Distinguishability of maximally entangled states

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In $2\otimes 2$, more than two orthogonal Bell states with a single copy can never be discriminated with certainty if only local operations and classical communication (LOCC) are allowed. We show here that more than d numbers of pairwise orthogonal maximally entangled states in $d\otimes d$, which are in canonical form, used by Bennett *et al.* [Phys. Rev. Lett. **70**, 1895 (1993)], can never be discriminated with certainty by LOCC, when single copies of the states are provided. Interestingly we show here that all orthogonal maximally entangled states, which are in canonical form, can be discriminated with certainty by LOCC if and only if two copies of each of the states are provided. We provide here a conjecture regarding the highly nontrivial problem of local distinguishability of any d or fewer numbers of pairwise orthogonal maximally entangled states in $d\otimes d$ (in the single copy case).

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I. INTRODUCTION

In quantum mechanics, any set of orthogonal states can be discriminated. But for multipartite systems, local information of the density matrices, and even local operations and classical communication (LOCC), are not sufficient to distinguish among orthogonal states. Recently some interesting studies have shown that pairwise orthogonal multipartite states cannot always be discriminated with certainty in a single copy case if only local operations and classical communication (LOCC) are allowed [1–4]. But any two multipartite orthogonal states can always be distinguished with certainty by LOCC, and, in general, d pairwise orthogonal multipartite states can be perfectly discriminated by LOCC if $(d-1)$ copies of each state are provided [2]. But there are sets of pairwise orthogonal states that can be discriminated with less than $(d-1)$ copies. One such example is that if *two copies* of a state are provided which is known to be one of the four (pairwise orthogonal) Bell states

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad (1)$$

one can discriminate between them using LOCC only [2]. In a recent paper Ghosh *et al.* [4] have shown, using some properties of entanglement measure, that more than two Bell states cannot be discriminated with certainty by LOCC, if a single copy is provided.

In this paper, we consider the problem of reliable local distinguishability of pairwise orthogonal maximally entangled states in $d\otimes d$, each of which is in canonical form, given by Eq. (2) below. Let $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ be the standard orthonormal basis of a d-dimensional Hilbert space. A *maximally entangled state* in $d\otimes d$ is defined to be a pure state for which both reduced density matrices are equal to the maximally mixed state $(1/d)I$, in a d-dimensional Hilbert space. Using the above-mentioned standard basis for both sides, one particular set of d^2 number of pairwise orthogonal maximally entangled states can be taken as

$$|\Psi_{nm}^{(d)}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} \exp\left[\frac{2\pi ijn}{d}\right]|j\rangle \otimes |(j+m)\mathrm{mod}\ d\rangle, \quad (2)$$

for $n,m=0,1,\ldots,d-1$.

Before going into the main results, we discuss in Sec. II some useful properties of the maximally entangled states, given in Eq. (2), to use them for later purposes. In Sec. III, we show that more than d pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in Eq. (2), can never be perfectly discriminated by LOCC in a single copy case. As mentioned above, in $2\otimes 2$, the set of four (or three) orthogonal Bell states can be discriminated with certainty, using LOCC only, if at least two copies of each state are given. Interestingly, this is universal, i.e., (we show here in Sec. IV that) any number of mutually orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in Eq. (2), can be discriminated by LOCC only, if two copies of the states are provided. This is definitely surprising as one would be inclined to think that the minimum number of copies needed for discrimination would be an increasing function of the dimension d. These are the two conclusive results of this paper. Next, in Sec. V, we discuss the problem of reliable local discrimination of any d or fewer number of pairwise orthogonal maximally entangled states in $d\otimes d$, taken from the set given by Eq. (2), in the single copy case [5], based on a particular type of one-way LOCC, namely, teleportation. We shall see here that there are some examples of sets of pairwise orthogonal maximally entangled states of $d\otimes d$ [all taken from (2)], for each of which, not

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only the above-mentioned local discrimination by teleportation does not work, but whether the states of the set are at all reliably locally distinguishable is still an unresolved issue. In Sec. VI, we discuss a necessary condition (in the form of a conjecture) for reliable distinguishability via LOCC. Finally, in Sec. VII, we draw the conclusion.

II. SOME USEFUL PROPERTIES OF $|\Psi_{nm}^{(d)}\rangle$

It is easy to verify, using Eq. (2), that

$$(U_{nm}^{(d)} \otimes I)|\Psi_{00}^{(d)}\rangle = |\Psi_{nm}^{(d)}\rangle,$$

$$(I \otimes V_{nm}^{(d)})|\Psi_{00}^{(d)}\rangle = |\Psi_{nm}^{(d)}\rangle, \tag{3}$$

where

$$U_{nm}^{(d)} = \sum_{j=0}^{d-1} \exp\left[\frac{2\pi ijn}{d}\right]|j\rangle\langle(j+m)\mathrm{mod}\ d| \tag{4}$$

and

$$V_{nm}^{(d)} = (U_{[(d-n)\mathrm{mod}\ d]m}^{(d)})^\dagger, \tag{5}$$

for $n,m=0,1,\ldots d-1$. From Eqs. (3)–(5), one gets that

$$|\Psi_{nm}^{(d)}\rangle = (U_{nm}^{(d)}(U_{n'm'}^{(d)})^\dagger \otimes I)|\Psi_{n'm'}^{(d)}\rangle,$$

$$|\Psi_{nm}^{(d)}\rangle = (I \otimes V_{nm}^{(d)}(V_{n'm'}^{(d)})^\dagger)|\Psi_{n'm'}^{(d)}\rangle, \tag{6}$$

where

$$U_{nm}^{(d)}(U_{n'm'}^{(d)})^\dagger = \exp\left[\frac{2\pi i(m'-m)n'}{d}\right]U_{[(d+n-n')\mathrm{mod}\ d][(d+m-m')\mathrm{mod}\ d]}^{(d)},$$

$$V_{nm}^{(d)}(V_{n'm'}^{(d)})^\dagger = \exp\left[\frac{2\pi i(n'-n)m'}{d}\right]V_{[(d+n-n')\mathrm{mod}\ d][(d+m-m')\mathrm{mod}\ d]}^{(d)}, \tag{7}$$

$n,m,n',m'=0,1,\ldots,d-1$.

