Capacity of nonlinear bosonic systems

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We analyze the role of nonlinear Hamiltonians in bosonic channels. We show that the information capacity as a function of the channel energy is increased with respect to the corresponding linear case, although only when the energy used for driving the nonlinearity is not considered as part of the energetic cost and when dispersive effects are negligible.

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INTRODUCTION

Noninteracting massless bosonic systems have been the object of extensive analysis. Their maximum capacity in transmitting information was derived for the noiseless case both in the narrowband regime (where only a few frequency modes are employed) and in the broadband regime [1,2]. However, it is still an open question whether nonlinearities in the system may increase these bounds: linearity appears to be the most important assumption in all previous derivations [2]. Up to now nonlinear effects have been used in fiber optics communications to overcome practical limitations, such as using solitons to beat dispersion or traveling wave amplifiers to beat loss [3].

The approach adopted in this paper is fundamentally different from these and from others where nonlinearities and squeezing are employed at the coding stage when using linear channels [4]. We follow the cue of a recent proposal [5] where interactions were exploited in increasing the capacity of a qubit-chain communication line. In the case of linear bosonic systems, the information storage capacity of a signal divided by the time it takes for it to propagate through the medium gives the transmission capacity of the channel. In the presence of nonlinearities, dispersion can affect the propagation of the signal complicating the analysis, but an increase in the capacity can be shown, at least when the dispersive effects are negligible. A complete analysis of dispersion in nonlinear materials is impossible at this stage, since the quantization of these systems has been solved only perturbatively. The basic idea behind the enhancement we find is that the modification of the system spectrum due to nonlinear Hamiltonians may allow one to better employ the available energy in storing the information: we will present some examples that exhibit this effect. Excluding the down-conversion channel (a model sufficiently accurate to include the propagation issue; see Sec. II C), all these examples are highly idealized systems but are still indicative of the possible nonlinearity-induced enhancements in the communication rates. An important caveat is in order. In the physical implementations that we have analyzed, there is no capacity enhancement if we include in the energy balance also the energy required to create the nonlinear Hamiltonians. This is a general characteristic of any system: if one considers the possibility of employing all the available degrees of freedom to encode information, then one cannot do better than the bound obtained in the noninteracting case [6,7]. However, the enhancement discussed here is not to be underestimated since in most situations many degrees of freedom are not usable to encode information, but can still be employed to augment the capacity of other degrees of freedom. A typical example is when the sender is not able to modulate the signals sufficiently fast to employ the full bandwidth supported by the channel: an external pumping (such as the one involved in the parametric down-conversion case) may allow an increase in the energy devoted to the transmission modes. The distinction between the signal energy and the communication energy is a subtle but practically relevant issue. As the above example shows, the signal energy accounts only for the energy needed by the information-carrying degrees of freedom in order to propagate. Since the information transmission is an intrinsically dynamical process, this contribution cannot be null (even though it is well known that no energy needs to be dissipated in the communication [8]). On the other hand, the communication energy accounts for all the employed energy (including that needed to drive the nonlinearity), which, can boost the information propagation.

We start by describing a general procedure to evaluate the capacity of a system and we apply it to a linear bosonic system to reobtain some known results (see Sec. I). Such a procedure is instructive since it emphasizes the role of the system spectrum in the capacity calculation. We then analyze a collection of examples of nonlinear bosonic systems in the narrowband and wideband regimes (see Sec. II): for each case we describe the capacity enhancements over the corresponding linear systems. General considerations of the energy balance in information storage and on information propagation conclude the paper (see Sec. III).

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I. INFORMATION CAPACITY

The information capacity of a noiseless channel is defined as the maximum number of bits that can be reliably sent per channel use. From the Holevo bound (see [9] and [1] for its infinite-dimensional extension), we know that it is given by the maximum of the von Neumann entropy $S(\varrho) = -\mathrm{Tr}[\varrho \log_2 \varrho]$ over all the possible input states ϱ of the channel. In our case, since ϱ is the state of a massless bosonic field, the associated Hilbert space is infinite dimensional and the maximum entropy is infinite. However, for all realistic scenarios a cutoff must be introduced by constraining the energy required in the storage or in the transmission, e.g., requiring the entropy $S(\varrho)$ to be maximized only over those states that have an average energy E, i.e.,

