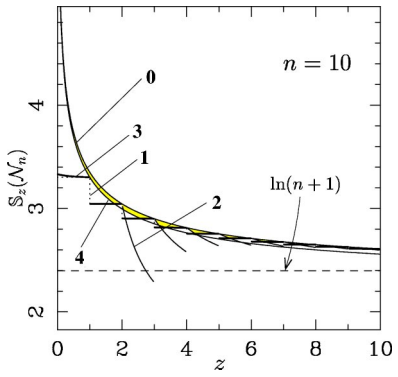

FIG. 2. Same as Fig. 1, but for a different value of the noise parameter n. Note that the lower bounds approach the upper bound 0 for high values of n, indicating that our conjecture is asymptotically true.

of (26) over all the possible inputs, we obtain

$$\frac{z-1}{z}\mathbb{S}_z(\mathcal{N}_n)\geqslant\frac{z'-1}{z'}\mathbb{S}_{z'}(\mathcal{N}_n). \tag{27}$$

When $z'=k\geqslant 2$ is an integer, this relation provides the lower bound

$$\mathbb{S}_z(\mathcal{N}_n)\geqslant\frac{z}{z-1}\frac{\ln[(n+1)^k-n^k]}{k} \tag{28}$$

for $z\geqslant k$, which is shown as curve 2 in Figs. 1 and 2.

3. Lower bound 3

Using the relation between different measures of entropy established in [13,14], the following inequality can be derived (see Appendix C):

$$\mathbb{S}_z(\mathcal{N}_n)\geqslant-\frac{1}{z-1}\ln\left\{h_z\left[h_k^{-1}\left(\frac{1}{(n+1)^k-n^k}\right)\right]\right\}, \tag{29}$$

for all $z\leqslant k$ and integers $k\geqslant 2$. Here, $h_z(x)$ is the function defined in Eq. (C2) and $h_z^{-1}(x)$ its inverse. For $z\leqslant 1$ a further lower bound can be obtained from $\overline{\mathbb{S}}(\mathcal{N}_n)$, the best of the lower bounds on the von Neumann output entropy given in [7]:

$$\mathbb{S}_z(\mathcal{N}_n)\geqslant-\frac{1}{z-1}\ln\{h_z[v^{-1}(\overline{\mathbb{S}}(\mathcal{N}_n))]\}, \tag{30}$$

where $v^{-1}(x)$ is the inverse of the function $v(x)$ defined in Eq. (C6). Curve 3 of Figs. 1 and 2 has been obtained by considering the maximum of all the functions on the right-hand sides of Eqs. (29) and (30).

4. Lower bound 4

Our final lower bound can be derived from the inequality [7]

$$\mathrm{Tr}\{[\mathcal{N}_n(\rho)]^z\}\leqslant\frac{\mathrm{Tr}[\mathcal{N}_{n/z}(\rho)]}{zn^{z-1}}=\frac{1}{zn^{z-1}}, \tag{31}$$

for $z\geqslant 1$, which implies

$$S_z(\mathcal{N}_n(\rho))\geqslant\frac{\ln z}{z-1}+\ln n \tag{32}$$

for any input ρ and $z\geqslant 1$. Inequality (31) was derived in [7] from the *convexity* of x^z for $z\geqslant 1$. For $z\leqslant 1$, the function x^z is concave and we obtain

$$\mathrm{Tr}\{[\mathcal{N}_n(\rho)]^z\}\geqslant\frac{\mathrm{Tr}[\mathcal{N}_{n/z}(\rho)]}{z\geqslant n^{z-1}}=\frac{1}{zn^{z-1}}. \tag{33}$$

The sign change associated with the $1/(z-1)$ factor in the Rényi entropy definition then shows that Eq. (32) also applies for $z\leqslant 1$. Lower bound (32) is plotted as curve 4 in Figs. 1 and 2.

