General criterion for the entanglement of two indistinguishable particles

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We relate the notion of entanglement for quantum systems composed of two identical constituents to the impossibility of attributing a complete set of properties to both particles. This implies definite constraints on the mathematical form of the state vector associated with the whole system. We then analyze separately the cases of fermion and boson systems, and we show how the consideration of both the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition of the global state vector and the von Neumann entropy of the one-particle reduced density operators can supply us with a consistent criterion for detecting entanglement. In particular, the consideration of the von Neumann entropy is particularly useful in deciding whether the correlations of the considered states are simply due to the indistinguishability of the particles involved or are a genuine manifestation of the entanglement. The treatment leads to a full clarification of the subtle aspects of entanglement of two identical constituents which have been a source of embarrassment and of serious misunderstandings in the recent literature.

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I. INTRODUCTION

Quantum entanglement, considered by Schrödinger "the characteristic trait of Quantum Mechanics, the one that enforces its entire departure from classical lines of thoughts" [1], has played a central role in the historical development of quantum mechanics and nowadays it constitutes an essential resource for many aspects of quantum information and quantum computation theory. In fact, the possibility of performing reliable teleportation processes, of generating unconditionally secure private keys in cryptography, or of devising quantum algorithms allowing us to solve certain computational problems in a more efficient way than the best known classical methods, is essentially based on the peculiar properties of entangled states. However, in spite of the fact that entangled states involving identical constituents are widely used in the experimental implementations of the above mentioned (and many other) processes, the very notion of entanglement for such ubiquitous physical systems seems to be too often misunderstood, or not understood at all, in the current scientific literature on the subject. The most frequent misinterpretations arise in connection with the symmetrization postulate of quantum mechanics, which requires definite symmetry properties for the state vectors associated with systems of identical particles. Their nonfactorized form seems to suggest,[1] when compared with the well-known case of systems composed of distinguishable particles, the occurrence of an unavoidable form of entanglement when identical particles enter into play.

In a recent paper [2] (see also [3]) we analyzed in great detail the problem of entanglement, dealing with systems of two (or more) both distinguishable and identical particles. In accordance with the position of the founding fathers of quantum mechanics [1,4], we strictly related the nonoccurrence of entanglement to the possibility of attributing complete sets of properties with both constituents of the composite system. In this way we were able to formulate an unambiguous criterion for deciding whether a given state vector is entangled or not, which works for the cases of both distinguishable and identical constituents. It has to be stressed that, contrary to what has sometimes been stated, nonentangled states involving identical constituents can actually occur.

Obviously, in the case of distinguishable particles, our criterion is equivalent (as we showed in Ref. [2]) to the commonly used criteria to identify whether a system is entangled or not, which involve consideration of the Schmidt number of the biorthonormal decomposition of the state or, equivalently, the evaluation of the von Neumann entropy of the reduced statistical operators.

The situation is radically different in the case of composite systems involving identical constituents. In the literature, various authors [5–7] have suggested identifying the entangled or nonentangled nature of a system of two identical constituents by resorting to natural generalizations of the above mentioned criteria. In so doing they have met various difficulties which emerge when one compares the results obtained with the analogous ones for the case of distinguishable particles. Moreover, the procedure yields apparently contradictory results for the fermion and boson cases [6–8].

In this paper, we show that the source of the problems rests in not having appropriately taken into account the fact

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[1]The only important exception being represented by the peculiar case of identical case of identical bosons in the same state.

that, even in the case in which it is physically legitimate and correct to consider a state as nonentangled, so that one knows the properties of the constituents, there is an unavoidable lack of information about the actual situation of the constituents, arising from their identity. Furthermore, we prove that by resorting to the general analysis of Refs. [2,3] one can get a complete clarification of the matter and we present a unified criterion for detecting entanglement in systems of identical particles such that (i) it involves *both* the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition of the state vectors *and* the von Neumann entropy of the reduced single-particle statistical operators; (ii) it applies equally well for fermions and bosons; and (iii) it is in complete accordance with our original criterion.

II. ENTANGLEMENT AND PROPERTIES

In this section we briefly review the arguments of Ref. [2], which show that the correct way to identify the entanglement is to relate it to the impossibility of attributing precise properties to the (identical) constituents of a two-particle system $S=S_1+S_2$. As is well known, in the case of nonentangled distinguishable particles the factorized nature of the state vector $|\psi(1,2)\rangle=|\phi\rangle_1\otimes|\chi\rangle_2$ is a necessary and sufficient condition for being allowed to claim that subsystem S_1 objectively possesses the (complete set of) properties associated with the state vector $|\phi\rangle_1$ and subsystem S_2 those associated with $|\chi\rangle_2$. We stress that in the case considered we know not only the properties that are possessed but also to which system they refer. Obviously, in the case of identical constituents one cannot resort to the factorizability criterion to claim that the two systems are nonentangled, otherwise one would be led to conclude (mistakenly) that nonentangled states cannot exist (an exception being made for two bosons in the same state), since the necessary symmetry requirements forbid the occurrence of factorized states.

However, this naive and inappropriate conclusion derives from taking a purely formal attitude about the problem, without paying due attention to the physically meaningful conditions which, when satisfied, allow one to legitimately state that two systems are nonentangled. Such conditions are, primarily, those of being allowed to claim that one particle possesses a precise and complete set of properties and the other one exhibits analogous features. Obviously, one must always keep clearly in mind that it is absurd to pretend to individuate the particles, i.e., to identify which one possesses one set of properties and which one the other set (in the case in which such sets are different).

Considerations of this kind have led us to identify the following physically appropriate criterion characterizing nonentangled states of two identical particles.

Definition 2.1. The identical constituents S_1 and S_2 of a composite quantum system $S=S_1+S_2$ are nonentangled when both constituents possess a complete set of properties.

Obviously, we still have to make fully precise the meaning of the expression "both constituents possess a complete set of properties." To this end we resort, first of all, to the following definition.

