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Motion of a heavy impurity through a Bose-Einstein condensate

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We study the motion of a pointlike impurity in a Bose-Einstein condensate at $T=0$. By solving the Gross-Pitaevskii (GP) equation in a perturbative manner we calculate the induced mass of the impurity and the drag force on the impurity in three-, two-, and one-dimensional (1D) cases. The relationship between the induced mass and the normal mass of fluid is found, and coincides with the result of the Bogoliubov theory. The drag force appears for the supersonic motion of the impurity. In 1D the drag force is investigated also on the basis of the exact Lieb-Liniger theory, using the dynamic form factor, which has been evaluated by the Haldane method of the calculation of correlation functions. In this theory the force appears for an arbitrarily small velocity of the impurity. The possibility of measuring the form factor in existing experiments is noted.

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I. INTRODUCTION

One of the most important peculiarities of Landau theory superfluidity is the existence of a finite critical velocity. If a body moves in a superfluid at $T=0$ with velocity V less then v_c, the motion is dissipationless. At $V > v_c$ a drag force arises because of the possibility of emission of elementary excitations. However, both theoretical and experimental investigation in superfluid ^{4}He are difficult. The critical velocity in ^{4}He is related to the creation of protons, for which one has no simple theoretical description. Further, an important role is played by complicated processes involving vortex ring production.

The situation in low-density weakly interacting Bose-Einstein condensed (BEC) gases is simpler. The Landau critical velocity in this case is due to Cherenkov emission of phonons, which can be described by mean-field theory. Due to the presence in the theory of an intrinsic length parameter—the correlation length ξ—the friction force for a small body does not depend on its structure. Vortex rings in the BEC cannot have a radius of less than ξ (see [1]) and it is reasonable to believe that probability of their creation by a small body is small. Thus, quantitative investigation of critical velocities in BEC is very interesting and can be used to probe the superfluidity of a quantum gas.

Recently, the existence of the critical velocity in a Bose-Einstein condensed gas was confirmed in a few experiments. At MIT a trapped condensate was stirred by a blue detuned laser beam [2] and the energy of dissipation was measured. The critical velocity was found to be smaller than the speed of sound due to the emission of vortices. The diameter of the laser spot in this experiment was of a macroscopic size and was large compared to the healing length. An improved technique allowed for the measurement of the drag force acting on the condensate in a subsequent experiment [3].

The analytical study of flow of the condensate over an impurity is highly nontrivial, due to the intrinsic nonlinearity of the problem arising from the interaction of the particles in the condensate. In one dimension the dissipation could occur at velocities smaller than predicted by Landau's approach, due to the emission of solitons [4]. The dependence of the critical velocity on the type of the potential was studied both by using a perturbative approach and numerical integration in [5,6]. The effective two-dimensional (2D) problem was considered in [7]. In this work generation of excitations in the oscillating condensate, in a time-dependent parabolic trap in the presence of a static impurity, was studied analytically. A three-dimensional (3D) flow of a condensate around an obstacle was calculated numerically by integration of the Gross-Pitaevskii (GP) equation and the emission of vortices was observed [8,9].

In this article we study a δ-function perturbation moving at a constant velocity in a condensate. We find analytically the depletion of the superfluid fraction and the drag force.

II. THREE-DIMENSIONAL SYSTEM

Let us consider an impurity moving through a 3D condensate at $T=0$. One of the possible realizations of this model could be the scattering of heavy neutral molecules by the condensate.