III. WEHRL ENTROPY

The Wehrl entropy is the continuous Boltzmann-Gibbs entropy of the Husimi probability function for the state ρ [15]

$$W(\rho)\equiv-\int d^2\mu Q(\mu)\ln[\pi Q(\mu)], \tag{34}$$

where $Q(\mu)\equiv\langle\mu|\rho|\mu\rangle/\pi$ with $|\mu\rangle$ a coherent state. The Wehrl entropy provides a measurement of the "localization" of the state ρ in the phase space: its minimum value is achieved on coherent states [15,16]. It is also useful in characterizing the statistics associated with heterodyne detection [17]. Here we study this minimum restricted to the output states from the classical-noise channel, i.e.,

$$\mathbb{W}(\mathcal{N}_n) \equiv \min_{\rho\in\mathcal{H}} W(\mathcal{N}_n(\rho)). \tag{35}$$

We will show that coherent-state inputs achieve this minimum, which is then given by

$$\mathbb{W}(\mathcal{N}_n) = 1 + \ln(n+1). \tag{36}$$

The output-state Husimi function $Q'(\mu)$ for the channel map $\mathcal{N}_n$ is the convolution of the input-state Husimi function $Q(\mu)$ with the Gaussian probability distribution P_n from Eq. (2),

$$Q'(\mu) = (P_n * Q)(\mu) = \int d^2\upsilon P_n(\upsilon) Q(\mu-\upsilon). \tag{37}$$

This property can be used to show that the right-hand side of Eq. (36) is an upper bound for $\mathbb{W}$, because it is the value achieved by a coherent-state input. In particular, the Husimi function of the coherent state $|\alpha\rangle$ is $Q_\alpha(\mu) \equiv |\langle\alpha|\mu\rangle|^2/\pi = \exp(-|\mu-\alpha|^2)/\pi$, which evolves into

$$Q'_\alpha(\mu) = \frac{\exp[-|\mu-\alpha|^2/(n+1)]}{\pi(n+1)}, \tag{38}$$

under Eq. (37). The resulting Wehrl output entropy is then

$$W(\mathcal{N}_n(|\alpha\rangle\langle\alpha|)) = \int d^2\mu Q'_\alpha(\mu)\frac{|\mu-\alpha|^2}{n+1} + \ln(n+1) = 1 + \ln(n+1). \tag{39}$$

(An analogous result was also given in [18].) To show that this quantity is also a lower bound for $\mathbb{W}$, we use Theorem 6 of [16], which states that for two probability distributions $f(\mu)$ and $h(\mu)$ on $\mathbb{C}$ we have

$$W((f*h)(\mu)) \geq \lambda W(f(\mu)) + (1-\lambda)W(h(\mu)) - \lambda\ln\lambda - (1-\lambda)\ln(1-\lambda) \tag{40}$$

for all $\lambda\in[0,1]$, where $f*h$ is the convolution of f and h and where the Wehrl entropy of a probability distribution is found from Eq. (34) by replacing $Q(\mu)$ with the given distribution. Choosing $f=P_n$ and $h=Q$ makes $f*h$ the classical-noise channel's output-state Husimi function Q'. Hence, the inequality (40) implies that

$$W(\mathcal{N}_n(\rho)) \geq \lambda W(P_n) + (1-\lambda)W(\rho) - \lambda\ln\lambda - (1-\lambda)\ln(1-\lambda), \tag{41}$$

where $W(P_n)=1+\ln n$ is the Wehrl entropy of the distribution P_n. Because $W(\rho)\geq 1$ for any ρ [16], Eq. (41) gives

$$W(\mathcal{N}_n(\rho)) \geq \lambda\ln n + 1 - \lambda\ln\lambda - (1-\lambda)\ln(1-\lambda), \tag{42}$$

which for $\lambda = n/(n+1)$ becomes

$$W(\mathcal{N}_n(\rho)) \geq 1 + \ln(n+1). \tag{43}$$

Inasmuch as this relation applies for all ρ, Eq. (36) then follows.