In the present paper, we shall repeatedly use teleportation of some state $|\phi^{(d)}\rangle$ of a d-dimensional Hilbert space, via some shared maximally entangled state $|\Psi_{max}^{(d)}\rangle$ of $d\otimes d$ [which is not necessarily of the form given in (2)], using complete von Neumann measurement in a maximally entangled basis [which is not necessarily of the form given in (2)] $\{|\Phi_i^{(d)}\rangle : i=1,2,\ldots,d^2\}$ of $d\otimes d$ (and then using corresponding unitary operations). Thus, for the shared channel state $|\Psi_{00}^{(d)}\rangle$, if $|\Phi_i^{(d)}\rangle=(U_i\otimes I)|\Psi_{00}^{(d)}\rangle$ clicks in the measurement of Alice (U_i being an unitary operator), in order to have *exact* teleportation (as described in Ref. [7]), Bob will have to apply this unitary operator U_i on his system, so that the output state at Bob's side will be $|\phi^{(d)}\rangle$ [8]. Let us denote this (exact) teleportation protocol as $\mathcal{P}_{00}^{(d)}$.

Now consider a situation where Alice and Bob share the state $|\Psi_{nm}^{(d)}\rangle$, but to teleport a state $|\phi^{(d)}\rangle$, the teleportation protocol $\mathcal{P}_{00}^{(d)}$ is followed. The final state will definitely be a distorted one. Using the algebraic properties of (3) one can easily check that the final teleported state will remain pure and will be $V_{nm}^{(d)}|\phi^{(d)}\rangle$, where $V_{nm}^{(d)}$'s are given by (5).

III. LOCAL INDISTINGUISHABILITY OF THE $(d+1)$ NUMBER OF MAXIMALLY ENTANGLED STATES

We are now going to show that no $(d+1)$ number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in (2), can be reliably discriminated by LOCC, in the single copy case. For this, we shall use the notion of the relative entropy of entanglement $E_r(\sigma)$ for a bipartite quantum state σ [9,10]:

$$E_r(\sigma) = \min_{\rho\in D} S(\sigma\,\|\,\rho),$$

where D is the set of all separable states on the Hilbert space on which σ is defined, and $S(\sigma\|\rho)\equiv tr\{\sigma(\log_2\sigma-\log_2\rho)\}$ is the relative entropy of σ with respect to ρ.

Consider now the following four party state:

$$\rho^{(d+1)} = \frac{1}{(d+1)}\sum_{i=1}^{(d+1)} P[|\Psi_{n_im_i}^{(d)}\rangle_{AB}|\Psi_{n_im_i}^{(d)}\rangle_{CD}],$$

shared between Alice (A), Bob (B), Charlie (C), and Darlie (D) with all four being at distant locations, where $|\Psi_{n_im_i}^{(d)}\rangle$ (for $i=1,2,\ldots,d+1$) are given any $(d+1)$ number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in Eq. (2). Consider now another four party state

$$\rho^{(S)} = \frac{1}{d^2}\sum_{n,m=0}^{d-1} P[|\Psi_{nm}^{(d)}\rangle_{AB}|\Psi_{[(d-n)\mathrm{mod}\ d]\ m}^{(d)}\rangle_{CD}],$$

shared among A, B, C, and D, where $|\Psi_{nm}^{(d)}\rangle$'s are given by Eq. (2). By construction, $\rho^{(S)}$ is separable across $AB:CD$ cut. One can check that $\rho^{(S)}$ has the same form in $AC:BD$ cut also (see, for example, Ref. [6]). Thus $\rho^{(S)}$ is separable in $AB:CD$ cut as well as in $AC:BD$ cut. Let $E_r(\rho_{AC:BD}^{(d+1)})$ be the relative entropy of entanglement of the state $\rho^{(d+1)}$ in the $AC:BD$ cut. Then

$$E_r(\rho_{AC:BD}^{(d+1)}) \leqslant S\left(\rho_{AC:BD}^{(d+1)}\,\|\,\frac{1}{d^2}\sum_{n,m=0}^{d-1} P[|\Psi_{nm}^{(d)}\rangle_{AC}|\Psi_{[(d-n)\mathrm{mod}\ d]\ m}^{(d)}\rangle_{BD}]\right) = S(\rho_{AC:BD}^{(d+1)}\,\|\,\rho^{(S)}) < \log_2 d.$$

Why does the last inequality hold good? This is because of the fact that the support of $\rho_{AC:BD}^{(d+1)}$ is contained in the subspace (of $d^2\otimes d^2$), spanned by the d^2 number of pairwise orthogonal states $|\Psi_{nm}^{(d)}\rangle_{AC}\otimes|\Psi_{[(d-n)\mathrm{mod}\ d]\ m}^{(d)}\rangle_{BD}$ (see Ref. [6]); and so

$$S(\rho^{(d+1)}_{AC:BD} \| \rho^{(S)}) = -\log_2(d+1) + 2(\log_2 d) \sum_{n,m=0}^{d-1} \mathrm{Tr}(\rho^{(d+1)}_{AC:BD} P[|\Psi^{(d)}_{nm}\rangle_{AC} \otimes |\Psi^{(d)}_{[(d-n)\mathrm{mod}\ d]\ m}\rangle_{BD}]) \leqslant -\log_2(d+1) + 2\log_2 d < \log_2 d.$$

But distillable entanglement is bounded above by E_r [11,12]. Consequently, the distillable entanglement of $\rho^{(d+1)}$, in the $AC:BD$ cut, is strictly less than $\log_2 d$.