A. Effective mass and normal fraction

We start from the 3D energy functional of a homogeneous weakly interacting Bose gas in the presence of a δ-function perturbation (an impurity) moving with a constant velocity $\mathbf{V}$,

$$E = \int \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + (\mu - g_i\delta(\mathbf{r}-\mathbf{V}t))|\psi|^2 + \frac{g}{2}|\psi|^4 \right] d^3x, \tag{1}$$

where ψ is the condensate wave function, μ is the chemical potential, m the mass of a particle in the condensate, and $g = 4\pi\hbar^2 a/m$ and $g_i = 2\pi\hbar^2 b/m$ are particle-particle and particle-impurity coupling constants, with a and b being the respective scattering lengths [10]. We will assume that the interaction with impurity is small and we will use perturbation theory. By splitting the wave function into a sum of the unperturbed solution and a small correction $\psi(\mathbf{r},t)=\phi_0+\delta\psi(\mathbf{r},t)$, and linearizing the time-dependent GP equation with respect to $\delta\psi$, we obtain an equation describing the time evolution of $\delta\psi$,

$$i\hbar\frac{\partial}{\partial t}\delta\psi=\left(-\frac{\hbar^2}{2m}\Delta-\mu+2g|\phi_0|^2\right)\delta\psi+g\phi_0^2\delta\psi^* + g_i\delta(\mathbf{r}-\mathbf{V}t)\phi_0. \tag{2}$$

In a homogeneous system ϕ_0 is a constant fixed by the particle density $\phi_0=\sqrt{n}$ and $\mu=gn=mc^2$.

The perturbation follows the moving impurity, i.e., $\delta\psi$ is a function of $(\mathbf{r}-\mathbf{V}t)$, so the coordinate derivative is related to the time derivative $\partial\delta\psi(\mathbf{r}-\mathbf{V}t)/\partial t=-\mathbf{V}\vec{\nabla}\delta\psi(\mathbf{r}-\mathbf{V}t)$. We shall work in the frame moving with the impurity $\mathbf{r}'=\mathbf{r}-\mathbf{V}t$, and the subscript over $\mathbf{r}$ will be dropped.

Equation (2) for a perturbation in a homogeneous system can be conveniently solved in momentum space. In order to do this we introduce the Fourier transform of the wave function $\delta\psi_{\mathbf{k}}=\int e^{-i\mathbf{k}\cdot\mathbf{r}}\delta\psi(\mathbf{r})d^3x$. Equation (2) becomes

$$\left(\hbar\mathbf{k}\cdot\mathbf{V}-\frac{\hbar^2k^2}{2m}-mc^2\right)\delta\psi_{\mathbf{k}}-mc^2(\delta\psi_{-\mathbf{k}})^*-g_i\phi_0=0. \tag{3}$$

The substitution of $k\to -k$ and complex conjugation of (3) give the second equation. The obtained system of linear equations can be easily solved:

$$\delta\psi_{\mathbf{k}}=g_i\phi_0\frac{\hbar\mathbf{k}\cdot\mathbf{V}+\dfrac{\hbar^2k^2}{2m}}{(\hbar\mathbf{k}\cdot\mathbf{V})^2-\dfrac{\hbar^2k^2}{2m}\left(\dfrac{\hbar^2k^2}{2m}+2mc^2\right)}. \tag{4}$$

Let us calculate the energy of the perturbation. By neglecting terms of the order of $g_i|\delta\psi|^2$ the energy functional (1) becomes

$$E=E_0+g_i\phi_0^2+\int\hbar\mathbf{k}\cdot\mathbf{V}|\delta\psi_{\mathbf{k}}|^2\frac{d^3k}{(2\pi)^3}+\frac{g_i\phi_0}{2}(\delta\psi+\delta\psi^*)_{\mathbf{r}=\mathbf{0}}. \tag{5}$$

Here $E_0=Ngn/2$ is the energy of the system in the absence of the perturbation. We expand the third and fourth terms in powers of V. To avoid the large-k divergency, one must also introduce a renormalization of the scattering amplitude. It is sufficient to express the coupling constant in the second term of Eq. (5) in terms of the scattering amplitude b, using the second-order Born approximation,

$$g_i=\frac{2\pi\hbar^2 b}{m}\left[1+\frac{2\pi\hbar^2 b}{m}\int\left(\frac{\hbar^2k^2}{2m}\right)^{-1}\frac{d^3k}{(2\pi)^3}\right]. \tag{6}$$