A. Rényi-Wehrl entropies

The z–Rényi-Wehrl entropies are defined by [19]

$$W_z(\rho) \equiv -\frac{1}{z-1}\ln[m_z(\rho)], \tag{44}$$

$$m_z(\rho) \equiv \int \frac{d^2\mu}{\pi}[\pi Q(\mu)]^z, \tag{45}$$

where $Q(\mu)$ is the Husimi function of ρ and $z\geq 1$. Thus, the Wehrl entropy $W(\rho)$ is the limit as $z\to 1$ of $W_z(\rho)$, and $W_z(\rho)$ achieves its minimum value $\ln(z)/(z-1)$, when ρ is a coherent state $|\alpha\rangle$, for which $m_z(|\alpha\rangle\langle\alpha|)=1/z$. For arbitrary ρ, Theorem 3 of [16] implies

$$m_z(\rho) = \int \frac{d^2\mu}{\pi}[\pi Q(\mu)]^z \leq \frac{1}{z}. \tag{46}$$

We now show that $\mathbb{W}_z(\mathcal{N}_n) \equiv \min_\rho(W_z(\mathcal{N}_n(\rho))$ is achieved by coherent-state inputs. From Eq. (38), the classical-noise channel's Rényi-Wehrl output entropy for the coherent-state input $|\alpha\rangle$ can be shown to be

$$W_z(\mathcal{N}_n(|\alpha\rangle\langle\alpha|)) = \frac{\ln z}{z-1} + \ln(n+1). \tag{47}$$

To show that the right-hand side of this equation is the global minimum, we observe that, for an arbitrary state ρ and for all $p,q\geq 1$ such that $1/p+1/q=1+1/z$, the sharp form of Young's inequality (Lemma 5 of Ref. [16]) together with Eq. (37) give

$$m_z(\mathcal{N}_n(\rho)) = \int \frac{d^2\mu}{\pi}[\pi Q'(\mu)]^z \leq \left(\frac{C_pC_q}{C_z}\right)^{2z} \times \left[\int \frac{d^2\mu}{\pi}[\pi Q(\mu)]^p\right]^{z/p}\left[\int \frac{d^2\mu}{\pi}\frac{e^{-q|\mu|^2/n}}{n^q}\right]^{z/q} = \left(\frac{C_pC_q}{C_z}\right)^{2z}[m_p(\rho)]^{z/p}\left[\frac{n}{qn^q}\right]^{z/q}, \tag{48}$$

where C_p, C_q, and C_z are the Young's inequality constants

$$C_x \equiv \left[\frac{x^{1/x}}{(x')^{1/x'}}\right]^{1/2}, \quad x' \equiv x/(x-1). \tag{49}$$

Choosing $p=(n+1)z/(nz+1)$ and, hence, $q=(n+1)z/(z+n)$, we then obtain

$$m_z(\mathcal{N}_n(\rho)) \leq \frac{1}{z(n+1)^{z-1}}, \tag{50}$$

which, via Eq. (44), completes the proof.

IV. THERMAL-NOISE CHANNEL

Thus far we have limited our attention to the CP map $\mathcal{N}_n$ associated with the classical-noise channel. This channel is a limiting case of the thermal-noise channel, in which the signal mode a and a thermal-reservoir mode b couple to the channel output through a beam splitter [7,8]. The thermal-

noise channel's CP map $\mathcal{E}_\eta^N$ is obtained by tracing away the noise mode—which initially is in a thermal state with average photon number N—from the evolution

$$a \to \sqrt{\eta}\, a + \sqrt{1-\eta}\, b, \tag{51}$$

where η is the coupling parameter (the channel's quantum efficiency). A detailed characterization of the two maps $\mathcal{N}_n$ and $\mathcal{E}_\eta^N$ is given in [7], where, in particular, it is shown that they are related through the composition rule

$$\mathcal{E}_\eta^N(\rho) = (\mathcal{N}_{(1-\eta)N} \circ \mathcal{E}_\eta^0)(\rho) \equiv \mathcal{N}_{(1-\eta)N}(\mathcal{E}_\eta^0(\rho)). \tag{52}$$

This means that the thermal-noise channel $\mathcal{E}_\eta^N$ can be regarded as the application of the map $\mathcal{N}_n$ to the output of the pure-loss channel $\mathcal{E}_\eta^0$, with the latter being a zero-temperature ($N=0$) thermal-noise channel.