Definition 2.2. Given a composite quantum system $S=S_1+S_2$ of two identical particles described by the normalized state vector $|\psi(1,2)\rangle$, we will say that one of the constituents possesses a complete set of properties if and only if there exists a one-dimensional projection operator P, defined on the single particle Hilbert space $\mathcal{H}$, such that

$$\langle\psi(1,2)|\mathcal{E}_P(1,2)|\psi(1,2)\rangle = 1 \tag{1}$$

where

$$\mathcal{E}_P(1,2) = P^{(1)}\otimes[I^{(2)}-P^{(2)}]+[I^{(1)}-P^{(1)}]\otimes P^{(2)} + P^{(1)}\otimes P^{(2)}. \tag{2}$$

Condition (1) gives the probability of finding *at least* one of the two identical particles (of course we cannot say which one) in the state associated with the one-dimensional projection operator P.[2] Since any state vector is a simultaneous eigenvector of a complete set of commuting observables, condition (1) allows us to attribute to at least one of the particles the complete set of properties (eigenvalues) associated with the considered set of observables.

At this point we must distinguish two cases. If the two identical particles are fermions, then one can immediately prove (see Ref. [2]) that Eq. (1) implies that there exists another one-dimensional projection operator Q, which is orthogonal to P, such that the operator $\mathcal{E}_Q$, which has the expression (2) with Q replacing P, also satisfies

$$\langle\psi(1,2)|\mathcal{E}_Q(1,2)|\psi(1,2)\rangle = 1. \tag{3}$$

Obviously, in such a case, since it is simultaneously true that there is at least one particle having the properties associated with P and there is at least one particle having the properties associated with Q and, moreover, such properties are mutually exclusive due to the orthogonality of the projection operators, we can legitimately claim that, in the state $|\psi(1,2)\rangle$, one particle (it is meaningless to ask which one) has the complete set of properties associated with P and one the complete set associated with Q. As we showed in Ref. [2] our request for nonentanglement leads, in the fermion case, to the following theorem.

Theorem 2.1. The identical fermions S_1 and S_2 of a composite quantum system $S=S_1+S_2$ described by the pure normalized state $|\psi(1,2)\rangle$ are nonentangled if and only if $|\psi(1,2)\rangle$ is obtained by antisymmetrizing a factorized state.

In the boson case the situation is slightly different since the two particles can be in the same state. To be completely general let us begin by assuming that at least one of the constituents possesses a complete set of properties, so that there exists a one-dimensional single-particle projection operator P such that the associated $\mathcal{E}_P$ satisfies Eq. (1). At this

[2]We remark that one could drop the last term in the expression of Eq. (2), getting an operatoe whose expectation value gives the probability of finding precisely one particle in the state onto which P projects. In the case of identical fermions this makes no difference, but for bosons it would not cover the case of both particles being in the same state.

point we take into account the operator $P^{(1)} \otimes P^{(2)}$ and we consider its expectation value in the state $|\psi(1,2)\rangle$. Three possible cases can occur.

(1) If $\langle\psi(1,2)|P^{(1)} \otimes P^{(2)}|\psi(1,2)\rangle = 1$, then we can say that both particles possess the complete set of properties associated with P, and $|\psi(1,2)\rangle$ is the tensor product of two identical state vectors. This is the only situation in which the unavoidable ambiguities ensuing from the identity of the constituents disappear.

(2) If $\langle\psi(1,2)|P^{(1)} \otimes P^{(2)}|\psi(1,2)\rangle = 0$ then the condition Eq. (1) implies that the state is obtained by symmetrizing the product of two orthogonal states, one of which is the one on which P projects, and, consequently, there is another one-dimensional projection operator Q orthogonal to P such that Eq. (3) is satisfied, and the mean value of $Q^{(1)} \otimes Q^{(2)}$ also vanishes. The situation is perfectly analogous to the fermion case, and one can conclude that there is precisely one particle possessing the properties identified by P and one possessing the properties identified by Q.

(3) If $\langle\psi(1,2)|P^{(1)} \otimes P^{(2)}|\psi(1,2)\rangle \in (0,1)$, since condition Eq. (1) implies that the state is obtained by symmetrizing the product of two states, one of which is the one on which P projects, the second state cannot be orthogonal to the first one. If we denote as Q the one-dimensional projection operator on such a state, then we can immediately verify that Eq. (3) holds and that $\langle\psi(1,2)|Q^{(1)} \otimes Q^{(2)}|\psi(1,2)\rangle \in (0,1)$. In such a situation, in spite of the fact that the two statements "there is at least one particle with the properties associated with P" and "there is at least one particle with the properties associated with Q" are true, one cannot conclude that "there is precisely one particle possessing the properties identified by P and one possessing the properties identified by Q."

The above considerations, as the reader can easily grasp and as has been proved in Ref. [2], lead to the following theorem.

Theorem 2.2. The identical bosons of a composite quantum system $S=S_1+S_2$ described by the pure normalized state $|\psi(1,2)\rangle$ are nonentangled if and only if either the state is obtained by symmetrizing a factorized product of two orthogonal states or it is the product of the same state for the two particles.

Concluding, we have shown that, when one deals with the problem of entanglement using the necessary logical rigor and making appeal to the physical meaning of entanglement itself, then the process of (anti)symmetrizing a state vector does not necessarily lead to an entangled state.

In addition to the physical motivations we have presented, there are other compelling reasons which show that our criterion for nonentanglement is the correct one. It is in fact easy to show that, when the conditions of Theorems 2.1 or 2.2 are satisfied, it is not possible to take advantage of the form of the state vector to perform teleportation processes or to violate Bell's inequality. These facts further strengthen our conclusion that the state is nonentangled.

