Unambiguous discrimination between mixed quantum states

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We prove that the states secretly chosen from a mixed state set can be perfectly discriminated if and only if they are orthogonal. The sufficient and necessary condition under which nonorthogonal mixed quantum states can be unambiguously discriminated is also presented. Furthermore, we derive a series of lower bounds on the inconclusive probability of unambiguous discrimination of states from a mixed state set with *a priori* probabilities.

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Quantum state discrimination is an essential problem in quantum information theory. Perfect discrimination among nonorthogonal pure states is, however, forbidden by the laws of quantum mechanics. Nonetheless, if a nonzero probability of inconclusive answer is allowed, one can distinguish with certainty linearly independent pure states. This strategy is usually called *unambiguous discrimination*. Unambiguous discrimination among two equally probable nonorthogonal quantum pure states was originally addressed by Ivanovic [1], and then Dieks [2] and Peres [3]. Jaeger and Shimony [4] extended their result to the case of two nonorthogonal pure states with unequal *a priori* probabilities. Chefles [5] showed that n quantum pure states can be unambiguously discriminated if and only if they are linearly independent. For the general case of unambiguous discrimination between n pure states with *a priori* probabilities, it was shown in Refs. [6] and [7] that the problem of optimal discrimination, in the sense that the success probability is maximized, or equivalently, the inconclusive probability is minimized, can be reduced to a semidefinite programming (SDP) problem, which has only numerical solution in mathematics. On the other hand, Zhang *et al.* [8] and Feng *et al.* [9] derived two lower bounds on the inconclusive probability of unambiguous discrimination among n pure states.

Somewhat surprisingly, it is only recently that the problem of unambiguous discrimination between mixed states is considered. In Ref. [10], the optimal unambiguous discrimination between a pure state and a mixed state with rank 2 was examined. Then the result was generalized to the case of a pure state and an arbitrary mixed states in Ref. [11]. Rudolph *et al.* [12] derived a lower bound and an upper bound on the maximal probability of successful discrimination of two mixed states. Raynal *et al.* [13] presented two reduction theorems to reduce the optimal unambiguous discrimination of two mixed states to that of other two mixed states which have the same rank. In the general case of n mixed state discrimination, Fiurasek and Jezek [14] and Eldar [15] gave some sufficient and necessary conditions on the optimal unambiguous discrimination and some numerical methods were discussed.

In this paper, we consider first the distinguishability of any mixed state set. We prove that any state chosen from a mixed state set can be perfectly discriminated if and only if the set is orthogonal, in the sense that any state in the set has its support orthogonal to the supports of the others. For the case of nonorthogonal mixed state, the sufficient and necessary condition for them to be unambiguously distinguishable is that any state among them has support not totally included in the supports of the others. Furthermore, we consider the problem of unambiguous discrimination between n mixed states with *a priori* probabilities and present a series of lower bounds on the inconclusive probability.

Suppose a quantum system is prepared in a state secretly drawn from a known set $\rho_1,\ldots,\rho_n$, where each ρ_i is a mixed state in the Hilbert space $\mathcal{H}$. The task of discrimination is to obtain as much information about the identification of the state as possible. In what follows, by perfect discrimination we mean that one can always get the correct answer while by unambiguous discrimination we mean that except a maybe nonzero inconclusive probability, one can identify the state without error. It is obvious that perfect discrimination is necessarily an unambiguous one, but the reverse is not true in general. To unambiguously discriminate $\rho_1,\ldots,\rho_n$, one can construct a most general positive-operator valued measurement (POVM) comprising $n+1$ elements $\Pi_0,\Pi_1,\ldots,\Pi_n$ such that

$$\Pi_i \geqslant 0, \quad i=0,1,\ldots,n, \qquad \sum_{i=0}^{n}\Pi_i = I, \tag{1}$$

where I denotes the identity matrix in $\mathcal{H}$. Each POVM element Π_i, $i=1,\ldots,n$ corresponds to identification of the corresponding state ρ_i, while Π_0 corresponds to the inconclusive answer. For the sake of simplicity, we often specify only $\Pi_1,\ldots,\Pi_n$ for a POVM since the left element Π_0 is uniquely determined by $\Pi_0=I-\sum_{i=1}^{n}\Pi_i$. It is then straightforward that a POVM $\Pi_1,\ldots,\Pi_n$, $\sum_{i=1}^{n}\Pi_i\leqslant I$, can perfectly discriminate $\rho_1,\ldots,\rho_n$ if and only if

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$$\mathrm{Tr}(\rho_i \Pi_j) = \delta_{ij},$$

while it can unambiguously discriminate $\rho_1, \ldots, \rho_n$ if and only if

$$\mathrm{Tr}(\rho_i \Pi_j) = p_i \delta_{ij}$$

for some $p_i > 0$, where $i, j = 1, \ldots, n$.