We can use Eq. (52) to extend all the analyses from the previous sections to the thermal-noise channel. Specifically, the minimum z-Rényi output entropy of the thermal-noise channel obeys

$$\mathbb{S}_z(\mathcal{E}_\eta^N) = \mathbb{S}_z(\mathcal{N}_{(1-\eta)N} \circ \mathcal{E}_\eta^0) \geqslant \mathbb{S}_z(\mathcal{N}_{(1-\eta)N}), \tag{53}$$

because the implicit minimization on the left is performed over a subset of the states considered in the implicit minimization on the right. Replacing n with $(1-\eta)N$ in this inequality, we immediately find that the lower bounds from Sec. II B also apply to the thermal-noise channel $\mathcal{E}_\eta^N$. Moreover, for $z \geqslant 2$ integer, Eq. (53) becomes an equality, because the implicit minimum on the left is achieved by the vacuum-state input $|0\rangle$, for which, according to Eq. (52),

$$\mathcal{E}_\eta^N(|0\rangle\langle 0|) = \mathcal{N}_{(1-\eta)N}(|0\rangle\langle 0|). \tag{54}$$

This proves that for integers $k \geqslant 2$ the minimum Rényi entropy at the output of the thermal-noise channel is

$$\mathbb{S}_k(\mathcal{E}_\eta^N) = \frac{\ln\{[(1-\eta)N+1]^k - [(1-\eta)N]^k\}}{k-1}. \tag{55}$$

Some preliminary results in this regard were obtained in [20], where it was shown that the linearized entropy of the thermal-noise channel—i.e., $S_2(\mathcal{E}_\eta^N(\rho))$—is minimized by the vacuum input in the limit of low coupling ($\eta \ll 1$) and high temperature ($N \gg 1$).

When the input to the thermal-noise channel is a Gaussian state ρ_G with covariance matrix Γ, the output state will be Gaussian with covariance matrix $\Gamma' = \eta\Gamma + (1-\eta)(N+1/2)\mathbb{1}$ [7]. We have previously shown that $\min_{\rho_G}(\sqrt{\det\Gamma'} - 1/2) = (1-\eta)N$ is achieved when the input is a coherent state, which shows that coherent-state inputs minimize $S_z(\mathcal{E}_\eta^N(\rho_G))$ for all $0 < z < \infty$, $z \neq 1$.

Finally, arguments identical to the ones given earlier for the minimum Wehrl and Rényi-Wehrl entropies at the output of the classical-noise channel also apply to the minimum Wehrl and Rényi-Wehrl entropies at the output of the thermal-noise channel. Because the minimum values $\mathbb{W}(\mathcal{N}_n)$ and $\mathbb{W}_z(\mathcal{N}_n)$ are achieved by coherent-state inputs, such as the vacuum, Eqs. (52) and (54) imply that

$$\mathbb{W}(\mathcal{E}_\eta^N) = 1 + \ln[(1-\eta)N + 1],$$

$$\mathbb{W}_z(\mathcal{E}_\eta^N) = \frac{\ln z}{z-1} + \ln[(1-\eta)N + 1]. \tag{56}$$

with these minima being realized by coherent-state inputs.

V. CONCLUSION

The minimum Rényi and Wehrl output entropies have been analyzed for bosonic channels in which the signal photons are disturbed by classical additive Gaussian noise, or by a combination of propagation loss and Gaussian noise. We conjectured that the Rényi output entropy is minimized by coherent-state inputs. Some arguments were provided to place this conjecture on solid ground. In particular, we have shown that it is true for integer orders greater than 1, and it is true when the input state is restricted to being Gaussian. For the general case—noninteger orders and arbitrary input states—we have provided entropic lower bounds that are compatible with the conjecture. In addition, we have shown that coherent-state inputs minimize the Wehrl and the Rényi-Wehrl output entropies for these two channels.