$$E = \mathrm{Tr}[\varrho H], \tag{1}$$

where H is the system Hamiltonian. The constrained maximization of $S(\varrho)$ can be solved by standard variational methods (see [11,12] for examples), which entail the solution of

$$\delta\left\{ S(\varrho) - \frac{\lambda}{\ln 2}\mathrm{Tr}[H\varrho] - \frac{\lambda'}{\ln 2}\mathrm{Tr}[\varrho] \right\} = 0, \tag{2}$$

where λ and λ' are Lagrange multipliers that take into account the energy constraint (1) and the normalization constraint $\mathrm{Tr}[\varrho]=1$, and where the ln 2 factor is introduced so that all subsequent calculations can be performed using natural logarithms. Equation (2) is solved by the density matrix $\varrho = \exp[-\lambda H]/Z(\lambda)$, where

$$Z(\lambda) \equiv \mathrm{Tr}[e^{-\lambda H}] \tag{3}$$

is the partition function of the system and λ is determined from the constraint (1) by solving the equation

$$E = -\frac{\partial}{\partial \lambda}\ln Z(\lambda). \tag{4}$$

The corresponding capacity is thus given by

$$C = S[\exp(-\lambda H)/Z(\lambda)] = [\lambda E + \ln Z(\lambda)]/\ln 2, \tag{5}$$

which means that we can evaluate the system capacity only from its partition function $Z(\lambda)$.

In general an explicit expression for $Z(\lambda)$ is difficult to derive, but it proves quite simple for noninteracting bosonic systems, such as the free modes of the electromagnetic field. In fact, in this case the Hamiltonian is given by

$$H = \sum_k \hbar\omega_k\, a_k^\dagger a_k, \tag{6}$$

where ω_k is the frequency of the kth mode and the mode operators a_k satisfy the usual commutation relations $[a_k, a_{k'}^\dagger] = \delta_{kk'}$. Hence, the partition function is

$$Z(\lambda) = \prod_k \sum_{n_k=0}^{\infty} e^{-\lambda\hbar\omega_k n_k} = \prod_k \frac{1}{1 - e^{-\lambda\hbar\omega_k}}. \tag{7}$$

From Eqs. (5) and (7) it is clear that the capacity C will be ultimately determined by the spectrum ω_k of the system. In particular, in the narrowband case (where only a mode of frequency ω is employed) Eq. (5) gives [2,4,11]

$$C_{nb} = g\left(\frac{E}{\hbar\omega}\right), \tag{8}$$

where $g(x) \equiv (1+x)\log_2(1+x) - x\log_2 x$ for $x \neq 0$ and $g(0) = 0$. On the other hand, in the homogeneous wideband case (where an infinite collection of equispaced frequencies $\omega_k = k\delta\omega$ is employed for $k \in \mathbb{N}$) Eq. (5) gives

$$C_{wb} \simeq \frac{\pi}{\ln 2}\sqrt{\frac{2E}{3\hbar\,\delta\omega}}, \tag{9}$$

which is valid in the limit $\hbar\delta\omega \ll E$. When applied to communication channels, this last equation is usually expressed in terms of the rate R (bits transmitted per unit time) and power P (energy transmitted per unit time) as [1,2,10–12]

$$R = \frac{1}{\ln 2}\sqrt{\frac{\pi P}{3\hbar}}, \tag{10}$$

by identifying the transmission time with $2\pi/\delta\omega$. In the next section we analyze how the capacities C_{nb} and C_{wb} are modified by introducing nonlinear terms in the system Hamiltonian.

II. NONLINEAR HAMILTONIANS

Nonlinear terms in the Hamiltonian of the electromagnetic field derive from the interactions between the photons and the medium in which they propagate. In this section we will employ the techniques described above to derive the narrowband and wideband capacities when quadratic nonlinearities are present. In Secs. II A–II C we discuss parametric down-conversion type Hamiltonians in the narrowband and broadband regimes. In Secs. II D and II E we discuss a mode swapping interaction. All these nonlinearities arise in real-world systems from $\chi^{(2)}$ type couplings, when one of the three fields involved in these types of interactions is a strong pump field that can be considered classical [13]. For all the cases analyzed we present the capacity enhancement over the corresponding noninteracting Hamiltonian.