Since the intersection of the kernels of all ρ_i is not useful for the purpose of unambiguous discrimination, sometimes we can assume without loss of generality that each Π_i, $i = 1, \ldots, n$, is in $\mathrm{supp}(\rho_1, \ldots, \rho_n)$. Here $\mathrm{supp}(\rho_1, \ldots, \rho_n)$ is defined by the Hilbert space spanned by eigenvectors of the matrices $\rho_1, \ldots, \rho_n$ with nonzero corresponding eigenvalues.

The following lemma is a necessary condition for a POVM to unambiguously discriminate a given mixed state set.

Lemma 1. Suppose $\Pi_1, \ldots, \Pi_n$ are POVM elements and $\sum_i \Pi_i \leqslant I$. If for any i, Π_i can unambiguously discriminate ρ_i, then $\Pi_j \rho_i = 0$ for any $i \neq j$.

Proof. Suppose for any i, Π_i can unambiguously discriminate ρ_i. Then we have $\mathrm{Tr}(\Pi_j \rho_i) = p_i \delta_{ij}$ for some $p_i > 0$. Let

$$\rho_i = \sum_{k=1}^{n_i} a_i^k |\psi_i^k\rangle\langle\psi_i^k| \tag{2}$$

for some $a_i^k > 0$ be the spectrum decomposition of ρ_i, then for any $i \neq j$,

$$0 = \mathrm{Tr}(\Pi_j \rho_i) = \sum_{k=1}^{n_i} a_i^k \langle\psi_i^k|\Pi_j|\psi_i^k\rangle \tag{3}$$

and $\langle\psi_i^k|\Pi_j|\psi_i^k\rangle = 0$ for $k = 1, \ldots, n_i$ from the fact $a_i^k > 0$. That implies $\Pi_j |\psi_i^k\rangle = 0$ and so $\Pi_j \rho_i = 0$. ∎

It is well known that perfect pure state discrimination is possible if and only if the states to be discriminated are orthogonal to each other. In the case of mixed state, we have a similar result as the following theorem.

Theorem 1. The mixed quantum states $\rho_1, \ldots, \rho_n$ can be perfectly discriminated if and only if they are orthogonal, that is, $\rho_i \rho_j = \delta_{ij} \rho_i^2$.

Proof. If $\rho_i \rho_j = \delta_{ij} \rho_i^2$, then $\mathrm{supp}(\rho_i) \perp \mathrm{supp}(\rho_j)$ for any $i \neq j$. We choose Π_i as the projector onto $\mathrm{supp}(\rho_i)$. Obviously $\sum_i \Pi_i = I_s$ and $\mathrm{Tr}(\Pi_i \rho_j) = \delta_{ij}$, where I_s is the identity matrix in supp $(\rho_1, \ldots, \rho_n)$. That indicates $\Pi_1, \ldots, \Pi_n$ can perfectly discriminate $\rho_1, \ldots, \rho_n$.

Conversely, if $\rho_1, \ldots, \rho_n$ can be discriminated perfectly, then there exist POVM elements $\Pi_1, \ldots, \Pi_n$, $\sum_k \Pi_k = I$, such that for any i, Π_i can perfectly (so unambiguously) discriminate ρ_i. From Lemma 1, we have $\Pi_j \rho_i = 0$ for any $i \neq j$. So $\rho_i \rho_j = \rho_i (\sum_k \Pi_k) \rho_j = \delta_{ij} \rho_i^2$. ∎

The above theorem gives us a sufficient and necessary condition under which mixed states can be discriminated perfectly. That is, they must be orthogonal to each other. In the case when the states are nonorthogonal, a strategy is, as in pure state situation, unambiguous discrimination. While a set of pure states can be unambiguously discriminated if and only if they are linearly independent [5], the unambiguous discrimination between mixed states has a stronger requirement, as the following theorem indicates.

Theorem 2. The mixed quantum states $\rho_1, \ldots, \rho_n$ can be unambiguously discriminated if and only if for any $i = 1, \ldots, n$, $\mathrm{supp}(S) \neq \mathrm{supp}(S_i)$, where $S = \{\rho_1, \ldots, \rho_n\}$ and $S_i = S \backslash \{\rho_i\}$.