ACKNOWLEDGMENTS

The authors thank P. W. Shor, H. P. Yuen, and P. Zanardi for useful discussions. This work was funded by the ARDA, NRO, NSF, and by ARO under a MURI program.

APPENDIX A: DERIVATION OF EQ. (18)

In this appendix we show that the operator Θ_j defined in Eq. (17) coincides with the right-hand side of Eq. (18). The easiest way to prove this assertion is to show that these operators have the same characteristic function. We take advantage of the interesting analysis in [9], where the maximal-entanglement teleportation fidelity is calculated for the classical-noise channel, and the $k=2$ version of Eq. (18) was implicitly demonstrated.

From Eq. (17), we immediately see that the symmetrical characteristic function [21] of the operator Θ_j is

$$\chi_j(v) \equiv \mathrm{Tr}[\Theta_j D_j(v)] = \frac{\exp(-d_j|\nu|^2/|e_j|^2)}{n|e_j|^2}. \tag{A1}$$

On the other hand, the characteristic function of the right-hand side of Eq. (18) is given by

$$\begin{aligned}\chi_j'(\nu) &= \frac{2/n}{2d_j + |e_j|^2} \sum_{m=0}^{\infty} \left(\frac{2d_j - |e_j|^2}{2d_j + |e_j|^2}\right)^m \langle m|D_{b_j}|m\rangle \\ &= \frac{2/n}{2d_j + |e_j|^2} \sum_{m=0}^{\infty} \left(\frac{2d_j - |e_j|^2}{2d_j + |e_j|^2}\right)^m e^{-|\nu|^2/2} L_m(|\nu|^2),\end{aligned} \tag{A2}$$

where $\{|m\rangle\}$ are the Fock states of the b_j mode and L_m is the Laguerre polynomial of order m. From the definition of the matrix C [see Eq. (13)] we know that

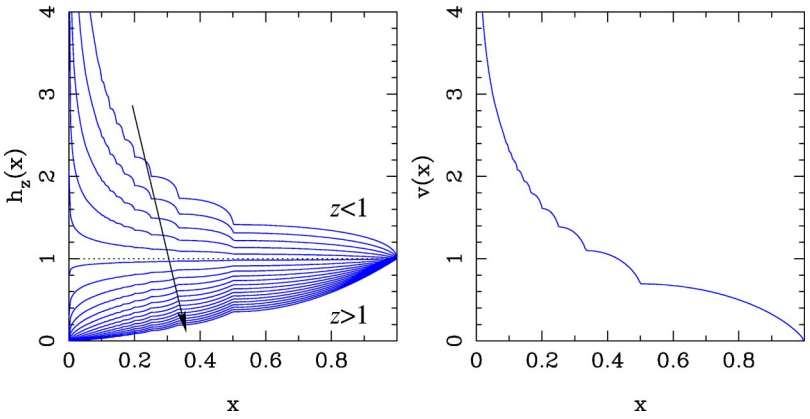

FIG. 3. Left: plot of the function $h_z(x)$ from Eq. (C2) as a function of x for different values of z; z increases from $1/2$ to $5/2$ in progressing along the direction of the arrow. For $z>1$, $h_z(x)$ is an increasing function, and for $z<1$ it is decreasing. Note that $h_z(1)=1$, $\lim_{x\to 0}h_z(x)=0$ for $z>1$, and $\lim_{x\to 0}h_z(x)=\infty$ for $z<1$. Right: plot of the function $v(x)$ from Eq. (C6).

$$d_j = 1/n + i\xi_j, \tag{A3}$$

where $\{i\xi_j\}$ are the imaginary eigenvalues of the real antisymmetric matrix A from Eq. (14). This implies that the $\{d_j\}$ have positive real parts, so that the absolute value of the parenthetical term in Eq. (A2), $(2d_j-|e_j|^2)/(2d_j+|e_j|^2)$, is less than one. The summation in Eq. (A2) can thus be performed using the formula [22]

$$\sum_{m=0}^{\infty} z^{\,m}L_m(x) = \frac{\exp[xz/(z-1)]}{1-z} \text{ for } |z|<1. \tag{A4}$$

With this relation Eq. (A2) yields χ_j, concluding the derivation.