To clarify our argument we resort to an extremely simple example. Let us consider the following state of two identical spin-1/2 fermions, in which we have denoted as $|R\rangle$ and $|L\rangle$ two precise orthonormal states of the single-particle configuration space having disjoint supports in two distant and nonoverlapping spatial regions right and left, respectively, and with $|z\uparrow\rangle$ and $|z\downarrow\rangle$ the eigenvectors of the spin observable σ_z:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|z\uparrow\rangle_1|R\rangle_1 \otimes |z\downarrow\rangle_2|L\rangle_2 - |z\downarrow\rangle_1|L\rangle_1 \otimes |z\uparrow\rangle_2|R\rangle_2]. \tag{4}$$

Such a state, which can be obtained by antisymmetrizing a factorized state (that is, $|z\uparrow\rangle_1|R\rangle_1 \otimes |z\downarrow\rangle_2|L\rangle_2$), makes perfectly legitimate a statement of the type[3] "there is one particle at the right having spin up along the z direction and the spatial properties associated with $|R\rangle$ and a particle at the left having spin down along the z direction and the spatial properties associated with $|L\rangle$." As a consequence, this state does not imply any (nonlocal) correlation between spin measurements performed in the two spacelike separated regions right and left and thus it cannot violate Bell's inequality. This follows immediately from the fact that, when the state is the one of Eq. (4), we have for the mean value $E(\vec{a},\vec{b})$ of the product of the outcomes of two spin measurements along the directions $\vec{a}$ and $\vec{b}$ in the two regions right and left the following expression:

$$\begin{aligned} E(\vec{a},\vec{b}) &= \langle\psi|[\vec{\sigma}^{(1)}\cdot\vec{a}P_R^{(1)} \otimes \vec{\sigma}^{(2)}\cdot\vec{b}P_L^{(2)} \\ &\quad + \vec{\sigma}^{(1)}\cdot\vec{b}P_L^{(1)} \otimes \vec{\sigma}^{(2)}\cdot\vec{a}P_R^{(2)}]|\psi\rangle \\ &= \langle z\uparrow|\vec{\sigma}\cdot\vec{a}|z\uparrow\rangle\langle z\downarrow|\vec{\sigma}\cdot\vec{b}|z\downarrow\rangle, \end{aligned} \tag{5}$$

where we have denoted as P_R and P_L the projection operators on the closed linear manifolds of the spatial wave functions with compact support in the right and left regions, respectively. The occurrence of a factorized product of two mean values implies that no choice of the unit vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ can lead to a violation of Bell's inequality [9]:

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| + |E(\vec{b},\vec{d}) + E(\vec{c},\vec{d})| \leq 2. \tag{6}$$

On the other hand, let us consider a state of the form

$$|\phi(1,2)\rangle = \frac{1}{2}[|z\uparrow\rangle_1|z\downarrow\rangle_2 - |z\downarrow\rangle_1|z\uparrow\rangle_2] \otimes [|R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2], \tag{7}$$

which is the one used in the Bohm version of the Einstein-Podolsky-Rosen argument of incompleteness. Since it cannot be obtained by antisymmetrizing a factorized product of two orthogonal states it is a genuine entangled state according to our criterion. Correspondingly, one can easily prove that such a state exhibits the nonlocal features leading to a violation of Bell's inequality for a proper choice of the orientations $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$.

In spite of their simplicity, the examples just considered show clearly why vectors displaying the apparent form of an entangled state, such as the one of Eq. (4), must be consid-

[3]Note that in making our statement we somehow individuate our constituents by making reference to the fact that they lie in different spatial regions. This remark is useful for the analysis that will follow.

ered as nonentangled, in perfect agreement with our criterion.

III. THE ENTANGLEMENT CRITERIA

As we have already stressed, when one deals with quantum systems composed of two distinguishable particles, the appearance of entanglement is equivalent to the impossibility of writing the state vector of the compound system $|\psi(1,2)\rangle$ as a tensor product of two single-particle states. This in turn implies two well-known formal facts: (i) by resorting to the Schmidt decomposition, the global state turns out to be nonentangled if and only if its associated Schmidt number (that is, the number of nonzero coefficients in such a decomposition) equals 1; (ii) the state is nonentangled if and only if the von Neumann entropy of the reduced statistical operator associated with both particles is equal to zero.[4]

Both facts have clear physical implications and a precise meaning: the first refers to the possibility of attributing a complete and precise set of properties to each constituent, the second ensures that we have the most complete and exhaustive information allowed by quantum theory about the situation of each constituent. In fact, in a factorized state each component subsystem is associated with a precise state vector and the reduced statistical operator for one of the two particles, for example, the one labeled by 1, i.e., $\rho^{(1)} = \mathrm{Tr}^{(2)}[|\psi(1,2)\rangle\langle\psi(1,2)|]$, turns out to be a projection operator onto a one-dimensional manifold. Correspondingly, its von Neumann entropy $S(\rho^{(1)}) \equiv -\mathrm{Tr}^{(1)}[\rho^{(1)}\log_2\rho^{(1)}]$ equals zero. This result is correct since such a quantity measures the lack of information about the single-particle subsystem and there is, in fact, no uncertainty at all concerning the state that must be attributed to it.

When passing to the more subtle case of interest, that is, to systems composed of two identical constituents, the relations between entanglement and both the Schmidt number and the von Neumann entropy of the reduced statistical operators become less clear and require a careful analysis. The purpose of this section is to clarify the matter and to present a criterion for determining whether a state is entangled or not which is (i) based on a consideration of *both* the Slater-Schmidt number and the von Neumann entropy; (ii) consistent with our original criterion summarized in Definition 2.1; (iii) equally applicable to fermion and boson systems; and finally (iv) able to unify in a consistent way various criteria which have appeared recently in the literature [5–7].

We limit our considerations to the case of a finite-dimensional single-particle Hilbert space and, in accordance with the above remarks, we deal separately with the fermion and the boson cases, since they exhibit quite different features.

A. The fermion case

The notion of entanglement for systems composed of two identical fermions has been discussed in Ref. [5] where a fermionic analog of the Schmidt decomposition was exhibited. Such a decomposition is based on a nice extension to the set of the antisymmetric complex matrices of a well-known theorem holding for antisymmetric real matrices (see, for example, [10]). The theorem of [5] states the following.

Theorem 3.1. For any antisymmetric $(N\times N)$ complex matrix A [that is, $A\in\mathcal{M}(N,\mathbb{C})$ and $A^T=-A$], there exists a unitary transformation U such that $A=UZU^T$, with Z a block-diagonal matrix of the sort

$$Z = \mathrm{diag}[Z_0, Z_1, \ldots, Z_M], \quad Z_0 = 0, \quad Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix}, \tag{8}$$

where Z_0 is the $(N-2M)\times(N-2M)$ null matrix and z_i are complex numbers. Equivalently, Z is the direct sum of the $(N-2M)\times(N-2M)$ null matrix and the M (2×2) complex antisymmetric matrices Z_i.