A. Squeezing Hamiltonian

Consider the single mode described by the Hamiltonian

$$H = \hbar\omega a^\dagger a + \hbar\xi[(a^\dagger)^2 + a^2]/2 + \hbar\Omega(\xi), \tag{11}$$

where ξ is the squeezing parameter. We employ $|\xi| < \omega$ to avoid Hamiltonians that are unbounded from below. In Eq. (11) the frequency $\Omega(\xi) \equiv \frac{1}{2}(\omega - \sqrt{\omega^2 - \xi^2})$ has been introduced so that the ground state of the system is null: with this choice, the average energy E is the energy associated with the mode a in the nonlinear medium. By applying the canonical transformation

$$a = A\cosh\theta - A^\dagger\sinh\theta, \quad \theta \equiv \frac{1}{4}\ln\left[\frac{\omega+\xi}{\omega-\xi}\right], \tag{12}$$

the Hamiltonian H is transformed to the free field form $\hbar\sqrt{\omega^2-\xi^2}A^\dagger A$, so that the derivation of the previous section

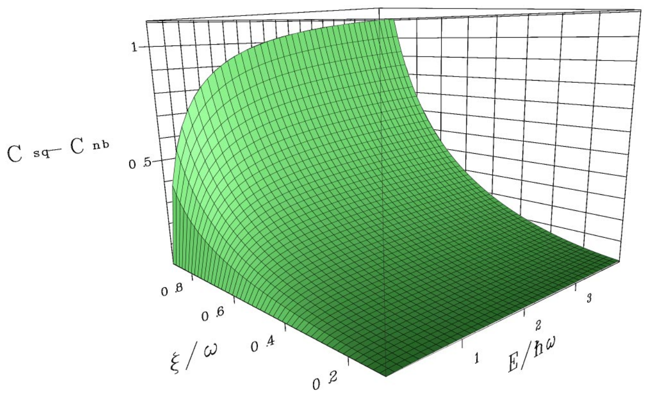

FIG. 1. Capacity increase of the squeezing Hamiltonian of Eq. (11) in bits per channel use as a function of the energy parameter $E/\hbar\omega$ and of the squeezing ratio ξ/ω. The increase is evident from the positivity of $C-C_{nb}$, where C is given in Eq. (13) and C_{nb} is the free space narrowband capacity of Eq. (8). Notice that, for high energy, the increase tends asymptotically to $-\log_2[1-\xi^2/\omega^2]/2$.

can be employed to calculate the capacity. It is thus immediate [see Eq. (8)] to find the capacity of this system as

$$C = g\left(\frac{E}{\hbar\sqrt{\omega^2-\xi^2}}\right). \tag{13}$$

This quantity measures the amount of information that can be encoded when E is the total energy associated with the mode a, and a nonlinear squeezing-generating term is present in the Hamiltonian: this result is quite different from the one obtained by using squeezed states as inputs to a linear system (see for example [4]). The capacity C of Eq. (13) is higher than the capacity C_{nb} of the linear case $\xi=0$ since $g(x)$ is an increasing function (see Fig. 1). The reason behind this enhancement is that the nonlinearity reduces the effective frequencies of the modes (from ω to $\sqrt{\omega^2-\xi^2}$), so that more energy levels can now be populated with the same energy.

B. Two-mode parametric down-conversion

Consider the two interacting modes a and b evolved by the Hamiltonian

$$H = \hbar\omega(a^\dagger a + b^\dagger b) + \hbar\xi(a^\dagger b^\dagger + ab) + \hbar\Omega(\xi), \tag{14}$$

where $|\xi|<\omega$ is the coupling constant and $\Omega(\xi)\equiv\omega-\sqrt{\omega^2-\xi^2}$ has again been introduced to ensure that the energy ground state is null. [This Hamiltonian, like the previous one, possesses the structure of the algebra of the SU(1,1) group, in the Holstein-Primakoff realization.] We can again morph the Hamiltonian to the free field form using the two-field canonical transformation

$$\begin{aligned} A &= a\cosh\theta + b^\dagger\sinh\theta \\ B &= a^\dagger\sinh\theta + b\cosh\theta \end{aligned}, \qquad \theta\equiv\frac{1}{4}\ln\left[\frac{\omega+\xi}{\omega-\xi}\right], \tag{15}$$