Examples

Here, for the sake of clarity, we carry out calculations of the $\{\Theta_j\}$ for the cases $k=2$ and $k=3$.

When $k=2$, the matrix A is null and G has eigenvalues $e_1=0$ and $e_2=-2$. The unitary transformation that diagonalizes A and G is then

$$Y = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \tag{A5}$$

so that $\Theta_1=1$ on the mode $b_1=(a_1+a_2)/\sqrt{2}$, and

$$\Theta_2 = \frac{1}{1+2n}\left(\frac{1-2n}{1+2n}\right)^{b_2^\dagger b_2} \tag{A6}$$

on the mode $b_2=(a_2-a_1)/\sqrt{2}$.

When $k=3$, the matrix A has eigenvalues $i\xi_1=0$, $i\xi_2=i\sqrt{3}/2$, and $i\xi_3=-i\sqrt{3}/2$. On the other hand, G has eigenvalues $e_1=0$, $e_2=i\sqrt{3}e^{i2\pi/3}$, and $e_3=-i\sqrt{3}e^{i2\pi/3}$. Now the unitary matrix Y is

$$Y = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ e^{2i\pi/3} & e^{4i\pi/3} & 1 \\ e^{4i\pi/3} & e^{2i\pi/3} & 1 \end{bmatrix}, \tag{A7}$$

so that $\Theta_1=\mathbb{1}$ on the mode $b_1=(a_1+a_2+a_3)/\sqrt{3}$,

$$\Theta_2 = \frac{2}{2+(3+i\sqrt{3})n}\left(\frac{2+(-3+i\sqrt{3})n}{2+(3+i\sqrt{3})n}\right)^{b_2^\dagger b_2} \tag{A8}$$

on the mode $b_2=(e^{4i\pi/3}a_1+e^{2i\pi/3}a_2+a_3)/\sqrt{3}$, and

$$\Theta_3 = \frac{2}{2+(3-i\sqrt{3})n}\left(\frac{2-(3+i\sqrt{3})n}{2+(3-i\sqrt{3})n}\right)^{b_3^\dagger b_3} \tag{A9}$$

on the mode $b_3=(e^{2i\pi/3}a_1+e^{4i\pi/3}a_2+a_3)/\sqrt{3}$.

APPENDIX B: ENTROPY-MINIMIZING INPUT STATES

Even though it was already proven in Sec. II [see Eqs. (7) and (9)], it is instructive to use a different method to explicitly show that the upper bound (19) on the integer-order Rényi output entropy can be achieved by employing a vacuum-state input, $\rho=|0\rangle\langle 0|$. By construction, the vacuum state for the b_j modes, $R_0=|0\rangle_{b_1}\langle 0|\otimes\cdots\otimes|0\rangle_{b_k}\langle 0|$, saturates this bound. Because $\vec{a}$ is obtained from $\vec{b}$ through the unitary matrix Y, the state R_0 is also the vacuum state of the $\vec{a}$ modes. Indeed, from the symmetric characteristic function decomposition, we find

$$\begin{aligned} R_0 &= \int \frac{d^2\vec{\nu}}{\pi^k}\exp[-|\vec{\nu}|^2/2 + \vec{\nu}\cdot\vec{b}^\dagger - \vec{b}\cdot\vec{\nu}^\dagger] \\ &= \int \frac{d^2\vec{\mu}}{\pi^k}\exp[-|\vec{\mu}|^2/2 + \vec{\mu}\cdot\vec{a}^\dagger - \vec{a}\cdot\vec{\mu}^\dagger] = |0\rangle_{a_1}\langle 0| \\ &\quad\otimes\cdots\otimes|0\rangle_{a_k}\langle 0|, \end{aligned} \tag{B1}$$