Suppose now that it is possible to discriminate (with certainty) any $(d+1)$ number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{n_i m_i}\rangle$'s in $d\otimes d$, using only LOCC and when only a single copy each of the state is provided. So if Alice, Bob, Charlie, and Darlie share the state $\rho^{(d+1)}$, then Alice and Bob, without meeting, would again be able to distill between Charlie and Darlie, $\log_2 d$ ebit of entanglement, by using this state-discrimination LOCC (together with possible unitary operations, to be applied by Charlie and/or Darlie, locally). Therefore distillable entanglement of $\rho^{(d+1)}$ in the $AC:BD$ cut is at least $\log_2 d$ ebit. But here, as the relative entropy of entanglement of $\rho^{(d+1)}$ in the $AC:BD$ cut is less than $\log_2 d$, so the distillable entanglement of $\rho^{(d+1)}$, in the $AC:BD$ cut, should be less than $\log_2 d$, and hence a contradiction. Therefore no $(d+1)$ number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in (2), are distinguishable by LOCC with certainty if only a single copy of each state is provided.

What would be the case if we consider local distinguishability of *any* $(d+1)$ number of pairwise orthogonal maximally entangled states $|\psi^{(max)}_i\rangle$ of $d\otimes d$, instead of considering only states from the set of states given in (2)? The above-mentioned argument will go through if we can find out two unitary operators U and V such that $(U\otimes V)|\psi^{(max)}_i\rangle = |\Psi^{(d)}_{n_i m_i}\rangle$ for $i=1,2,\ldots,d+1$. But, for most general dimension d, we do not know whether such U and V exist.

IV. LOCAL DISTINGUISHABILITY OF MAXIMALLY ENTANGLED STATES, SUPPLIED WITH TWO COPIES

It has been shown by Horodecki *et al.* [13] that a complete orthonormal basis of $d\otimes d$ can be distinguished by LOCC, deterministically (i.e., reliably) or probabilistically, in the single copy case, if and only if all the states are product. Fan [6] has shown that the total d^2 number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{nm}\rangle$ $(n,m=0,1,\ldots,d-1)$, given in (2), in the single copy case, can never (i.e., neither deterministically nor probabilistically) be distinguished by using LOCC only.

We are now going to show that any given set of k (where $1\leqslant k\leqslant d^2$) number of pairwise orthogonal maximally entangled states in $d\otimes d$, taken from the set given in (2), can be reliably discriminated by LOCC, if *two* copies of each of these states are provided. To show this we employ the following protocol: We teleport the following state $|0\rangle$ through the first copy of each of the shared (between Alice and Bob) unknown channel state $|\Psi^{(d)}_{nm}\rangle$, and also we teleport the state $(1/\sqrt{d})(|0\rangle+|1\rangle+\cdots+|d-1\rangle)$ through the second copy of this shared channel state, by using the standard teleportation protocol $\mathcal{P}^{(d)}_{00}$ of Bennett *et al.* [7], used for each of the above-mentioned two channel states, separately. Now, after this teleportation protocol is over, the final two-qudit state at Bob's side, corresponding to two copies of the unknown channel state $|\Psi^{(d)}_{nm}\rangle$, is given by (modulo a phase)

$$(V^{(d)}_{nm}\otimes V^{(d)}_{nm})|0\rangle \otimes \frac{1}{\sqrt{d}}(|0\rangle + |1\rangle + \cdots + |d-1\rangle)$$

$$=|m\rangle \otimes \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} \exp\left[\frac{2\pi i j n}{d}\right]|(j+m)\mathrm{mod}\ d\rangle,$$

where $V^{(d)}_{nm}$'s are given in Eq. (5), for $n,m=0,1,\ldots,d-1$. Bob now first does measurement in the computational basis $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$, on his first qudit. If $|m\rangle$ is the outcome, Bob will then distinguish the following d number of pairwise orthogonal states $(1/\sqrt{d})\Sigma_{j=0}^{d-1}\exp[2\pi i j n/d]|(j+m)\mathrm{mod}\ d\rangle$, where $n=0,1,\ldots,d-1$. And from both the measurement results finally Alice and Bob will be able to discriminate the d^2 number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{nm}\rangle$, given by Eq. (2).

One can proceed in the same way, for local discrimination of *any* d^2 number of pairwise orthogonal maximally entangled states $\{|\psi^{(max)}_i\rangle\}_{i=1}^{d^2}$ in $d\otimes d$, if two unitary operators U, V can be found, for which $(U\otimes V)|\psi^{(max)}_i\rangle=|\Psi^{(d)}_{n_i m_i}\rangle$ for $i=1,2,\ldots,d^2$. As has been mentioned in the last section, existence of such U, V is yet to be established. Had it been the case that the most general form $\{|\psi^{(max)}_i\rangle\}_{i=1}^{d^2}$ of any set of d^2 number of pairwise orthogonal maximally entangled states of $d\otimes d$ was known, one might think of searching for above-mentioned U and V. But what would be the most general form of any set of d^2 number of pairwise orthogonal maximally entangled states in $d\otimes d$ is not yet known [14].