which transforms the Hamiltonian (14) to the form $\hbar\sqrt{\omega^2-\xi^2}(A^\dagger A+B^\dagger B)$. The capacity in this case is

$$C = 2\,g\left(\frac{E}{2\hbar\sqrt{\omega^2-\xi^2}}\right), \tag{16}$$

which is higher than the capacity of two independent single-mode bosonic systems, given by Eq. (16) for $\xi=0$. [The factors 2 in Eq. (16) derive from the presence of the two modes A and B.] The enhancement is again a consequence of the fact that the nonlinearity reduces the effective frequency of the two modes: $\omega\to\sqrt{\omega^2-\xi^2}$.

C. Broadband parametric down-conversion

Here we apply the above results to the case of two wide-band modes (signal and idler) coupled by a parametric down-conversion interaction. The interaction is mediated by a nonlinear crystal with second-order susceptibility $\chi^{(2)}$ pumped with an intense coherent field of amplitude $\mathcal{E}_p$ at frequency ω_p. Assuming undepleted pumping and perfect phase matching, the Hamiltonian up to the first order in the interaction is given by

$$H = \sum_k [\hbar\omega_k a_k^\dagger a_k + \hbar(\omega_p-\omega_k)b_k^\dagger b_k + \hbar\xi_k(a_k^\dagger b_k^\dagger + a_k b_k) + \hbar\Omega_k], \tag{17}$$

where a_k and b_k are the mode operators of the down-converted modes of frequency ω_k and $\omega_p-\omega_k$, respectively. Their interaction is described by the coupling parameter

$$\xi_k = \frac{\chi^{(2)}\pi\hbar\omega_k\mathcal{E}_p}{c\epsilon_0\, n_a(\omega_k)n_b(\omega_p-\omega_k)}\Phi(\omega_k), \tag{18}$$

with n_a and n_b the refractive indices of the signal and idler, and $\Phi(\omega_k)$ the phase matching function that takes into account the spatial matching of the modes in the crystal [13]. As usual, the frequency $\Omega_k=[\omega_p-\sqrt{\omega_p^2-4\xi_k^2}]/2$ has been introduced in the Hamiltonian to appropriately rescale the ground state energy. Notice that, in contrast to the case described in Sec. II B, the Hamiltonian (17) couples nondegenerate modes whose frequencies sum up to the pump frequency ω_p. Canonical transformations analogous to Eq. (15) allow us to rewrite the Hamiltonian in the free field form

$$H = \sum_k [\hbar(\omega_k-\Omega_k)A_k^\dagger A_k + \hbar(\omega_p-\omega_k-\Omega_k)B_k^\dagger B_k]. \tag{19}$$

With this Hamiltonian, the partition function is given by

$$\ln Z(\lambda) = \sum_k \ln\left[\frac{1}{1-e^{-\lambda\hbar(\omega_k-\Omega_k)}}\right] + \sum_k \ln\left[\frac{1}{1-e^{-\lambda\hbar(\omega_p-\omega_k-\Omega_k)}}\right], \tag{20}$$

where the two contributions are due to the signal and idler modes, respectively, and the sum over k is performed on all the frequencies up to ω_p. For ease of calculation, we will assume that the coupling ξ_k is acting only over a frequency band $\zeta\omega_p$ $(\zeta<1)$ centered around $\omega_p/2$, where it assumes the constant value ξ. This choice gives a rough approximation of the crystal phase matching function $\Phi(\omega_k)$ that prevents the