Finally, by carrying out the integration over momentum space and considering N_{imp} impurities with a concentration given by $\chi=N_{\mathrm{imp}}/N$ we obtain the energy per particle,

$$\frac{E}{N}=\left\{2\pi na^3\left(1+\chi\frac{b}{a}\right)+8\pi^{3/2}(na^3)^{3/2}\chi\left(\frac{b}{a}\right)^2\right\}\frac{\hbar^2}{ma^2} + \frac{2\sqrt{\pi}}{3}(na^3)^{1/2}\chi\left(\frac{b}{a}\right)^2\frac{mV^2}{2}. \tag{7}$$

If we set $V=0$ we recover Bogoliubov's corrections to the energy in the presence of quenched impurities [11,12]. Note that even if the "mean-field" energy obtained from the GP equation in the absence of impurities $(\chi=0)$ leaves out terms of the order of $(na^3)^{3/2}$, the equations we obtain in the presence of impurities in a perturbative manner still correctly describe the effect of the disorder up to the terms of the order of $(na^3)^{3/2}$.

If $V\neq 0$ a quadratic term in the impurity contribution to the energy is present. It can be denoted as $\chi m^*V^2/2$, with

$$m^*=\frac{2\sqrt{\pi}}{3}(na^3)^{1/2}\left(\frac{b}{a}\right)^2 m \tag{8}$$

being the induced mass, i.e., the mass of particles dragged by an impurity [13]. Applicability of the perturbation theory demands m^* to be small compared to m. This gives the condition $(na^3)^{1/2}(b/a)^2\ll 1$. At zero temperature the interaction between particles does not lead to depletion of the superfluid density, and the suppression of the superfluidity comes only from the interaction of particles with impurities. Thus (8) defines the normal density $\rho_n=\chi\rho m^*/m$.

This result is in agreement with the one obtained by the means of Bogoliubov transformation starting from the Hamiltonian written in the second-quantized form in the presence of disorder [11,12].

The normal density of a superfluid is an observable quantity. It was evaluated in liquid ^{4}He by measuring of the moment of inertia of a rotating liquid or by measuring of the second-sound velocity. Both methods can be, in principle, developed for BEC gases.

B. Drag force and energy dissipation

The force with which the impurity acts on the system is

$$\mathbf{F}=-\int|\psi(\mathbf{r})|^2\vec{\nabla}[g_i\delta(\mathbf{r})]d^3x=g_i[\vec{\nabla}|\psi(\mathbf{r})|^2]_{\mathbf{r}=\mathbf{0}}. \tag{9}$$

Expanding the wave function into the sum of ϕ_0 and $\delta\psi$ and neglecting terms of order $\delta\psi^2$, we obtain

$$\mathbf{F}=g_i\phi_0\int i\mathbf{k}[\delta\psi_{\mathbf{k}}+(\delta\psi_{-\mathbf{k}})^*]\frac{d^3k}{(2\pi)^3} =\int\frac{2(g_i\phi_0)^2 i\mathbf{k}(\hbar^2k^2/2m)}{(\hbar\mathbf{k}\cdot\mathbf{V}+i0)^2-\dfrac{\hbar^2k^2}{2m}\left(\dfrac{\hbar^2k^2}{2m}+2mc^2\right)}\frac{d^3k}{(2\pi)^3}, \tag{10}$$

where we added an infinitesimal positive imaginary part $+i0$ to the frequency $\mathbf{k}\cdot\mathbf{V}$, according to the usual Landau causality rule. The drag force is obviously directed along to the velocity $\mathbf{V}$.