V. SUFFICIENT CONDITION FOR RELIABLE LOCAL DISTINGUISHABILITY OF *d* OR LESS NUMBER OF MAXIMALLY ENTANGLED STATES IN THE SINGLE COPY CASE

Next we discuss the problem of reliable local distinguishability of d (or less than d) number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set of states given by Eq. (2), in the single copy case. Thus the problem is to test the possibility of reliable local distinguishability of d (or less than d) number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{n_k m_k}\rangle$ (in the single copy case), chosen at random from the set of d^2 number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{nm}\rangle$, given in (2), where $n_k, m_k \in \{0,1,\ldots,d-1\}$ for $k=1,2,\ldots,L$, and L is a positive integer less than or equal to d. The answer to the question whether any given L (where $0<L\leqslant d$) number of pairwise orthogonal maximally entangled states, taken at random from (2), can be reliably distinguished by using LOCC only in the single copy case is not yet known with full generality. In this section, we first provide a sufficient con-

dition (Theorem 1, below) towards answering this question. In the later part of this section, we discuss this sufficient condition with examples of particular dimensions, where either this condition is violated or it is satisfied.

Theorem 1. Single copies of L number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{n_i m_i}\rangle$ (for $i=1,2,\ldots,L$), taken from the set given in Eq. (2), can be reliably discriminated by LOCC if there exists at least one state $|\phi^{(d)}\rangle$ for which the states $V^{(d)}_{n_1m_1}|\phi^{(d)}\rangle$, $V^{(d)}_{n_2m_2}|\phi^{(d)}\rangle$, $\ldots$, $V^{(d)}_{n_Lm_L}|\phi^{(d)}\rangle$ are pairwise orthogonal, where $V^{(d)}_{nm}$'s are given by Eq. (5).

Proof. Let us assume that the state $|\phi^{(d)}\rangle$ is being teleported via one of the unknown channel states $|\Psi^{(d)}_{n_im_i}\rangle$ (for $i=1,2,\ldots,L$), taken from the set given in Eq. (2), by using the teleportation protocol $\mathcal{P}^{(d)}_{00}$. After completion of this teleportation, the corresponding output states will be one of $V^{(d)}_{n_1m_1}|\phi^{(d)}\rangle$, $V^{(d)}_{n_2m_2}|\phi^{(d)}\rangle$, $\ldots$, $V^{(d)}_{n_Lm_L}|\phi^{(d)}\rangle$ depending on the channel state. As these L numbers of output states are orthogonal to each other, therefore they can be reliably discriminated, and, hence, the initially shared states $|\Psi^{(d)}_{n_im_i}\rangle$ (for $i=1,2,\ldots,L$) can be reliably discriminated by using LOCC only.

Below, we use this theorem to discuss the possibilities of local distinguishability of d number of pairwise orthogonal maximally entangled states of $d\otimes d$, taken from (2), in the single copy case, for particular as well as general values of d.

The case for d=2 and d=3. Although we are dealing with maximally enatngled states, in the case of $2\otimes 2$ (and, in fact, in the case of any bipartite system), Walgate *et al.* [2] have already shown that any two orthogonal states can always be reliably discrminated.

In $3\otimes 3$, Walgate *et al.*'s result [2] is not going to help us directly, as we are interested here in local distinguishability of three states. Consider the following three mutually orthogonal maximally entangled states:

$$|\Psi^{(3)}_{00}\rangle,\ |\Psi^{(3)}_{10}\rangle,\quad \text{and}\quad |\Psi^{(3)}_{01}\rangle.$$

Alice first teleports the state $1/\sqrt{3}(|0\rangle+w|1\rangle+|2\rangle)$ (where $w=\exp[2\pi i/3]$) through the unknown channel state (taken from the above-mentioned three states) using the protocol $P^{(3)}_{00}$, and Bob will have (respectively) one of the following three mutually orthogonal states (up to some phases), after finishing the teleportation protocol:

$$V^{(3)}_{00}\frac{1}{\sqrt{3}}(|0\rangle + w|1\rangle + |2\rangle) = \frac{1}{\sqrt{3}}(|0\rangle + w|1\rangle + |2\rangle),$$

$$V^{(3)}_{10}\frac{1}{\sqrt{3}}(|0\rangle + w|1\rangle + |2\rangle) = \frac{1}{\sqrt{3}}(|0\rangle + w^2|1\rangle + w^2|2\rangle),$$

and

$$V^{(3)}_{01}\frac{1}{\sqrt{3}}(|0\rangle + w|1\rangle + |2\rangle) = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + w|2\rangle).$$

So, in this case, in Theorem 1, $|\phi^{(3)}\rangle=(1/\sqrt{3})(|0\rangle+w|1\rangle+|2\rangle)$. By discriminating these states, Alice and Bob will come up with certainty to know which was the state, they were initially sharing.

Similarly, one can always show any three pairwise orthogonal maximally entangled states in $3\otimes 3$, taken from the set of states given in Eq. (2), can be reliably discriminated by LOCC, using our Theorem 1.