where $\vec{\nu}=\vec{\mu}\cdot Y^\dagger$. From Eq. (7) we know that all coherent-state inputs produce the same Rényi output entropy. This means that every coherent state $|\beta\rangle_{a_1}\langle\beta|\otimes\cdots\otimes|\beta\rangle_{a_k}\langle\beta|$ must saturate the bound (19). To show that this is so, we note that for any integer k the matrices G and A have a null eigenvalue (say for $j=1$), associated with the common eigenvector $(1,1,\ldots,1)$. In this case $e_1=0$ and $d_1=1/n$, so that $\Theta_1=\mathbb{1}_{j=1}$. This means that for arbitrary $|\varphi\rangle_{b_1}$ any state of the form $R_\varphi\equiv|\varphi\rangle_{b_1}\langle\varphi|\otimes|0\rangle_{b_2}\langle 0|\otimes\cdots\otimes|0\rangle_{b_k}\langle 0|$ saturates the bound (19). If $|\varphi\rangle$ is not a coherent state, then it corresponds to an entangled state of the a_j modes, so it cannot be written in the form $\rho\otimes\cdots\otimes\rho$. Thus $\mathrm{Tr}[R_\varphi\Theta]$ cannot be an output k purity of the classical-noise channel. If, instead, we repeat the same analysis of Eq. (B1) with $|\varphi\rangle=|\sqrt{k}\beta\rangle$ being a coherent state, we find that the resulting R_φ is a tensor product of coherent states $|\beta\rangle$ in the a_j modes, so that $\mathrm{Tr}[R_\varphi\Theta]$ is the classical-noise channel's output k purity relative to the coherent-state input $|\beta\rangle$.

APPENDIX C: DERIVATION OF LOWER BOUND 3

In this appendix we derive the lower bound 3, given by Eqs. (29) and (30).

The z purity $\mathrm{Tr}[\rho^z]$ for $z\neq 1$ belongs to the class of entropic measures defined in [14]. Hence, for $1<z'\leqslant z$, the state that minimizes $\mathrm{Tr}[\rho^z]$ over the family of states having constant $\mathrm{Tr}[\rho^z]=c$ is known [14] to have a q-times degenerate eigenvalue λ_1, and a nondegenerate eigenvalue $\lambda_0=1-q\lambda_1\leqslant\lambda_1$. The values of the parameters λ_1 and q are determined by the constraint

$$\lambda_0^{z'}+q\lambda_1^{z'}=c, \tag{C1}$$

which, for $1\geqslant\lambda_1\geqslant\lambda_0\geqslant 0$, gives $q=\lfloor 1/\lambda_1\rfloor$ and can be written as

$$h_{z'}(\lambda_1)\equiv\left(1-\left\lfloor\frac{1}{\lambda_1}\right\rfloor\lambda_1\right)^{z'}+\left\lfloor\frac{1}{\lambda_1}\right\rfloor\lambda_1^{z'}=c, \tag{C2}$$

where $\lfloor x\rfloor$ is the integer part of x. The function $h_z(x)$ can be shown to be continuous and monotonically increasing (see Fig. 3), so that Eq. (C2) has only one solution in the range $c\in[0,1]$. Hence, following [14], we can establish the inequality

$$\mathrm{Tr}[\rho^z]\geqslant h_z[h_{z'}^{-1}(Tr[\rho^{z'}])], \tag{C3}$$

which applies for all ρ and $z\geqslant z'>1$ (h^{-1} being the inverse of the function h). Because $h_z(h_{z'}^{-1}(x))$ is monotonically increasing, Eq. (C3) can be recast as

$$S_{z'}(\rho)\geqslant-\frac{\ln\{h_{z'}(h_z^{-1}(\mathrm{Tr}[\rho^z]))\}}{z'-1}. \tag{C4}$$

Evaluating this expression on the output states $\mathcal{N}_n(\rho)$, we can obtain a lower bound for $\mathbb{S}_{z'}(\mathcal{N}_n)$ by minimizing both terms. Moreover, we can replace the term $\mathrm{Tr}[\rho^z]$ in Eq. (C4) with its maximum value, because it is the argument of a decreasing function. For $z=k$ an integer, we can then use the results of Sec. II. A (where the maximum value of $\mathrm{Tr}\{[N_n(\rho)]^k\}$ was calculated) to derive Eq. (29) from Eq. (C4).