We can carry out the integration with respect to $\cos\vartheta$, where ϑ is the angle between the momentum $\mathbf{k}$ and velocity $\mathbf{V}$, using the formula $1/x+i0=\mathrm{P}(1/x)-i\pi\delta(x)$, where P denotes the principal value. Due to the integration between symmetric limits, only the imaginary part contributes to the integral with respect to $\cos\vartheta$. The poles in the integration over $\cos\vartheta$ appear not for all values of momentum, but only for

$$|k| \leq k_{\max} = 2m(V^2 - c^2)^{1/2}/\hbar. \quad (11)$$

Thus the energy dissipation takes place only if the impurity moves with a speed larger than the speed of sound. Integration with respect to k, taking into account restriction (11), finally gives

$$F_V = 4\pi n b^2 m V^2 (1 - c^2/V^2)^2. \quad (12)$$

The energy dissipation, $\dot{E}=-F_V V$, can be evaluated by measuring the heating of the gas.

For large V the force is proportional to V^2. The energy dissipation per unit time can then be presented as $\dot{E}=-\gamma E$ with the damping rate $\gamma \sim nb^2 V$.

Note in conclusion that our perturbative calculations cannot describe processes involving the dissipation of energy due to the creation of quantized vortex rings. Such a creation is possible at $V<c$, but has a small probability for low velocity and for a weak pointlike impurity.

III. LOW-DIMENSIONAL SYSTEMS

In this type of experiment the role of the impurity can also be played by a laser beam with small enough size and intensity. The Fourier components of the perturbed wave function $\delta\psi_{\mathbf{k}}$ are given by the formula (4), which is derived in an arbitrary number of dimensions. The only difference is in the substitution of $d^3k/(2\pi)^3$, with $d^Dk/(2\pi)^D$ in the integrals.

A. Two-dimensional system

There are different possible geometries of the experiment. One can create a 2D perturbation in the 3D condensate. Such a 2D impurity can be created, analogously to the MIT experiment [2,3], by a thin laser beam. Such a beam creates a cylindrical hole in the condensate, which is stirred by moving the position of the laser beam. Another possibility is to fix the position of the laser beam along the long axis of an elongated condensate, so that the dissipation can be studied by shaking the trap and exciting the breathing modes. The problem is to create a beam with a diameter that is small compared to the correlation length. The theory can be easily generalized for beams of finite diameter. The intensity of the beam can be tuned to satisfy the condition of a weak perturbation.

The more interesting possibility is the investigation of true 2D condensates, which can be created in plane optical traps, produced by a standing light wave. If the light intensity is large enough, tunneling between planes is small and the condensates behave as independent 2D systems. The impurity can again be created by a laser beam perpendicular to the condensate plane. Another possibility is to use impurity atoms, which can be driven by a laser beam, with a frequency close to the atomic resonance of the impurity.

In two dimensions one has $d^2k=kdkd\vartheta$. The drag force $\mathbf{F}$ is different from zero only if the denominator has poles, which means that the velocity V must be larger than the speed of sound c. Only momenta smaller than $k_{\max}$ [see Eq. (11)] contribute to the integral.

Integration for the velocities $V>c$ gives us the drag force in the 2D case,

$$F_V^{2D} = g_i^2 n_2 m^2 (V^2 - c^2)/(\hbar^3 V), \quad (13)$$

where n_2 is 2D density.

In a quasi-two-dimensional system, i.e., when the gas is confined in the z direction by the harmonic potential $(1/2)m\omega_z^2 r^2$, the 2D coupling constant equals $g_i=\sqrt{2\pi}(\hbar^2 b/ma_z)$, where $a_z=\sqrt{\hbar/m\omega_z}$ is the oscillator length and b is the 3D scattering length. (We consider here only the mean-field 2D situation. See [14], Sec. 17 for a more detailed discussion.)

The calculation of the effective mass gives

$$m^* = g_i^2 n_2 m/(4\pi^2\hbar^2 c^2). \quad (14)$$

Notice again that our calculations do not take into account the creation of vortex pairs which is possible at $V<c$.