The fermionic analog of the Schmidt decomposition follows from an application of Theorem 3.1 to systems composed of identical fermions.

Theorem 3.2. Any state vector $|\psi(1,2)\rangle$ describing two identical fermions of spin s and, consequently, belonging to the antisymmetric manifold $\mathcal{A}(\mathbb{C}^{2s+1}\otimes\mathbb{C}^{2s+1})$ can be written as

$$|\psi(1,2)\rangle = \sum_{i=1}^{(2s+1)/2} a_i \frac{1}{\sqrt{2}}[|2i-1\rangle_1 \otimes |2i\rangle_2 - |2i\rangle_1 \otimes |2i-1\rangle_2], \tag{9}$$

where the states $\{|2i-1\rangle, |2i\rangle\}$ with $i=1,\ldots,(2s+1)/2$ constitute an orthonormal basis of $\mathbb{C}^{2s+1}$, and the complex coefficients a_i (some of which may vanish) satisfy the normalization condition $\sum_i |a_i|^2=1$.

Following the authors of Ref. [5], the number of nonzero coefficients a_i appearing in the decomposition (9) is called the *Slater number* of $|\psi(1,2)\rangle$. The relation of such a number to the notion of entanglement has been made explicit in the papers [6,7], where a state displaying the form of Eq. (9) is called entangled if and only if its Slater number is strictly greater than 1. It is worth noticing that this condition turns out to be totally equivalent to our Definition 2.1. In fact, given an arbitrary two-particle state $|\psi(1,2)\rangle$, suppose that its fermionic Schmidt decomposition has Slater number equal to 1. According to Theorem 3.2, this means that there exist two orthonormal vectors $|1\rangle$ and $|2\rangle$ belonging to $\mathbb{C}^{2s+1}$ such that

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|1\rangle_1 \otimes |2\rangle_2 - |2\rangle_1 \otimes |1\rangle_2], \quad \langle 1|2\rangle = 0. \tag{10}$$

Since the state can be obtained by antisymmetrizing the product state $|1\rangle_1\otimes|2\rangle_2$, the state must be considered as nonentangled in accordance with our criterion. Vice versa, any state obtained by antisymmetrizing a factorized state has Slater number equal to 1. On the contrary, if the Slater number is greater than (or equal to) two, the form Eq. (9) of the state shows immediately that it cannot be obtained by antisymmetrizing a product of two orthonormal vectors, so that

[4]As is well known, given a statistical operator ρ, its von Neumann entropy is defined as $S(\rho)\equiv -\mathrm{Tr}[\rho \ln \rho]$ where the base of the logarithm function is the number e. However, in the present paper, we will rescale this quantity and follow the information theory convention of using all logarithms in base 2.

the state must be considered as a genuinely entangled one.

So far the criterion of Refs. [5–7] and ours agree completely; however, some problems arise when one calculates the von Neumann entropy of the reduced density operator associated with one of the two particles.[5] In fact from Eq. (9) one gets

$$\rho^{(1)} \equiv \mathrm{Tr}^{(2)}[|\psi(1,2)\rangle\langle\psi(1,2)|] = \sum_i \frac{|a_i|^2}{2}[|2i-1\rangle_1\langle 2i-1| + |2i\rangle_1\langle 2i|]. \quad (11)$$

The von Neumann entropy for such an operator, which is already in its diagonal form, can be easily calculated:

$$S(\rho^{(1)}) \equiv -\mathrm{Tr}^{(1)}[\rho^{(1)}\log_2 \rho^{(1)}] = -\sum_i |a_i|^2 \log_2 \frac{|a_i|^2}{2} = 1 - \sum_i |a_i|^2 \log_2 |a_i|^2. \quad (12)$$

It follows trivially that

$$S(\rho^{(1)}) \geqslant 1 \quad \forall\, |a_i|^2 \in [0,1], \quad \sum_i |a_i|^2 = 1. \quad (13)$$

In analogy with the case of distinguishable particles, one could be tempted to regard this quantity as a measure of entanglement. But, according to the authors of Refs. [6,8], this naturally raises the following two puzzling issues: (i) $S(\rho^{(1)})$ of Eq. (12) attains its minimum value $S_{min}=1$ in correspondence with a state with Slater number equal to 1 (which, in accordance with their position, is assumed to identify a nonentangled state), contrary to what happens for a (nonentangled) state of distinguishable particles with Schmidt number equal to 1, for which the value of the entropy still takes its minimum value which, however, equals 0; (ii) moreover, in the case of two identical bosons, the minimum of the analogous quantity, as we will show later, is null. This seems to imply that the von Neumann entropy is inappropriate to deal with our problem since it gives different measures of entanglement for boson and fermion states.

The problematic aspects of this situation derive entirely from not having taken correctly into account the real meaning of the von Neumann entropy as a measure of the uncertainty about the state of a quantum system. To be more precise, let us consider a state $|\psi(1,2)\rangle$ with Slater number equal to 1, i.e., a nonentangled state according to our and the Slater number criterion, like that of Eq. (10). As we stressed in Sec. I, in this situation we can attribute definite quantum states $|1\rangle$ and $|2\rangle$ to the particles but, since they are totally indistinguishable, we cannot know whether, for example, particle 1 is associated with the state $|1\rangle$ or with the state $|2\rangle$. As a natural consequence, the reduced density operator of each particle and its associated von Neumann entropy reflect such an unavoidable ignorance.[6] Actually, they are equal to

$$\rho^{(1\ or\ 2)} = \frac{1}{2}[|1\rangle\langle 1| + |2\rangle\langle 2|] \Rightarrow S(\rho^{(1\ or\ 2)}) = 1. \quad (14)$$

It should be obvious that we cannot pretend that the operator $\rho^{(1\ or\ 2)}$ of Eq. (14) describes the properties of *precisely* the first or of the second particle of the system: once again, due to the subtle implications of the identity in quantum mechanics, such an operator describes correctly the properties of a randomly chosen particle (a particle that cannot be better *identified*). Accordingly, in this case, the quantity $S(\rho^{(1\ or\ 2)})=1$ correctly measures the uncertainty concerning the quantum state to attribute to each of the two identical physical subsystems, and, in this situation, it cannot be regarded as a measure of the entanglement of the whole state.