Proof. Suppose $\rho_1, \ldots, \rho_n$ can be unambiguously discriminated, then there exist POVM elements $\Pi_1, \ldots, \Pi_n$ such that $\sum_i \Pi_i \leqslant I$ and $\mathrm{Tr}(\Pi_i \rho_j) = p_i \delta_{ij}$ for some $p_i > 0$. Let $|\psi_i^k\rangle, k = 1, \ldots, n_i$, be the eigenvectors of ρ_i with the corresponding eigenvalues larger than 0. Then there exists $1 \leqslant h_i \leqslant n_i$ such that $\langle\psi_i^{h_i}|\Pi_i|\psi_i^{h_i}\rangle > 0$ and from Lemma 1, for any $1 \leqslant j \leqslant n_i$, $\langle\psi_i^j|\Pi_r|\psi_i^j\rangle = 0$ provided that $i \neq r$.

In what follows, we prove that for any i, $|\psi_i^{h_i}\rangle$ cannot be written as a linear combination of the states $|\psi_r^j\rangle$ for $r \neq i$ and $j = 1, \ldots, n_r$, that will imply the result $\mathrm{supp}(S_i) \neq \mathrm{supp}(S)$. Suppose

$$|\psi_i^{h_i}\rangle = \sum_{r \neq i;k} a_{i,r}^k |\psi_r^k\rangle$$

for some $a_{i,r}^k$, then

$$\Pi_i |\psi_i^{h_i}\rangle = \sum_{r \neq i;k} a_{i,r}^k \Pi_i |\psi_r^k\rangle = 0, \tag{4}$$

which contradicts with $\langle\psi_i^{h_i}|\Pi_i|\psi_i^{h_i}\rangle > 0$.

Suppose $\mathrm{supp}(S) \neq \mathrm{supp}(S_i)$, then $\mathrm{supp}(\rho_i) \subsetneq \mathrm{supp}(S_i)$. It follows that there exists a state $|\phi_i\rangle$ such that $|\phi_i\rangle \not\perp \mathrm{supp}(\rho_i)$ but $|\phi_i\rangle \perp \mathrm{supp}(S_i)$. That is $\langle\phi_i|\rho_i|\phi_i\rangle > 0$ but $\langle\phi_i|\rho_r|\phi_i\rangle = 0$ for any $r \neq i$. Let $\Pi_i = q_i |\phi_i\rangle\langle\phi_i|$, where q_i is positive but sufficiently small such that $\sum_{i=1}^n \Pi_i \leqslant I$. It is easy to check that the POVM elements $\Pi_1, \ldots, \Pi_n$ can unambiguously discriminate ρ_i with a positive probability $p_i = q_i \langle\phi_i|\rho_i|\phi_i\rangle > 0$ for any $i = 1, \ldots, n$. ∎

When $\rho_1, \ldots, \rho_n$ are all pure states, the requirement for them to be unambiguously distinguishable presented in the above theorem is exactly that they should be linearly independent, just as we all know. This is because if $\rho_i = |\psi_i\rangle\langle\psi_i|$ for some state $|\psi_i\rangle$ then $\mathrm{supp}(S) \neq \mathrm{supp}(S_i)$ for any i if and only if $|\psi_1\rangle, \ldots, |\psi_n\rangle$ are linearly independent.

In general, however, the requirement of $\rho_1, \ldots, \rho_n$ to be unambiguously distinguishable is more strict than just linear independence. To see this, for any $i = 1, \ldots, n$, suppose $\mathrm{supp}(S) \neq \mathrm{supp}(S_i)$, we show that ρ_i cannot be written as a linear combination of the other states. In fact, if $\rho_i = \sum_{j \neq i} a_i^j \rho_j$ for some a_i^j, let $|\phi_i\rangle$ be a state orthogonal to $\mathrm{supp}(S_i)$ but not orthogonal to $\mathrm{supp}(\rho_i)$, then

$$0 < \langle\phi_i|\rho_i|\phi_i\rangle = \sum_{j \neq i} a_i^j \langle\phi_i|\rho_j|\phi_i\rangle = 0.$$

This contradiction indicates that $\rho_1, \ldots, \rho_n$ are linearly independent. The converse, however, does not necessarily hold. That is, the linear independence of $\rho_1, \ldots, \rho_n$ cannot guarantee that $\mathrm{supp}(S) \neq \mathrm{supp}(S_i)$ for any i. To see this, let us give a simple example. Suppose ρ_1 and ρ_2 are two different density matrices with ranks m in an m-dimensional Hilbert space. It is obvious that ρ_1 and ρ_2 are linearly independent but $\mathrm{supp}(\rho_1) = \mathrm{supp}(\rho_2) = \mathrm{supp}(\rho_1, \rho_2)$. So in general the linear independence of certain mixed states cannot ensure the existence of a POVM to unambiguously discriminate them.