Examples for general d. We now discuss local distinguishability of particular sets of d number of pairwise orthogonal maximally entangled states of $d\otimes d$, all taken from (2), by using the teleportation protocol $\mathcal{P}^{(d)}_{00}$, for general values of d. Consider, for example, the following set of d number of pairwise orthogonal maximally entangled states: $|\Psi^{(d)}_{0m}\rangle$, $|\Psi^{(d)}_{1m}\rangle$, $\ldots$, $|\Psi^{(d)}_{(d-1)m}\rangle$. The states of this set can be reliably discriminated by LOCC, in the single copy case, for any given value of m from the set $\{0,1,\ldots,d-1\}$. This can be achieved by sending the state $(1/\sqrt{d})(|0\rangle+|1\rangle+\cdots+|d-1\rangle)$ through the unknown channel state, using $\mathcal{P}^{(d)}_{00}$, and corresponding output states will be one of the following d number of pairwise orthogonal states $V^{(d)}_{nm}(1/\sqrt{d})(|0\rangle+|1\rangle+\cdots+|d-1\rangle)=(1/\sqrt{d})\Sigma_{j=0}^{d-1}\exp[2\pi ijn/d]|(j+m)\text{mod } d\rangle$ (for $n=0,1,2,\ldots,d-1$) depending on the channel state. Similarly the following set of d number of pairwise orthogonal maximally entangled states $|\Psi^{(d)}_{n0}\rangle$, $|\Psi^{(d)}_{n1}\rangle$, $\ldots$, $|\Psi^{(d)}_{n(d-1)}\rangle$ can be reliably discriminated by LOCC, in the single copy case, for any given value of n from the set $\{0,1,\ldots,d-1\}$, by sending the state $|0\rangle$ using $\mathcal{P}^{(d)}_{00}$; the output state will be one of the mutually orthogonal states, $V^{(d)}_{nm}|0\rangle=|m\rangle$ (for $m=0,1,2,\ldots,d-1$) depending on the channel.

Counterexamples for d=4. When we come across the problem of reliable local distinguishability in $4\otimes 4$, we will encounter one example of having four mutually orthogonal maximally entangled states, taken from the cannonical set (2), which cannot be discriminated reliably by using the teleportation protocol $\mathcal{P}^{(4)}_{00}$. Let Alice and Bob share a single copy of one of the following four states:

$$|\Psi^{(4)}_{00}\rangle,\ |\Psi^{(4)}_{10}\rangle,\ |\Psi^{(4)}_{20}\rangle,\quad \text{and}\quad |\Psi^{(4)}_{02}\rangle.$$

There exists *no* input state $|\phi^{(4)}\rangle$ for which

$$V^{(4)}_{00}|\phi^{(4)}\rangle,\ V^{(4)}_{10}|\phi^{(4)}\rangle,\ V^{(4)}_{20}|\phi^{(4)}\rangle,\ V^{(4)}_{02}|\phi^{(4)}\rangle$$

are pairwise orthogonal. This failure does not depend on the choice of the teleportation protocol. In fact, we would like to mention here that if we allow only one-way protocols for discriminating the above-mentioned four states, the reliable discrimination of these four states will be then *impossible* [15]. So there will be *no* local basis transformation via which these four states can be rewritten as

$$|\Psi^{(4)}_{00}\rangle = |0'\alpha_1\rangle + |1'\beta_1\rangle + |2'\gamma_1\rangle + |3'\delta_1\rangle,$$

$$|\Psi^{(4)}_{10}\rangle = |0'\alpha_2\rangle + |1'\beta_2\rangle + |2'\gamma_2\rangle + |3'\delta_2\rangle,$$

$$|\Psi^{(4)}_{20}\rangle = |0'\alpha_3\rangle + |1'\beta_3\rangle + |2'\gamma_3\rangle + |3'\delta_3\rangle,$$

$$|\Psi^{(4)}_{02}\rangle = |0'\alpha_4\rangle + |1'\beta_4\rangle + |2'\gamma_4\rangle + |3'\delta_4\rangle, \tag{8}$$

where $|0'\rangle$, $|1'\rangle$, $|2'\rangle$, and $|3'\rangle$ are pairwise orthogonal states of a four-dimensional Hilbert space, and $\langle\alpha_i|\alpha_j\rangle=\langle\beta_i|\beta_j\rangle$

$=\langle\gamma_i|\gamma_j\rangle=\langle\delta_i|\delta_j\rangle=0$ if $i\neq j$—a sufficient condition for reliable local discrimination [2].

$\{|\Psi_{10}^{(4)}\rangle,|\Psi_{20}^{(4)}\rangle,|\Psi_{30}^{(4)}\rangle,|\Psi_{12}^{(4)}\rangle\}$ is another set of four locally indistinguishable states (by the teleportation protocol, described above) in $4\otimes 4$, like the one described above.

$d=5$ and beyond. For the case when $d=5$, there are nontrivial (i.e., sets of states which are not of the two general forms, described above, for any d) sets of five states, taken from the set of states given in Eq. (2), which can be reliably distinguished by using the teleportation protocol. One such example is the following set of five states: $|\Psi_{00}^{(5)}\rangle$, $|\Psi_{10}^{(5)}\rangle$, $|\Psi_{20}^{(5)}\rangle$, $|\Psi_{30}^{(5)}\rangle$, $|\Psi_{03}^{(5)}\rangle$. On the other hand, there are sets of states (e.g., $|\Psi_{00}^{(5)}\rangle$, $|\Psi_{11}^{(5)}\rangle$, $|\Psi_{21}^{(5)}\rangle$, $|\Psi_{13}^{(5)}\rangle$, $|\Psi_{23}^{(5)}\rangle$) which cannot be reliably distinguished by using the teleportation protocol $\mathcal{P}_{00}^{(5)}$. Similarly, for $d=6$, $\{|\Psi_{00}^{(6)}\rangle,|\Psi_{10}^{(6)}\rangle,|\Psi_{20}^{(6)}\rangle,|\Psi_{30}^{(6)}\rangle,|\Psi_{40}^{(6)}\rangle,|\Psi_{03}^{(6)}\rangle,\}$ is a set of six pairwise orthogonal maximally entangled states in $6\otimes 6$, which cannot be reliably distinguished (in the single copy case) by the above-mentioned teleportation method.