The same analysis can be repeated for $z'<1$; in this case $hz'(x)$ is monotonically decreasing, which is compensated by the sign change of the factor $1/(z'-1)$ in Eq. (C4).

In order to derive Eq. (30), we apply the analysis of [14] to the von Neumann entropy $S(\rho)$ and $\mathrm{Tr}[\rho^{z'}]$ with $z'<1$. Maximizing $S(\rho)$ over the family of states that have constant $\mathrm{Tr}[\rho^{z'}]=c$, we find that the optimal state has the same eigenvalue structure $\{\lambda_0,\lambda_1\}$ encountered above. Equation (C3) is thus replaced by

$$S(\rho)\leqslant v[h_{z'}^{-1}(\mathrm{Tr}[\rho^{z'}])], \tag{C5}$$

where

$$v(x)\equiv-\left(1-\left\lfloor\frac{1}{x}\right\rfloor x\right)\ln\left(1-\left\lfloor\frac{1}{x}\right\rfloor x\right)-\left\lfloor\frac{1}{x}\right\rfloor x\ln x \tag{C6}$$

is the decreasing function plotted in Fig. 3. Because $v[h_{z'}^{-1}(x)]$ is monotonically increasing, Eq. (C5) can be used to derive Eq. (30).

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[2] C. H. Bennett and P. W. Shor, IEEE Trans. Inf. Theory **44**, 2724 (1998).

[3] P. W. Shor, e-print quant-ph/0305035; A. S. Holevo, e-print quant-ph/0306196.

[4] G. G. Amosov, A. S. Holevo, and R. F. Werner, Probl. Inf. Transm. **36**, 305 (2000); C. King and M. B. Ruskai, IEEE Trans. Inf. Theory **47**, 192 (2001); C. King, *ibid.* **49** 221 (2003).

[5] K. Życzkowski, Open Syst. Inf. Dyn. **10**, 297 (2003); C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, England, 1993).

[6] A. Anderson and J. J. Halliwell, Phys. Rev. D **48**, 2753 (1993).

[7] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and J. H. Shapiro, e-print quant-ph/0404005, Phys. Rev. A (to be published).

[8] V. Giovannetti and S. Lloyd, Phys. Rev. A **69**, 062307 (2004).

[9] C. Caves and K. Wódkiewicz, e-print quant-ph/0401149.

[10] M. J. W. Hall and M. J. O'Rourke, Quantum Opt. **5**, 161 (1993); M. J. W. Hall, Phys. Rev. A **50**, 3295 (1994).

[11] In the case $z=1$, Eq. (9) is trivially verified. However, this does not imply the conjecture in (8) owing to the presence of the $1/(z-1)$ factor in the Rényi entropy.

[12] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications* (University of California Press, Berkeley, 1958).

[13] X. Wang, B. C. Sanders, and D. W. Berry, Phys. Rev. A **67**, 042323 (2003); V. M. Kendon, K. Życzkowski, and W. J. Munro, *ibid.* **66**, 062310 (2002).

[14] D. W. Berry and B. C. Sanders, J. Phys. A **36**, 12255 (2003); P. Harremoës and F. Topsøe, IEEE Trans. Inf. Theory **47**, 2944 (2001).

[15] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978); Rep. Math. Phys. **16**, 353 (1979).

[16] E. H. Lieb, Commun. Math. Phys. **62**, 35 (1978).

[17] H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **26**, 78 (1980).

[18] M. J. W. Hall, Phys. Rev. A **55**, 100 (1997).

[19] F. Mintert and K. Życzkowski, e-print quant-ph/0307169.

[20] W. H. Zurek, S. Habib, and J. P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).

[21] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).

[22] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 2000), Chap. 8.957.