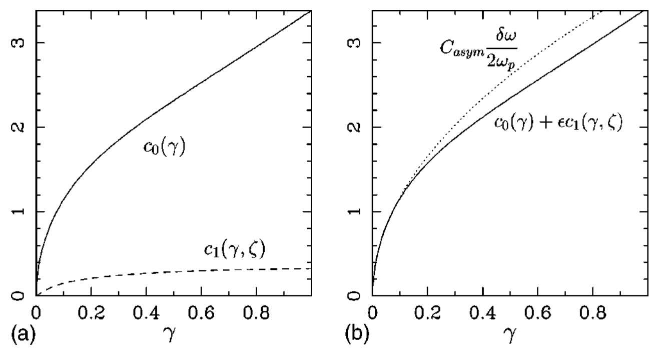

FIG. 2. (a) Capacity function $c_0(\gamma)$ (continuous line) and its correction $c_1(\gamma,\zeta)$ (dashed line) of Eq. (22) with fractional coupling bandwidth $\zeta=.5$ and $\gamma\equiv E\delta\omega/(2\hbar\omega_p^2)$. The capacity increase accomplished by the nonlinearity is evident from the positivity of the term c_1. (b) Comparison between the capacity C of Eq. (22) with $\epsilon\equiv 4\xi^2/\omega_p^2=0.1$ (continuous line) and the asymptotic two-mode rate $C_{asym}\equiv(2\pi/\ln 2)(\omega_p/\delta\omega)\sqrt{2\gamma/3}$, obtained using an infinite-frequency band (dotted line).

coupling of signal and idler photons when their frequencies are too mismatched [13]. In the high-energy regime $E \gg \hbar\delta\omega$, the sums in Eq. (20) can be replaced by frequency integrals, obtaining

$$\ln Z(\lambda) = \frac{2}{\delta\omega}\int_0^{\omega_p(1-\zeta)/2} d\omega \ln\left[\frac{1}{1-e^{-\lambda\hbar\omega}}\right] + \frac{2}{\delta\omega}\int_{\omega_p(1-\zeta)/2}^{\omega_p(1+\zeta)/2} d\omega \ln\left[\frac{1}{1-e^{-\lambda\hbar(\omega-\Omega)}}\right] + \frac{2}{\delta\omega}\int_{\omega_p(1+\zeta)/2}^{\omega_p} d\omega \ln\left[\frac{1}{1-e^{-\lambda\hbar\omega}}\right], \quad (21)$$

where $\Omega\equiv[\omega_p-\sqrt{\omega_p^2-4\xi^2}]/2$. The integrals in Eq. (21) have no simple analytical solution, but we can give a perturbative expansion in the low-interaction regime, i.e., $\epsilon\equiv 4\xi^2/\omega_p^2 \ll 1$. In this limit, the result is derived in the Appendix and is given by

$$C = \frac{2\omega_p}{\delta\omega}\left[c_0\left(\frac{E\delta\omega}{\hbar\omega_p^2}\right) + \epsilon\, c_1\left(\frac{E\delta\omega}{\hbar\omega_p^2},\zeta\right) + O(\epsilon^2)\right], \quad (22)$$

where the functions c_0 and c_1 are plotted in Fig. 2. The zeroth-order term c_0 in Eq. (22) gives the capacity of two broadband noninteracting modes with cutoff frequency ω_p: in the limit of infinite bandwidth ($\omega_p\to\infty$), this function reaches the asymptotic behavior $C\to C_{asym} \equiv(2\pi/\ln 2)\sqrt{E/(3\hbar\delta\omega)}$ that corresponds to C_{wb} of Eq. (9) for two noninteracting wideband bosonic systems. To first order in ϵ, the increase in capacity due to the interaction ξ is given by the value of c_1, which is a positive quantity (see the Appendix). It can be shown that $c_1\to 0$ in the limit $\omega_p\to\infty$, so that in the infinite-bandwidth regime no improvement is obtained from the interaction. In fact, an infinite continuous spectrum is invariant under the transformation $\omega_k \to\sqrt{\omega_k^2-\xi_k^2}$. As in the noninteracting broadband case of Eqs. (9) and (10), the transmission time τ of the signal can be estimated as $2\pi/\delta\omega$. The validity of this assumption (which implies negligible dispersion) rests on the fact that the Hamiltonian (17) is valid to first order in the interaction term ξ_k, and the small dispersion which derives from this term does not play any role in the capacity [10,11].

In this calculation E is the energy devoted to the signal and idler modes: it does not include the energy stored in the nonlinear medium and the energy spent to pump the crystal. This last quantity is proportional to $|\mathcal{E}_p|^2$ (apart from corrections of order ϵ which derive from the coupling). In the limit of undepleted pumping we have considered, this is a very large contribution which overshadows E and is not directly used to encode information. In this respect, the system is very inefficient in using the total available energy. However, this is not the correct attitude in evaluating such a system: our process is analogous to the amplification of signals where, if one considers the total energy employed in the process (amplification plus input energy), no gain is obtained. The correct attitude is, of course, to consider the signals alone.