With this, we end our discussions about satisfiability of Theorem 1, for particular values of d.

Local distingushability of less than d number of states. Let us now mention another sufficiency condition on local distinguishability of L number of pairwise orthogonal maximally entangled states, taken from (2), where $2\leqslant L\leqslant d$, and d is a prime number (see Ref. [6]):

Any set of L number of pairwise orthogonal maximally entangled states of $d\otimes d$, taken from Eq. (2), *can be reliably distinguished by LOCC, in the single copy case, if* $L(L-1)\leqslant 2d$, *where we have to take d to be prime*.

This shows that any three states in $5\otimes 5$, taken from Eq. (2), can be distinguished. Then there is a possibility of finding a set of four states in $5\otimes 5$ which cannot be locally distinguished. One such possible example is the set consisting of $|\Psi_{11}^{(5)}\rangle$, $|\Psi_{21}^{(5)}\rangle$, $|\Psi_{13}^{(5)}\rangle$, $|\Psi_{23}^{(5)}\rangle$ [given in Eq. (2)]. One can check that there exists no pure state $|\phi^{(5)}\rangle$ for which $V_{11}^{(5)}|\phi^{(5)}\rangle, V_{21}^{(5)}|\phi^{(5)}\rangle, V_{13}^{(5)}|\phi^{(5)}\rangle, V_{23}^{(5)}|\phi^{(5)}\rangle$ are mutually orthogonal [$V_{nm}^{(5)}$'s are given in Eq. (5)].

VI. NECESSARY CONDITION FOR RELIABLE DISTINGUISHABILITY

We have seen in Sec. V that there are sets of d number of pairwise orthogonal maximally entangled states of $d\otimes d$, all taken in the form of Eq. (2) (we have seen it for $d=4,5,6$), the states of none of which can be reliably discriminated by using the teleportation protocol $\mathcal{P}_{00}^{(d)}$. This is because of the fact that in none of these cases is the condition of Theorem 1 satisfied. Does it mean that *none* of these sets of states can be reliably discriminated by LOCC only, in the single copy case? Or (i.e., contrapositively), we want to check whether *reliable local distinguishability of single copies (or multiple copies) of L number of pairwise orthogonal maximally entangled states* $|\Psi_{n_im_i}^{(d)}\rangle$ (*for* $i=1,2,\ldots,L$) *implies the existence of at least one state* $|\phi^{(d)}\rangle$ *for which the states* $V_{n_1m_1}^{(d)}|\phi^{(d)}\rangle$, $V_{n_2m_2}^{(d)}|\phi^{(d)}\rangle,\ldots,$ $V_{n_Lm_L}^{(d)}|\phi^{(d)}\rangle$ *are pairwise orthogonal, where* $V_{nm}^{(d)}$'s *are given by (5)*. Later, in this section, we discuss about this implication (written in italics); and if it turns out to be true, this *necessary* condition will become the same as the sufficient condition (given by Theorem 1) for local distinguishability of the states $|\Psi_{n_im_i}^{(d)}\rangle$ (for $i=1,2,\ldots,L$).

Let us now give two examples where the above-mentioned implication (or similar to this) is satisfied.

(i) *Distinguishability of any two orthogonal maximally entangled states.* Let $|\psi_1^{(d)}\rangle$ and $|\psi_2^{(d)}\rangle$ be given as any two pairwise orthogonal maximally entangled states of $d\otimes d$, which are not necessarily of the form given in (2). These two states are reliably distinguishable by LOCC only [2]. For these two states, we can always find an orthonormal basis $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$ for Alice's system, and another orthonormal basis $\{|0'\rangle,|1'\rangle,\ldots,|(d-1)'\rangle\}$ for Bob's system such that

$$|\psi_1^{(d)}\rangle=(I\otimes I)|\psi_1^{(d)}\rangle=\frac{1}{\sqrt{d}}(|00'\rangle+\exp[i\theta_1]|11'\rangle+\cdots+\exp[i\theta_{d-1}]|(d-1)(d-1)'\rangle),$$

$$|\psi_2^{(d)}\rangle=(I\otimes V)|\psi_1^{(d)}\rangle=\frac{1}{\sqrt{d}}(|00''\rangle+\exp[i\delta_1]|11''\rangle+\cdots+\exp[i\delta_{d-1}]|(d-1)(d-1)''\rangle),\quad(9)$$

where $\{|0''\rangle,|1''\rangle,\ldots,|(d-1)''\rangle\}$ is an orthonormal basis of Bob's system, $\langle j'|j''\rangle=0$ for $j=0,1,2,\ldots,d-1$, and V is an unitary operator, acting on Bob's system, such that $V|0'\rangle=|0''\rangle$, $V|1'\rangle=\exp[i\delta_1]|1''\rangle,\ldots,$ $V|(d-1)'\rangle=\exp[i\delta_{d-1}]|(d-1)''\rangle$ [2]. Thus there are d states $|0'\rangle$, $|1'\rangle,\ldots,$ $|(d-1)'\rangle$ (of Bob's system) for which $\{I|0'\rangle,V|0'\rangle\}$, $\{I|1'\rangle,V|1'\rangle\},\ldots,$ $\{I|(d-1)'\rangle,V|(d-1)'\rangle\}$ are d pairs of orthogonal states.