A counterpart of this is the fact that the only quantum correlations exhibited by the state $|\psi(1,2)\rangle$ of Eq. (10) are those related to the exchange properties of the indistinguishable fermions. As we have stressed before, such correlations cannot be used to violate Bell's inequality or to perform any teleportation process. Accordingly, they cannot be considered as a manifestation of entanglement.

Continuing our analysis, if we consider a state vector with Slater number strictly greater than 1, i.e., an entangled state according to our and to the Slater number criterion, the von Neumann entropy of the associated reduced density operator turns out to be strictly greater than 1 [see Eq. (13)]. In addition to the minimum amount of uncertainty deriving from the indistinguishability of the particles, there is now the *additional* ignorance naturally connected with the genuine entanglement of the state. In this case, in fact, we cannot identify two quantum states that can be attributed[7] to the pair of particles (contrary to the previously described situation).

To conclude this subsection, we exhibit the criteria for detecting entanglement involving the Slater number or the von Neumann entropy, in the case of two identical fermions. Such criteria are totally consistent with our original requirement that definite properties can be attributed to both component subsystems of a nonentangled state:

Theorem 3.3. A state vector $|\psi(1,2)\rangle$ describing two identical fermions is nonentangled if and only if its Slater number is equal to 1 or equivalently if and only if the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1\ or\ 2)})$ is equal to 1.

In full agreement with the previous considerations we can say that the von Neumann entropy is a measure both of the amount of uncertainty deriving from the indistinguishability of the involved particles *and* of the possible uncertainty re-

[5]Since the state vector of the compound system $|\psi(1,2)\rangle$ is antisymmetric under the exchange of the two particles, its associated density operator $\rho^{(1+2)} \equiv |\psi(1,2)\rangle\langle\psi(1,2)|$ is a symmetric operator and the reduced density operators of the two particles are equal, that is, $\rho^{(1)} \equiv \mathrm{Tr}^{(2)}[\rho^{(1+2)}] = \rho^{(2)} \equiv \mathrm{Tr}^{(1)}[\rho^{(1+2)}]$.

[6]To avoid any misunderstanding we stress that with the expression "ignorance" we do not intend any lack of knowledge about the system. In fact in the considered case this ignorance derives entirely from the fact that, within quantum mechanics, there is no conceivable way to individuate identical particles.

[7]With this expression we mean that a measurement aimed to test whether there is one particle in state $|1\rangle$ and one in state $|2\rangle$ gives with certainty a positive outcome.

lated to the entangled nature of the states (that is, the impossibility of attributing two definite state vectors to the pair of particles). This quantity, in the case of fermions, is strictly greater than 1, and the greater it is, the larger is the amount of entanglement of the state.

Before coming to the boson case, a last remark is appropriate. Let us consider a nonentangled state of two fermions which has the form of Eq. (10). Suppose we choose two arbitrary orthonormal vectors $|\bar{1}\rangle$ and $|\bar{2}\rangle$ belonging to the two-dimensional manifold spanned by the states $|1\rangle$ and $|2\rangle$, such that

$$|1\rangle = \alpha|\bar{1}\rangle + \beta|\bar{2}\rangle, \quad |2\rangle = -\beta^*|\bar{1}\rangle + \alpha^*|\bar{2}\rangle, \tag{15}$$

with $|\alpha|^2+|\beta|^2=1$.

Substituting these expressions in Eq. (10) we have

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\bar{1}\rangle_1 \otimes |\bar{2}\rangle_2 - |\bar{2}\rangle_1 \otimes |\bar{1}\rangle_2]. \tag{16}$$

Therefore, when a state is obtained by antisymmetrizing the product of two orthogonal single-particle states, the same state can also be obtained by antisymmetrizing the product of any pair of orthogonal states belonging to the two-dimensional manifold spanned by the original states. This is not surprising but it raises some questions concerning the problem of the assignment of definite properties to the constituents. In fact, Eq. (16) shows that, just as we can claim that in the state of Eq. (10) there is one particle with the properties associated with $|1\rangle$ and one with the properties associated with $|2\rangle$, we can make an analogous statement with reference to any two arbitrary orthogonal states $|\bar{1}\rangle$ and $|\bar{2}\rangle$ of the same manifold. However, this peculiar feature is only apparently problematic. In fact, it seems so because here we are confining our consideration to the spin degrees of freedom. When one takes into account also the space degrees of freedom one realizes that the only physically interesting situations are those associated with states like the one of Eq. (4), that is, those in which one wants, for example, to investigate the spin properties of a particle that has a precise location.[8] As we have seen, in such a state one can make precise claims about the spin state of the particle in the right or left, respectively, contrary to what happens for a state like the one of Eq. (7) which is genuinely entangled. This point has been exhaustively discussed in Ref. [2], to which we refer the reader for a more detailed analysis.

B. The boson case

Let us now pass to the more delicate case of two identical bosons. In such a case, dealing with the problem of their entanglement requires great care in order to avoid possible misunderstandings. Let us start, as before, by considering the bosonic Schmidt decomposition for an arbitrary state vector $|\psi(1,2)\rangle$ belonging to the symmetric manifold $S(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$ describing two identical bosons, and let us recall a well-known theorem of matrix analysis (the so-called Takagi factorization theorem [11], a particular instance of a more general theorem named the singular value decomposition) which is particularly useful for this case.

Theorem 3.4. For any symmetric $(N\times N)$ complex matrix B [that is, $B \in \mathcal{M}(N,\mathbb{C})$ and $B^T=B$] there exists a unitary transformation U such that $B=U\Sigma U^T$, where Σ is a real nonnegative diagonal matrix $\Sigma=\mathrm{diag}[b_1,\ldots,b_N]$. The columns of U are an orthonormal set of eigenvectors of $BB^\dagger$ and the diagonal entries of Σ are the nonnegative square roots of the corresponding eigenvalues.