We now turn to consider the problem of unambiguous discrimination between n mixed quantum states with *a priori* probabilities. The aim is to optimize the discrimination by choosing appropriate measurements to maximize the success probability, or equivalently, minimize the inconclusive probability. For the general case of unambiguous discrimination between n pure states, the optimization problem can be reduced to a semidefinite programming problem [6], which has no analytic solution. So the bound on the success (or inconclusive) probability for any unambiguous discrimination becomes very important. A lot of work, such as Refs. [8] and [9] is dedicated to this field. In the following, we derive a lower bound on the inconclusive probability of unambiguous discrimination between n mixed states using a method similar to that in Ref. [9].

Theorem 3. Suppose a quantum system is prepared in one of the n mixed states $\rho_1,\ldots,\rho_n$ with a priori probabilities $\eta_1,\ldots,\eta_n$. Then a lower bound on the inconclusive probability P_0 of unambiguous discrimination between these states is

$$P_0 \geqslant \sqrt{\frac{n}{n-1}\sum_{i\neq j}\eta_i\eta_j F(\rho_i,\rho_j)^2},$$

where $F(\rho_i,\rho_j)$ is the fidelity of ρ_i and ρ_j.

Proof. For any POVM elements $\Pi_1,\ldots,\Pi_n$, $\Sigma_i \Pi_i \leqslant I$, which can unambiguously discriminate $\rho_1,\ldots,\rho_n$, we have $\mathrm{Tr}(\Pi_i\rho_j)=p_i\delta_{ij}$ for $i,j=1,\ldots,n$. Define $\Pi_0=I-\Sigma_{i=1}^n \Pi_i \geqslant 0$, then $P_0=\Sigma_i\,\eta_i\mathrm{Tr}(\Pi_0\rho_i)$. So

$$P_0^2=\sum_i \eta_i^2(\mathrm{Tr}(\Pi_0\rho_i))^2+\sum_{i\neq j}\eta_i\eta_j\mathrm{Tr}(\Pi_0\rho_i)\mathrm{Tr}(\Pi_0\rho_j). \quad (5)$$

By the Cauchy inequality, we have

$$\sum_i \eta_i^2(\mathrm{Tr}(\Pi_0\rho_i))^2 \geqslant \frac{1}{n-1}\sum_{i\neq j}\eta_i\eta_j\mathrm{Tr}(\Pi_0\rho_i)\mathrm{Tr}(\Pi_0\rho_j). \quad (6)$$

Substituting Eq. (6) into Eq. (5) we have

$$P_0^2 \geqslant \frac{n}{n-1}\sum_{i\neq j}\eta_i\eta_j\mathrm{Tr}(\Pi_0\rho_i)\mathrm{Tr}(\Pi_0\rho_j). \quad (7)$$

Furthermore, using the Cauchy inequality again, we have

$$\begin{aligned}\mathrm{Tr}(\Pi_0\rho_i)\mathrm{Tr}(\Pi_0\rho_j) &= \mathrm{Tr}(U\sqrt{\rho_i}\sqrt{\Pi_0}\sqrt{\Pi_0}\sqrt{\rho_i}U^\dagger)\\ &\quad\times\mathrm{Tr}(\sqrt{\rho_j}\sqrt{\Pi_0}\sqrt{\Pi_0}\sqrt{\rho_j})\\ &\geqslant(\mathrm{Tr}(U\sqrt{\rho_i}\Pi_0\sqrt{\rho_j}))^2\\ &=(\mathrm{Tr}(U\sqrt{\rho_i}(I-\sum_{k=1}^n\Pi_k)\sqrt{\rho_j}))^2 \end{aligned} \quad (8)$$

for any unitary matrix U. From Lemma 1, we have $\sqrt{\rho_i}\Pi_k\sqrt{\rho_j}=0$ for any $i\neq j$ and $k=1,\ldots,n$. Notice also that

$$F(\rho_i,\rho_j)=\max_U \mathrm{Tr}(U\sqrt{\rho_i}\sqrt{\rho_j}),$$

where the maximum is taken over all unitary matrix U. It follows that for any $i\neq j$,

$$\mathrm{Tr}(\Pi_0\rho_i)\mathrm{Tr}(\Pi_0\rho_j)\geqslant F(\rho_i,\rho_j)^2. \quad (9)$$

Taking Eq. (9) back into Eq. (7) we derive the lower bound on P_0 as

$$P_0 \geqslant \sqrt{\frac{n}{n-1}\sum_{i\neq j}\eta_i\eta_j F(\rho_i,\rho_j)^2}. \quad (10)$$

That completes the proof of the theorem. ■

When $n=2$, the lower bound we presented above reduces to $P_0 \geqslant 2\sqrt{\eta_1\eta_2}F(\rho_1,\rho_2)$, which partially coincides with the bound given in Ref. [12]. On the other hand, when $\rho_1,\ldots,\rho_n$ are all pure states, the lower bound reduces to the one derived in Ref. [9].