D. Swapping Hamiltonian

Consider N modes that are pairwise coupled through the Hamiltonian

$$H=\sum_{j=1}^{N}\hbar\omega a_j^\dagger a_j + \sum_{j\neq j'}\Lambda_{jj'}a_j^\dagger a_{j'} = \hbar\vec{a}^\dagger\cdot(\omega\mathbb{1}+\Lambda)\cdot\vec{a}, \quad (23)$$

where $\vec{a}$ is the column vector containing the annihilation operators a_j of the N modes and Λ is an $N\times N$ symmetric real matrix with null diagonal. This Hamiltonian describes N modes a_j whose photons have a probability amplitude $\Lambda_{jj'}$ to be swapped into the mode $a_{j'}$. By performing a canonical transformation on all the mode operators, it is possible to rewrite Eq. (23) in the free field form

$$H=\sum_{j=1}^{N}\hbar(\omega+\lambda_j)A_j^\dagger A_j, \quad (24)$$

where the λ_j's are the N eigenvalues of Λ. Two conditions must be satisfied: the positivity of H requires that $-\lambda_j\le\omega$ for all j, and, since the diagonalization of Λ must preserve its trace, $\Sigma_j\,\lambda_j=0$. The capacity of this system can now be easily computed as [1,2,11]

$$C=\max_{e_1,e_2,\ldots,e_N}\left[\sum_{j=1}^{N} g\left(\frac{e_j}{\hbar(\omega+\lambda_j)}\right)\right], \quad (25)$$

where the maximum must be evaluated under the requirement that $\Sigma_j\, e_j=E$. In the simple case of $N=2$ (where $\lambda_1 =-\lambda_2\equiv\xi$), the maximization (25) can be easily performed numerically: the increase in capacity over the two-mode noninteracting case is presented in Fig. 3. In this case, in the strong coupling regime ($|\xi|\to\omega$) the capacity diverges as $\log_2\{E/[\hbar(\omega-|\xi|)]\}$: this corresponds to employing for the information storage only the lowest-frequency mode among A_1 and A_2.

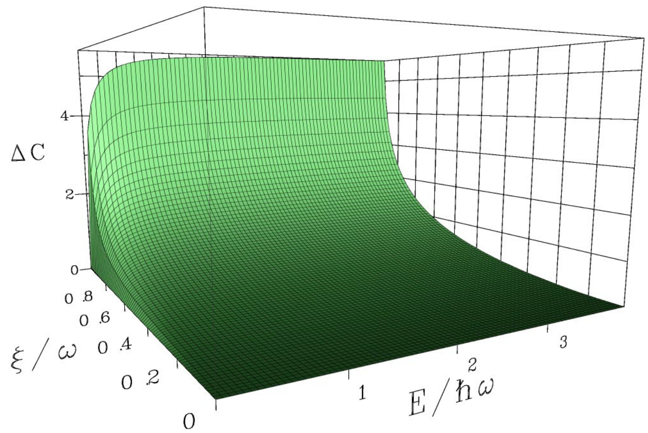

FIG. 3. Capacity increase of the swapping Hamiltonian (in bits per channel use) as a function of the energy parameter $E/\hbar\omega$ and of the coupling ratio ξ/ω, given by plotting $\Delta C \equiv C - 2g[E/(2\hbar\omega)]$ [the first term being the capacity C of Eq. (25) and the second the capacity of two noninteracting modes that share an energy E].