It is worth noticing that the columns of the unitary operator U must be built by choosing a precise set[9] of eigenvectors of $BB^\dagger$ (to be determined in an appropriate manner, see [11]). The bosonic Schmidt decomposition turns out to be a trivial consequence of the just mentioned theorem 3.4.

Theorem 3.5. Any state vector describing two identical s-spin boson particles $|\psi(1,2)\rangle$ and, consequently, belonging to the symmetric manifold $\mathcal{S}(\mathbb{C}^{2s+1}\otimes\mathbb{C}^{2s+1})$ can be written as

$$|\psi(1,2)\rangle = \sum_{i=1}^{2s+1} b_i|i\rangle_1 \otimes |i\rangle_2, \tag{17}$$

where the states $\{|i\rangle\}$, with $i=1,\ldots,2s+1$, constitute an orthonormal basis for $\mathbb{C}^{2s+1}$, and the real nonnegative coefficients b_i are the diagonal elements of the matrix Σ and satisfy the normalization condition $\Sigma_i\, b_i^2=1$.

The Schmidt decomposition of Eq. (17) is always unique when the eigenvalues of the operator $BB^\dagger$ are nondegenerate, as happens for the biorthonormal decomposition of states describing distinguishable particles.

In analogy with the case of two distinguishable particles and of two identical fermions, one could naively be inclined to term entangled any bosonic state whose Schmidt number [that is, the number of nonzero coefficients in the decomposition (17)] is strictly greater than 1 (as clearly stated, for example, in Ref. [6]). However, as we will show, this criterion turns out to be inappropriate because the states with Schmidt number equal to 1 (that is, factorized states in which the two bosons are in the same state) do not exhaust the class of nonentangled pairs of identical bosons.

In order to identify a satisfactory and unambiguous criterion for detecting entanglement in systems of identical bosons, we resort to the decomposition of Eq. (17) and we analyze separately the cases of Schmidt number equal to 1, 2, and greater than or equal to 3.

Schmidt number=1. In this case the factorized state $|\psi(1,2)\rangle=|i^\star\rangle\otimes|i^\star\rangle$ describes two identical bosons in the same state $|i^\star\rangle$. It is obvious that such a state must be considered as nonentangled because one knows precisely the properties of both constituents and, consequently, no uncer-

[8] Obviously, due to the previous remarks concerning the arbitrariness of the states appearing in Eq. (10), also with reference to Eq. (4) one might claim, e.g., that there is with certainty a particle in the state $1/\sqrt{2}[|z\uparrow\rangle|R\rangle+|z\downarrow\rangle|L\rangle]$ and one in the state $1/\sqrt{2}[|z\uparrow\rangle|R\rangle-|z\downarrow\rangle|L\rangle]$. However, such states are not physically interesting and the corresponding measurements are extremely difficult (if not impossible) to perform.

[9] With this expression we mean that one has to appropriately choose the phase factors of the eigenstates.

tainty remains concerning which particle has which property. This fact perfectly agrees with the von Neumann entropy of the single-particle reduced statistical operators $S(\rho^{(1\ or\ 2)})$ being null.

Schmidt number=2. According to Eq. (17), the most general state with Schmidt number equal to 2 has the following form:

$$|\psi(1,2)\rangle = b_1|1\rangle_1 \otimes |1\rangle_2 + b_2|2\rangle_1 \otimes |2\rangle_2, \tag{18}$$

where $b_1^2+b_2^2=1$. The single-particle reduced density operator and its associated von Neumann entropy are

$$\rho^{(1\ or\ 2)} = b_1^2|1\rangle\langle 1| + b_2^2|2\rangle\langle 2|, \tag{19}$$

$$S(\rho^{(1\ or\ 2)}) = -b_1^2 \log_2 b_1^2 - b_2^2 \log_2 b_2^2. \tag{20}$$

We can now distinguish two cases, depending on the values of the nonnegative coefficients b_1 and b_2.

First we consider the case where the two coefficients are equal, that is, $b_1=b_2=1/\sqrt{2}$, and we prove the following theorem.

Theorem 3.6. The condition $b_1=b_2=1/\sqrt{2}$ is necessary and sufficient in order that the state $|\psi(1,2)\rangle=b_1|1\rangle_1\otimes|1\rangle_2+b_2|2\rangle_1\otimes|2\rangle_2$ might be obtained by symmetrizing the factorized product of two orthogonal states.

Proof. Suppose that the state $|\psi(1,2)\rangle$ is obtained by symmetrizing the tensor product of two orthogonal states $|\phi\rangle$ and $|\chi\rangle$. If we define the orthogonal states $|1\rangle=1/\sqrt{2}(|\phi\rangle+|\chi\rangle)$ and $|2\rangle=i/\sqrt{2}(|\phi\rangle-|\chi\rangle)$, we get

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\phi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\phi\rangle_2] = \frac{1}{\sqrt{2}}[|1\rangle_1 \otimes |1\rangle_2 + |2\rangle_1 \otimes |2\rangle_2]. \tag{21}$$

On the contrary, if $b_1=b_2=1/\sqrt{2}$ in Eq. (18), then by defining the orthogonal states $|\phi\rangle=1/\sqrt{2}(|1\rangle-i|2\rangle)$ and $|\chi\rangle=1/\sqrt{2}(|1\rangle+i|2\rangle)$, one gets

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\phi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\phi\rangle_2], \tag{22}$$

thus showing that $|\psi(1,2)\rangle$ can be obtained by symmetrizing the factorized product of two orthogonal states. ■

In this situation, and in full accordance with our Theorem 2.2, one must consider this state as a nonentangled one since it is possible to attribute definite state vectors to both particles (of course, once again, we cannot say which particle is associated with the state $|\phi\rangle$ or $|\chi\rangle$ because of their indistinguishability). Actually, such a state has precisely *the same conceptual status and exhibits the same physical features* as the nonentangled states of a pair of fermions and the reduced density operator $\rho^{(1\ or\ 2)}$ describes the physical state of one randomly chosen boson (given that one of them is associated with the state $|\phi\rangle$ while the other with the state $|\chi\rangle$):

$$\rho^{(1\ or\ 2)} = \frac{1}{2}[|\phi\rangle\langle\phi| + |\chi\rangle\langle\chi|]. \tag{23}$$

Moreover, since the state contains no correlations but those descending from the exchange of the two identical particles, the von Neumann entropy $S(\rho^{(1\ or\ 2)})$, according to Eq. (20), is correctly equal to 1 and our ignorance concerns only which particle has to be associated with which state.