(ii) *Distinguishability of all* $|\Psi_{nm}^{(d)}\rangle$'s *when two copies are given*. We have seen that the d^2 number of pairwise orthogonal maximally entangled states, given by Eq. (2), are reliably distinguishable by using LOCC only, if two copies of each of these states are given. Let

$$|\chi_{nm}\rangle_{AC:BD}=|\Psi_{nm}^{(d)}\rangle_{AB}\otimes|\Psi_{nm}^{(d)}\rangle_{CD},$$

for $n,m=0,1,\ldots,d-1$ and where Alice possesses the two systems A, C, and Bob possesses the other two systems B, D. Thus we see that when Alice and Bob share a single copy of one of the d^2 number of pairwise orthogonal maximally entangledstates $|\chi_{nm}\rangle_{AC:BD}$ of $d^2\otimes d^2$, they can reliably distinguish these states, using LOCC only. Also here

$$(I^{AC}\otimes W_{nm}^{BD})|\chi_{nm}\rangle_{AC:BD}=|\chi_{00}\rangle_{AC:BD},$$

where I^{AC} is the identity operator on the d^2-dimensional Hilbert space of Alice, while $W_{nm}^{BD}=V_{nm}^{(d)}\otimes V_{nm}^{(d)}$ is a unitary operator acting on the d^2-dimensional Hilbert space of Bob, where $V_{nm}^{(d)}$ is given by (5). Let us consider the state $|\phi^{(d^2)}\rangle=|0\rangle\otimes(1/\sqrt{d})(|0\rangle+|1\rangle+\cdots+|d-1\rangle)$. It can be shown that (in fact, we have shown it earlier, in this paper) the states $W_{nm}|\phi^{(d^2)}\rangle$ (for $n,m=0,1,\ldots,d-1$) are pairwise orthogonal.

For each of the sets of d pairwise orthogonal maximally entangled states $|\Psi_{n_im_i}^{(d)}\rangle$ (for $i=1,2,\ldots,d$), discussed earlier for particular values of d, where the states (of the set) can be shown to be reliably distinguishable (using $\mathcal{P}_{00}^{(d)}$) by LOCC only, we are able to find an input state $|\phi^{(d)}\rangle$ such that the following d number of states $V_{n_i m_i}^{(d)}|\phi^{(d)}\rangle$ (for $i=1,2,\ldots,d$) are orthogonal to each other. On the other hand, for particular values of d, we have seen that there are examples of sets of d states $|\Psi_{n_i m_i}^{(d)}\rangle$ (for $i=1,2,\ldots,d$), where one can never find a pure state $|\phi^{(d)}\rangle$ such that the d number of states $V_{n_i m_i}^{(d)}|\phi^{(d)}\rangle$ (for $i=1,2,\ldots,d$) are orthogonal to each other. And in each of these later types of examples, the states are possibly reliably locally indistinguishable (e.g., the four pairwise orthogonal maximally entangled states $|\Psi_{00}^{(4)}\rangle$, $|\Psi_{10}^{(4)}\rangle$, $|\Psi_{20}^{(4)}\rangle$, $|\Psi_{02}^{(4)}\rangle$ of $4\otimes 4$ can be shown to be reliably indistinguishable by using one-way LOCC only, and there exists no state $|\phi^{(4)}\rangle$ for which the four states $V_{00}^{(4)}|\phi^{(4)}\rangle$, $V_{10}^{(4)}|\phi^{(4)}\rangle$, $V_{20}^{(4)}|\phi^{(4)}\rangle$, $V_{02}^{(4)}|\phi^{(4)}\rangle$ are orthogonal to each other). All these facts lead to the following conjecture, in terms of the above-mentioned necessary condition:

Conjecture. Let $S=\{|\Phi_i^{(d)}\rangle=(I\otimes V_i^{(d)})|\Psi_{00}^{(d)}\rangle : i=1,2,\ldots,L\}$ be any given set of L number of pairwise orthogonal maximally entangled states of $d\otimes d$, where $V_i^{(d)}$'s are unitary operators acting on the states of a d-dimensional Hilbert space, and $2\leqslant L\leqslant d^2$. If these L number of states are reliably distinguishable by using LOCC only in the single copy case, then there will always exist at least one state $|\phi^{(d)}\rangle$ of the d-dimensional Hilbert space, for which the L states $V_i^{(d)}|\phi^{(d)}\rangle$ (for $i=1,2,\ldots,L$) are pairwise orthogonal.

We have mentioned at the end of Sec. V, for the set of four states $|\Psi_{11}^{(5)}\rangle,|\Psi_{21}^{(5)}\rangle,|\Psi_{13}^{(5)}\rangle$, and $|\Psi_{23}^{(5)}\rangle$, one can show that there exists no state $|\phi^{(5)}\rangle$ for which $V_{11}^{(5)}|\phi^{(5)}\rangle$, $V_{21}^{(5)}|\phi^{(5)}\rangle, V_{13}^{(5)}|\phi^{(5)}\rangle, V_{23}^{(5)}|\phi^{(5)}\rangle$ [$V_{nm}^{(5)}$'s are given in Eq. (5)] are mutually orthogonal to each other. Thus if the above-mentioned conjecture turns out to be true, the set of four states $|\Psi_{11}^{(5)}\rangle$, $|\Psi_{13}^{(5)}\rangle$, $|\Psi_{21}^{(5)}\rangle$, and $|\Psi_{23}^{(5)}\rangle$ will be indistinguishable by LOCC, in the single copy case.