E. Broadband swapping Hamiltonian

A generalization of the Hamiltonian (23) given in the previous subsection can be obtained by considering N parallel broadband modes in which the coupling joins all the modes with the same frequency, namely,

$$H = \sum_k \sum_{j=1}^{N} \hbar\omega_k a_{jk}^\dagger a_{jk} + \sum_k \sum_{j\neq j'} \Lambda_{jj'}^{(k)} a_{jk}^\dagger a_{j'k}, \qquad (26)$$

where a_{jk} is the annihilation operator of the jth system with frequency ω_k. This Hamiltonian can be taken to a free field form using the procedure of the previous subsection, so that

$$H = \sum_k \sum_{j=1}^{N} \hbar(\omega_k + \lambda_{jk}) A_{jk}^\dagger A_{jk}, \qquad (27)$$

where λ_{jk} is the jth eigenvalue of the matrix $\Lambda^{(k)}$. Expressed in terms of the normal modes A_{jk}, the Hamiltonian (27) describes N independent wideband modes. Since in this case the maximization of the form (25) is complicated, we show an increase in capacity by considering the following choice of the coupling constants (which may well not be the optimal one):

$$\lambda_{jk} = -\,\omega_k\, r \quad \text{for } j = 1, \ldots, N-1, \qquad (28)$$

$$\lambda_{Nk} = \omega_k (N-1) r, \qquad (29)$$

where $0 \leqslant r \leqslant 1$. The choices (28) and (29) automatically guarantee both the positivity condition on the Hamiltonian and the trace preservation condition on the $\Lambda^{(k)}$ matrices. With this choice, the Hamiltonian (27) becomes

$$H = (1-r)\sum_k \sum_{j=1}^{N-1} \hbar\omega_k A_{jk}^\dagger A_{jk} + [1 + (N-1)r]\sum_k \hbar\omega_k A_{Nk}^\dagger A_{Nk}. \qquad (30)$$

Essentially we have chosen a coupling that contracts by a factor $1-r$ the frequencies of the first $N-1$ normal modes and stretches by a factor $[1+(N-1)r]$ the frequency of the Nth one. Choosing $r \to 1$, we can increase the capacity *ad libitum* over the case of N noninteracting systems. In fact, in this limit a straightforward application of the wideband calculation of Eq. (9) gives $C \sim \sqrt{(N-1)/(1-r)}C_{wb}$. This result must be compared with the case in which the N wideband modes are independent where the capacity scales as $\sqrt{N}C_{wb}$. Clearly, an arbitrary increase in capacity is gained as $r \to 1$.

III. GENERAL CONSIDERATIONS

In the previous sections we derived the capacity of various types of nonlinear bosonic systems. The common feature of all these results is that the nonlinearities were used to reshape the spectrum by compressing it to lower frequencies, where it is energetically cheaper to encode information. A couple of remarks are in order. First of all, in our calculations the mean energy E represents the energy of the modes *in* the system used to transmit information, whereas customarily one considers the input energy. Our choice is motivated by the fact that nonlinear systems dispense energy to the information-bearing modes, so that the input energy does not necessarily coincide with the amount of energy that the medium needs to sustain. This last quantity is a practically relevant one for the cases in which the degrees of freedom used to encode information cannot handle high energies (such as, for example, optical fibers [14]). Notice that, in some of the cases we have studied (the example of Sec. II C in particular), the nonlinearity is achieved by supplying the system with an external energy source in the form of an intense coherent beam. If one were to take into account also this contribution in the energy balance, then no capacity enhancement would be evident, since the pumping energy is used only indirectly to store the information. However, it is not unwarranted to exclude the pump from the energy balance, since it is used only to set up the required Hamiltonian and not directly employed in the information processing. Finally, our analysis is limited to the noiseless case. In the presence of noise, in place of the von Neumann entropy in Eq. (2), one would have to consider the Holevo information [15]. This is a highly demanding problem because of the yet unknown additivity properties of this quantity. At least in the case of linear bosonic systems, the capacity in the presence of noise was studied in [4,16].

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APPENDIX

In this appendix we derive formula (22) for the broadband parametric down-conversion Hamiltonian. In the low-interaction regime $\epsilon \ll 1$, Eq. (21) becomes

$$\ln Z(\lambda) = \frac{2\omega_p}{\delta\omega}[f_0(\beta) + \epsilon f_1(\beta,\zeta) + O(\epsilon^2)], \qquad (A1)$$

where $\beta \equiv \lambda\hbar\omega_p$ and

$$f_0(\beta) \equiv \frac{1}{\beta}\int_0^{\beta} dx \ln\left[\frac{1}{1-e^{-x}}\right], \tag{A2}$$

$$f_1(\beta,\zeta) \equiv \frac{1}{4}\ln\left[\frac{1-e^{-\beta(1+\zeta)/2}}{1-e^{-\beta(1-\zeta)/2}}\right]. \tag{A3}$$