Before proceeding a short digression is appropriate. First, if one chooses any pair of orthonormal vectors that are a linear combination, *with real coefficients*, of the above states $|1\rangle$ and $|2\rangle$:

$$|\overline{1}\rangle = \alpha|1\rangle + \beta|2\rangle, \quad |\overline{2}\rangle = -\beta|1\rangle + \alpha|2\rangle, \tag{24}$$

where $\alpha,\beta\in\mathbb{R}$ and $\alpha^2+\beta^2=1$, one has

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|1\rangle_1 \otimes |1\rangle_2 + |2\rangle_1 \otimes |2\rangle_2] = \frac{1}{\sqrt{2}}[|\overline{1}\rangle_1 \otimes |\overline{1}\rangle_2 + |\overline{2}\rangle_1 \otimes |\overline{2}\rangle_2]. \tag{25}$$

Thus, as in the case of distinguishable particles, when degeneracy occurs, the Schmidt decomposition is not unique.

This situation recalls the one we met in the case of two nonentangled fermions. However, the possibility of writing $|\psi(1,2)\rangle$ in different Schmidt forms, contrary to what happens with fermions, is absolutely unproblematic from the physically interesting point of view of the properties possessed by the two particles. Actually, the pair of orthonormal states whose symmetrization leads to the expression for the state of Eq. (22) is uniquely determined in the present case. Accordingly, for identical bosons there is no ambiguity concerning the properties, which allows us to claim that in the state of Eq. (22) there is one particle with the properties identified by $|\phi\rangle$ and one with the properties identified by $|\chi\rangle$. It is illuminating to stress the difference between two ways of looking at the properties of the constituents according to what emerges naturally when one expresses the state in the form of Eq. (22) or of Eq. (25). Actually, if one looks at the form of Eq. (25), without taking into account that it has been obtained by symmetrizing a factorized pair of orthogonal states, one is naturally led to make different statements, which, however, have a *probabilistic* character, i.e., to assert that with probability 1/2 both particle are (in the sense they will be found to be) in state $|1\rangle$ (or $|\overline{1}\rangle$) and with the same probability they are in state $|2\rangle$ (or $|\overline{2}\rangle$).

Let us now consider the second case, i.e., the one in which $b_1\neq b_2$, and prove the following theorem.

Theorem 3.7. The condition $b_1\neq b_2$ is necessary and sufficient in order that the state $|\psi(1,2)\rangle=b_1|1\rangle_1\otimes|1\rangle_2+b_2|2\rangle_1\otimes|2\rangle_2$ might be obtained by symmetrizing the factorized product of two nonorthogonal states.

Proof. Since the Schmidt number of $|\psi(1,2)\rangle$, and as a consequence the rank of the reduced statistical operators, is equal to 2, the state is obtained either by symmetrizing the factorized product of two orthogonal states or by symmetrizing the product of two nonorthogonal states. In fact, if at least three linearly independent states were involved in the symmetrization procedure, the rank of the single-particle statistical operator would consequently be strictly greater than 2. This theorem is then simply the logical negation of the previous Theorem 3.6. ■

As we have just proved, in this case the state $|\psi(1,2)\rangle$ can be obtained by symmetrizing a factorized product of two nonorthogonal states $|\phi\rangle$ and $|\chi\rangle$:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2(1+|\langle\chi|\phi\rangle|^2)}}[|\phi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\phi\rangle_2],$$

$$\langle\chi|\phi\rangle \neq 0. \tag{26}$$

Then, if one defines the vector $|\phi_\perp\rangle$ as the unique normalized vector orthogonal to $|\phi\rangle$ and lying in the two-dimensional manifold spanned by $|\phi\rangle$ and $|\chi\rangle$, the state of Eq. (26) can be written as[10]

$$|\psi(1,2)\rangle = a|\phi\rangle_1 \otimes |\phi\rangle_2 + \frac{b}{\sqrt{2}}[|\phi\rangle_1 \otimes |\phi_\perp\rangle_2 + |\phi_\perp\rangle_1 \otimes |\phi\rangle_2], \tag{27}$$

with certain complex coefficients $a, b \neq 0$ satisfying the normalization condition $|a|^2+|b|^2=1$ and depending on the modulus of the scalar product $|\langle\chi|\phi\rangle|$ in the following way: $|b|=(1-|\langle\chi|\phi\rangle|^2)^{1/2}(1+|\langle\chi|\phi\rangle|^2)^{-1/2}$.

In this case, the criteria adopted in Refs. [6–8] correctly lead one to declare the state of Eq. (27) as a genuine entangled one. Since this state is obtained by symmetrizing a factorized product of two *nonorthogonal* vectors, it is impossible to attribute to both particles definite properties, so that it has to be considered as entangled within our approach also. It is of some interest to exhibit explicitly the Schmidt decomposition of the state of Eq. (27):

$$|\psi(1,2)\rangle = \sqrt{\frac{1+\sqrt{1-|b|^4}}{2}}|1\rangle_1 \otimes |1\rangle_2 + \sqrt{\frac{1-\sqrt{1-|b|^4}}{2}}|2\rangle_1 \otimes |2\rangle_2, \quad |b| \in (0,1), \tag{28}$$

where the orthonormal states $|1\rangle$ and $|2\rangle$ are defined as

$$|1\rangle = \frac{|b|e^{i\theta_a/2}}{\sqrt{2-|a|^2-\sqrt{1-|b|^4}}}\left(|\phi\rangle + \frac{\sqrt{1-|b|^4}-|a|^2}{\sqrt{2}ab^\star}|\phi_\perp\rangle\right),$$