VII. CONCLUSION

In conclusion, we have shown that more than d number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in (2), cannot be reliably discriminated, in the single copy case, by using LOCC only, but they can be reliably discriminated, by using LOCC only, if two copies of each of the states are given. It has been shown here, using the standard teleportation protocol of Bennett *et al.* [7], that for $d\leqslant 3$, any d number of pairwise orthogonal maximally entangled states in $d\otimes d$ can be reliably discriminated, in the single copy case, by using LOCC only, when all the states are taken from the set given in (2) (the same result has also been obtained by Fan [6]). But for $d\geqslant 4$, our method of discrimination, by using the above-mentioned standard teleportation protocol, fails in some cases, and we are undecisive in this situation regarding reliable local distinguishability of d number of pairwise orthogonal maximally entangled states in $d\otimes d$, all taken from the set given in (2). Whether the most general type of d or fewer number of pairwise orthogonal maximally entangled states of $d\otimes d$ [i.e., maximally entangled states which are not necessarily of the form of equation (2)] are reliably locally distinguishable, in the single copy case, is yet to be settled fully. Fan [6] provided a partial answer to this question, by showing that if d is a prime number and if L is a positive integer such that $L(L-1)\leqslant d$, then L number of paiwise orthogonal maximally entangled states, each of which is of the form given in (2), can be reliably distinguished by LOCC only, in the single copy case.

If the above-mentioned conjecture is true, one can easily see that no L number of pairwise orthogonal maximally entangled states $|\Phi_i^{(d)}\rangle\equiv(I\otimes V_i^{(d)})|\Psi_{00}^{(d)}\rangle$ (for $i=1,2,\ldots,L$) of $d\otimes d$ can be reliably discriminated by using LOCC only, and in the single copy case, if $L\geqslant(d+1)$. This is so because there would be no room for the existence of L number of pairwise orthogonal states $V_i^{(d)}|\phi^{(d)}\rangle$ (for $i=1,2,\ldots,L$) in a d-dimensional Hilbert space, if $L\geqslant(d+1)$. It is to be noted here that the maximally entangled states $|\Phi_i^{(d)}\rangle$ are not necessarily of the form given in Eq. (2).

While giving the sufficient condition (in terms of Theorem 1) for reliable local discrimination of pairwise orthogonal maximally entangled states, we restricted ourselves to states which are of the form given in Eq. (2). This is so because there are examples of sets of pairwise orthogonal maximally entangled states for which the above-mentioned necessary condition (given by the conjecture) is satisfied (i.e., one can find at least one state $|\phi^{(d)}\rangle$ for which the states $V_i^{(d)}|\phi^{(d)}\rangle$ are pairwise orthogonal), but local discrimination, by using standard teleportation protocol, fails. One such example is the following set of three pairwise orthogonal maximally entangled states of $3\otimes 3$:

$$|\psi_1\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|11\rangle+|22\rangle),$$

$$|\psi_2\rangle=\frac{1}{\sqrt{3}}(|01\rangle+|12\rangle+|20\rangle),$$

$$|\psi_3\rangle=\frac{1}{\sqrt{3}}(|0\phi_0\rangle+|1\phi_1\rangle+|2\phi_2\rangle),$$

where $|\phi_0\rangle=(1/\sqrt{3})(|0\rangle+|1\rangle+|2\rangle)$, $|\phi_1\rangle=(1/\sqrt{3})(|0\rangle+w|1\rangle+w^2|2\rangle)$, $|\phi_2\rangle=(1/\sqrt{3})(|0\rangle+w^2|1\rangle+w|2\rangle)$, and where $w=\exp[2\pi i/3]$. Although failure of local discrimination by using the standard teleportation protocol does not guarantee the same for all other teleportation protocols, we are, still now, unable to reliably distinguish the above-mentioned three pairwise orthogonal maximally entangled states in $3\otimes 3$ by using any teleportation protocol.

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[15] Thus, we here assume that either Alice or Bob starts (say, Alice, i.e., "Alice going first" [3]) doing generalized measurements $\{A_i\}_{i=1}^N$, where $\Sigma_{i=1}^N A_i^\dagger A_i = I$, and the resulting states $Tr_A[(A_i \otimes I)P[|\Psi_{00}^{(4)}\rangle](A_i^\dagger \otimes I)]$, $Tr_A[(A_i \otimes I)P[|\Psi_{10}^{(4)}\rangle](A_i^\dagger \otimes I)]$, $Tr_A[(A_i \otimes I)P[|\Psi_{20}^{(4)}\rangle](A_i^\dagger \otimes I)]$, and $Tr_A[(A_i \otimes I)P[|\Psi_{02}^{(4)}\rangle](A_i^\dagger \otimes I)]$ at Bob's end (corresponding to each measurement outcome "i" of Alice's generalized measurement) will be pairwise orthogonal (including the case when one or more of these resulting states becomes a null state), so that Bob can then reliably distinguish these pairwise orthogonal states, and hence the discrimination protocol is over—no further operation has to be done by Alice or Bob, on their respective subsystems. Now choosing $A_i = \Sigma_{j=0}^3 |\phi_{ij}\rangle\langle j|$ (for $i=1,2,\ldots,N$), where $|\phi_{i0}\rangle$, $|\phi_{i1}\rangle$, $|\phi_{i2}\rangle$, $|\phi_{i3}\rangle$ are states of some d-dimensioanl Hilbert space, and they are not necessarily normalized, not necessarily orthogonal to each other, but $\Sigma_{i=1}^N \langle\phi_{ij}|\phi_{ik}\rangle = \delta_{jk}$, for $j,k=0,1,2,3$, one can verify that the above-mentioned orthogonality conditions will always give rise to a contradiction.