Notice that the zeroth-order term f_0 in the expansion (A1) corresponds to the partition function of two broadband modes (signal and idler) with cutoff frequency ω_p. Replacing Eq. (A1) into the energy constraint (4), we can find the value of the Lagrange multiplier λ, contained in the parameter β, by solving the equation

$$\frac{\partial f_0(\beta)}{\partial \beta} + \epsilon\frac{\partial f_1(\beta,\zeta)}{\partial \beta} = -\gamma, \tag{A4}$$

where $\gamma \equiv E\delta\omega/(2\hbar\omega_p^2)$ is a dimensionless quantity. By expanding the solution β for small ϵ as $\beta=\beta_0+\epsilon\beta_1$, it follows that

$$\frac{\partial f_0(\beta_0)}{\partial \beta} = -\gamma, \tag{A5}$$

$$\beta_1 = -\frac{\partial f_1(\beta_0,\zeta)}{\partial \beta} \bigg/ \frac{\partial^2 f_0(\beta_0)}{\partial \beta^2}. \tag{A6}$$

These two equations can be numerically solved for any value of γ. Replacing the solution in Eq. (A1) and using the capacity formula (5) we can evaluate the parametric down-conversion capacity as reported in Eq. (22), where

$$c_0(\gamma) \equiv [\beta_0\gamma + f_0(\beta_0)]/\ln 2, \tag{A7}$$

$$c_1(\gamma,\zeta) \equiv f_1(\beta_0,\zeta)/\ln 2. \tag{A8}$$

Both these functions depend on the system energy only through the quantity γ.

[1] H. P. Yuen and M. Ozawa, Phys. Rev. Lett. **70**, 363 (1992).

[2] C. M. Caves and P. D. Drummond, Rev. Mod. Phys. **66**, 481 (1994), and references therein.

[3] H. A. Haus, *Electromagnetic Noise and Quantum Optical Measurements* (Springer, Berlin, 2000); L. A. Lugiato, F. Casagrande, and L. Pizzuto, Phys. Rev. A **26**, 3438 (1982); F. Haake and M. Lewenstein, *ibid.* **27**, 1013 (1983); C. M. Caves and D. D. Crouch, J. Opt. Soc. Am. B **4**, 1535 (1987); L. G. Jonekis and J. H. Shapiro, *ibid.* **10**, 1102 (1993); P. D. Drummond *et al.*, Nature (London) **365**, 307 (1993); C. Fidani and P. Tombesi, Phys. Rev. A **65**, 033815 (2002).

[4] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001); A. S. Holevo, M. Sohma, and O. Hirota, *ibid.* **59**, 1820 (1999).

[5] S. Lloyd, Phys. Rev. Lett. **90**, 167902 (2003).

[6] S. Lloyd, V. Giovannetti, and L. Maccone (unpublished).

[7] M. Schiffer and J. D. Bekenstein, Phys. Rev. D **39**, 1109 (1989); J. D. Bekenstein, e-print gr-qc/0006003.

[8] R. Landauer, Nature (London) **335**, 779 (1988); Appl. Phys. Lett. **51**, 2056 (1987).

[9] A. S. Holevo, Probl. Inf. Transm. **9**, 177 (1973).

[10] J. B. Pendry, J. Phys. A **16**, 2161 (1983).

[11] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981); Phys. Rev. A **37**, 3437 (1988).

[12] V. Giovannetti, S. Lloyd, L. Maccone, and J. H. Shapiro, Phys. Rev. A **69**, 052310 (2004).

[13] R. W. Boyd, *Nonlinear Optics* (Academic Press, New York, 2002).

[14] P. P. Mithra and J. B. Stark, Nature (London) **411**, 1027 (2001).

[15] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. A **54**, 1869 (1996); B. Schumacher and M. D. Westmoreland, *ibid.* **56**, 131 (1997).

[16] V. Giovannetti, S. Lloyd, L. Maccone, and P. W. Shor, Phys. Rev. Lett. **91**, 047901 (2003); Phys. Rev. A **68**, 062323 (2003); V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. **92**, 027902 (2004).