B. One-dimensional system: Mean-field theory

In one dimension the integration is straightforward. The integration (10) over k gives $2\pi i$ if $V>c$ and zero otherwise. So, the force is $F_V^{1D}=2g_i^2 n_1 m/\hbar^2$, where n_1 is the linear density. In a quasi-one-dimensional system (i.e., a very elongated trap or a waveguide) there are no excitations in the radial harmonic confinement, and the coupling constant is given by $g_i=-\hbar^2/mb_{1D}$ with $b_{1D}=-a_\perp^2/b$, where $a_\perp=\sqrt{\hbar/m\omega_\perp}$ and one has $F^{1D}=2n_1\hbar^2/mb_{1D}^2$.

An interesting peculiarity is that the result does not depend on the velocity V (where, of course, the velocity must be larger than the speed of sound). The same result for the δ potential was found in [6]. Calculation of the effective mass gives $m^*=g_i^2 n_1/2\hbar c^3$.

In a 1D system energy dissipation is possible at $V<c$ due to the creation of the "gray solitons" first considered in [15]. Nonlinear calculations [4] show that the critical velocity for this process decreases with increasing coupling constant g_i.

This theory can be checked in an experiment in a 3D condensate. The 2D impurity can be presented by a moving light sheet.

C. One-dimensional system: Bethe-ansatz theory

We saw in Sec. III B that for a weakly interacting impurity the drag force appears only when the impurity velocity V is larger that the Landau critical velocity, which is equal to the velocity of sound c. The situation is, however, different in the Bethe-ansatz Lieb-Liniger theory of a 1D Bose gas [16]. According to this theory, excitations in the system actually have a fermionic nature. Even a low-frequency perturbation can create a particle-hole pair with a total momentum near $2p_F \equiv 2\hbar k_F = \hbar 2\pi n_1$. To calculate the drag force for this case we will use the dynamic form factor of the system $\sigma(\omega,k)$ (we follow the notation of [17], Sec. 87). The dissipated energy at $T=0$ can be calculated as

$$\dot{E} = -\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{0}^{\infty}\frac{d\omega}{\pi}\omega\frac{n_1}{2\hbar}\sigma(\omega,k)|U(\omega,k)|^2, \qquad (15)$$

where $U(\omega,k)=2\pi g_i\delta(\omega-kV)$ is the Fourier transform of the impurity potential $U(t,z)=g_i\delta(z-Vt)$. One has $|U(\omega,k)|^2 = 2\pi g_i^2 t\delta(\omega-kV)$, where t is the "time of observation." Thus the energy dissipation per unit of time is

$$\dot{E} = -F_V V = -\frac{g_i^2 n_1 V}{\hbar}\int_0^{\infty}\frac{dk}{2\pi}k\sigma(kV,k), \qquad (16)$$

where F_V is the drag force. We will try to estimate the velocity dependence of F_V.

For low-frequency dissipation the important values of k are near $2k_F$. According to [18],

$$\sigma(\omega,2k_F) \sim \omega^{(\eta-2)}, \quad \omega \to 0, \qquad (17)$$

where $\eta = 2\hbar k_F/mc = 2\pi\hbar n_1/mc \geqslant 2$ is the characteristic parameter of a 1D Bose gas. In the mean-field limit when $n_1 \to \infty$, the parameter $\eta \to \infty$. In the opposite case of a small density bosons behave as impenetrable particles (Girardeau limit [19]) and the dynamic form factor coincides with the one of an ideal Fermi gas. In this limit $\eta=2$.

In the general case one can calculate $\sigma(\omega,k)$ at small ω and $k \approx 2k_F$, generalizing the method of Haldane [20] for the case of time-dependent correlation functions. Calculations give

$$\sigma(\omega,k) = \frac{n_1 c}{\omega^2}\left(\frac{\hbar\omega}{mc^2}\right)^{\eta} f\left(\frac{c\Delta k}{\omega}\right), \quad \omega > 0, \ k > 0, \qquad (18)$$

where $k=2k_F+\Delta k$ and the function $f(x)$ is

$$f(x) = A(\eta)(1-x^2)^{\eta/2-1} \qquad (19)$$

in the interval $|x|<1$ and is equal to zero at $|x|\geqslant 1$ (see also [21]). The constant $A(\eta)$ can be calculated in two limiting cases: $A(\eta=2)=\pi/4$ (see [14], Sec. 17.3) and $A(\eta) \approx 8\pi^2/[(8C)^{\eta}\Gamma^2(\eta/2)]$, where $C=1.78\ldots$ is the Euler 's constant for $\eta \gg 1$ (details of the calculation will be published elsewhere).