What we would like to point out here is that from the proof of the above theorem, we can actually derive a series of lower bounds on the inconclusive probability. In fact, if we let

$$A_k=\sum_i \eta_i^{2k}(\mathrm{Tr}(\Pi_0\rho_i))^{2k}$$

and

$$B_k=\sum_{i\neq j}\eta_i^k\eta_j^k(\mathrm{Tr}(\Pi_0\rho_i))^k(\mathrm{Tr}(\Pi_0\rho_j))^k,$$

then by the Cauchy inequality, we have $A_k \geqslant B_k/(n-1)$. Using these notations, the key steps, Eqs. (5)–(7), in the proof of the above theorem can be reexpressed as

$$P_0^2=B_1+A_1\geqslant\frac{n}{n-1}B_1, \quad (11)$$

which implies the lower bound

$$P_0 \geqslant P_0^{(0)} \doteq \sqrt{\frac{n}{n-1}C_1} \quad (12)$$

as in Eq. (10). Here C_k is defined as

$$C_k=\sum_{i\neq j}\eta_i^k\eta_j^kF(\rho_i,\rho_j)^{2k}.$$

Now, if we notice the fact that $A_1^2=B_2+A_2$, then we can first consider the term A_1 and derive that $A_1=\sqrt{B_2+A_2}$, so Eq. (11) can be rewritten as

$$P_0^2=B_1+\sqrt{B_2+A_2}\geqslant B_1+\sqrt{\frac{n}{n-1}B_2},$$

which implies another lower bound

$$P_0 \geqslant P_0^{(1)} \doteq \sqrt{C_1+\sqrt{\frac{n}{n-1}C_2}}. \quad (13)$$

Using the Cauchy inequality, we can easily prove that $P_0^{(1)} \geqslant P_0^{(0)}$, which means that the bound $P_0^{(1)}$ is better than $P_0^{(0)}$ in general. These two lower bounds are equal if and only if

$$\sqrt{\eta_i\eta_j}F(\rho_i,\rho_j)\equiv C, \quad \forall\, i\neq j \quad (14)$$

for some constant C which is independent of i and j. Notice that in the special case of $n=2$, the condition (14) holds automatically. It follows that $P_0^{(1)}$ and $P_0^{(0)}$ coincide when discriminating two mixed states.

Similarly, for any $k \geqslant 0$, noticing that $A_r^2 = A_{2r} + B_{2r}$ for all r, we have the following equalities:

$$P_0^2 = B_1 + A_1 = B_1 + \sqrt{B_2 + A_2} = B_1 + \sqrt{B_2 + \sqrt{B_4 + A_4}} = \cdots = B_1 + \sqrt{B_2 + \sqrt{\ldots + \sqrt{B_{2^k} + A_{2^k}}}}. \tag{15}$$

Then applying the fact that $A_r \geqslant B_r/(n-1)$ and $B_r \geqslant C_r$ for any r, we derive a series of lower bounds on the inconclusive probability of unambiguous discrimination between n mixed states as follows:

$$P_0 \geqslant P_0^{(k)} \doteq \sqrt{C_1 + \sqrt{\cdots + \sqrt{\frac{n}{n-1} C_{2^k}}}}. \tag{16}$$

We can also prove that $P_0^{(0)} \leqslant P_0^{(1)} \leqslant \cdots$ using the Cauchy inequality, which means in general when k increases, the lower bounds become better and better in the sense that they are closer and closer to the real optimal inconclusive probability. Again, when the condition (14) holds, all these lower bounds coincide. On the other hand, since the increasing sequence $\{P_0^{(k)}, k = 0, 1, \ldots\}$ has an upper bound 1, they definitely converge at a limit denoted by

$$P_0^{(\infty)} \doteq \sqrt{C_1 + \sqrt{C_2 + \sqrt{C_4 + \cdots}}}, \tag{17}$$

which is the best lower bound we can derive using this method.

To summarize, we prove that any state chosen from a mixed state set can be perfectly discriminated if and only if the set is orthogonal. For the case of nonorthogonal mixed states, the sufficient and necessary condition for them to be unambiguously distinguishable is that any state has support not totally included in the supports of the others. We consider also the problem of unambiguous discrimination of n mixed states with *a priori* probabilities and present a series of lower bounds on the inconclusive probability.

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