Substituting (18) into (16) we finally find velocity dependence of the drag force,

$$F_V = \frac{\Gamma((\eta/2))}{2\sqrt{\pi}\Gamma((\eta+1/2))}A(\eta)\frac{g_i^2 n_i^2}{\hbar V}\left(\eta\frac{V}{c}\right)^{\eta}. \qquad (20)$$

Equation (20) is valid for the condition $V \ll c$.

Thus in the Girardeau strong-interaction limit $F_V \sim V$ and Bose gas behaves, from the point of view of friction, as a normal system, where the drag force is proportional to the velocity. On the contrary, in the mean-field limit the force is very small and the behavior of the system is analogous to a 3D superfluid. However, even in this limit the presence of the small force makes a great difference. Let us imagine that our system is twisted into a ring, and that the impurity rotates around the ring with a small angular velocity. If the system is superfluid in the usual sense of the word, the superfluid part must stay at rest. Presence of the drag force means that equilibrium will be reached only when the gas as a whole rotates with the same angular velocity. From this point of view the superfluid part of the 1D Bose gas is equal to zero even at $T=0$. Notice that in an earlier paper [22] the author concluded that $\rho_s=\rho$ at $T=0$ for arbitrary η. We believe that this difference results from different definitions of ρ_s and reflects the nonstandard nature of the system.

Equation (20) is equivalent to a result which was obtained by a different method in [23], with a model consisting of an impurity considered as a Josephson junction. Notice that the process of dissipation, which in the language of fermionic excitations can be described as creation of a particle-hole pair, corresponds in the mean-field limit to creation of a phonon and a small-energy soliton. It seems that such a process cannot be described in the mean-field approach in the linear approximation.

Experimental confirmation of these quite nontrivial predictions demands a true 1D condensate, where non-mean-field effects can be sufficiently large. Such condensates have been investigated in previous experiments [24,25]. In other experiments [26,27] condensates have been created in the form of elongated independent "needles" in optical traps, consisting of two perpendicular standing laser waves. The role of an impurity in this case must be played by a light sheet, perpendicular to the axis of condensates and moving along them.

Notice also that application of the additional light waves in experiments of this type allows one to create a harmonic perturbation of the form

$$U(t,z) = U_0\cos(\omega t - kz), \quad k = 2k_F + \Delta k, \qquad (21)$$

with small ω and Δk. Such potential was used in [26,27] for experiments with a 1D condensate in a periodic lattice. However, for a small amplitude U_0, measurement of the dissipation energy Q gives, according to (15), the dynamic form factor $S(\omega,k)$ directly.

IV. CONCLUSIONS

We have studied the motion of an impurity through the condensate at zero temperature by considering the perturbation of a stationary solution of the GP equation. We calculated the induced mass which contributes to the mass of a normal component. We find that the motion at small velocities is dissipationless in 1D, 2D, and 3D systems, although movement with velocities larger than the speed of sound leads to a nonzero drag force due to Cherenkov radiation of phonons. The expressions for the drag force are calculated. We used results for the dynamic form factor of the exact Lieb-Liniger theory to investigate the velocity dependence of the drag force in a 1D system. The form factor was calculated with the help of the Haldane method of calculations of correlation functions. The drag force exists at an arbitrarily small velocity of motion, but is very small in the mean-field limit. The dynamic form factor can be also directly measured by applying a harmonic time-dependent perturbation on 1D condensates [26,27].

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