$$|2\rangle = \frac{i|b|e^{i\theta_a/2}}{\sqrt{2-|a|^2+\sqrt{1-|b|^4}}}\left(|\phi\rangle - \frac{\sqrt{1-|b|^4}+|a|^2}{\sqrt{2}ab^\star}|\phi_\perp\rangle\right), \tag{29}$$

with $a=|a|e^{i\theta_a}$, a and $b\neq 0$, and $|a|^2+|b|^2=1$. It is worth noticing that, contrary to what happens for the nonentangled state of Eq. (22), the Schmidt decomposition of Eq. (27) is now uniquely determined. This reflects the fact that the two eigenvalues of the operator $BB^\dagger$, where B is the matrix of the coefficients of the decomposition of $|\psi(1,2)\rangle$ of Eq. (27) on a factorized single-particle basis including $|\phi\rangle$ and $|\phi_\perp\rangle$, are distinct.

Let us come now to a consideration, for the case under investigation, of the reduced density operator describing one randomly chosen particle and of its associated von Neumann entropy. They are given by Eqs. (19) and (20) which immediately show that the entropy belongs to the open interval $(0,1)$. Actually, assuming that $b_1^2 > 1/2$, in a measurement process there is a probability greater than $1/2$ of finding both bosons in the same physical state $|1\rangle$, and a probability less than $1/2$ of finding them both in the orthogonal state $|2\rangle$. Accordingly, in this situation we have more information about the single-particle state than in the case of the nonentangled state of Eq. (22) and, as a consequence, the von Neumann entropy is strictly less than 1.

Schmidt number⩾3. In this situation, the state is a genuine entangled one since it cannot be obtained by symmetrizing a factorized product of two orthogonal states,[11] and the von Neumann entropy of the reduced density operators is such that $S(\rho^{(1\ or\ 2)}) \in (0, \log_2(2s+1)]$. As before, the entropy is close to zero when, in the Schmidt decomposition of the state, one of its coefficients is very close to 1, implying that there is a very high probability of being right in claiming that both bosons are (in the usual quantum sense of "will be found to be if a measurement is performed") in the state associated with the largest coefficient. On the contrary, the entropy equals $\log_2(2s+1)$ when the decomposition involves all the basis states of $\mathbb{C}^{2s+1}$ with equal weights (a maximally entangled state). Moreover, as we will prove below, when the entropy lies within the interval $(0,1)$, there is a precise state such that the probability of finding a (randomly chosen) particle in it, in the appropriate measurement, is greater than $1/2$. Obviously, the identification of the privileged state mentioned requires a knowledge of the Schmidt decomposition, since this state is the one associated with the largest b_i appearing in Eq. (17). This feature is a consequence of the following theorem.

Theorem 3.8. Consider the statistical operator $\rho = \Sigma_i p_i |i\rangle\langle i|$ where at least three of its weights p_i are different from zero. If each nonzero weight p_i is strictly less than $1/2$, then the von Neumann entropy $S(\rho)$ is strictly greater than 1.

Proof. let us consider the following function $f(p_1,p_2)$ of two variables:

$$f(p_1,p_2) \equiv -\sum_{i=1}^{2} p_i \log_2 p_i - (1-p_1-p_2)\log_2(1-p_1-p_2) \tag{30}$$

with $p_1, p_2 \in (0,1/2)$ and $p_1+p_2 > 1/2$. The only stationary point of $f(p_1,p_2)$ in the considered (open) region is a maxi-

[10]We notice that one could have equivalently defined the vector $|\chi_\perp\rangle$ as the unique normalized vector orthogonal to $|\chi\rangle$ and lying in the two-dimensional manifold spanned by $|\phi\rangle$ and $|\chi\rangle$, without modifying the forthcoming conclusions.

[11]In fact, if this was true, the rank of the reduced density operator would be equal to 2, in contradiction with the fact that a Schmidt number greater than or equal to 3 implies a rank equal to or greater than 3.

mum, while the minimum value of f is attained on the boundary of such a region and it equals 1. As a consequence, given a rank-3 statistical operator $\rho=\Sigma_{i=1}^{3}p_i|i\rangle\langle i|$, its von Neumann entropy $S(\rho)=f(p_1,p_2)$ is strictly greater than 1 whenever $p_i\in(0,1/2), i=1,2,3$. Let us now suppose that the theorem holds true for an arbitrary rank $n>3$ statistical operator $\rho=\Sigma_{i=1}^{n}p_i|i\rangle\langle i|$ and let us prove that it also holds true for an arbitrary rank-$(n+1)$ statistical operator in the following way. Take any p_1, which by assumption is smaller than $1/2$, and split it into two positive weights p_{11} and p_{12}, so that $p_1=p_{11}+p_{12}$. A basic property of the Shannon entropy function $-\Sigma_i p_i \log_2 p_i$ tells us that

$$-(p_{11}+p_{12})\log_2(p_{11}+p_{12})-\sum_{i=2}^{n}p_i\log_2 p_i < -p_{11}\log_2 p_{11}-p_{12}\log_2 p_{12}-\sum_{i=2}^{n}p_i\log_2 p_i. \quad (31)$$

By hypothesis the left hand side (LHS) of Eq. (31) is strictly greater than 1 since it is the von Neumann entropy of a statistical operator of rank n with weights strictly less than $1/2$, while the RHS is the von Neumann entropy of a statistical operator of rank $n+1$. Since the weights $(p_{11},p_{12},p_2,\ldots,p_n)$, apart from being constrained to belong to the interval $(0,1/2)$, are totally arbitrary, we have proved that the von Neumann entropy of any rank-$(n+1)$ statistical operator, with all weights belonging to $(0,1/2)$, is strictly greater than 1. By induction the theorem holds true for all $n>3$. ■

Summarizing the previous analysis, we have seen that, in the case of two identical bosons, consideration of the Schmidt number alone to detect the entanglement of a state fails, since there exist bosonic states with Schmidt number equal to 2 which can be entangled as well as nonentangled. Similarly, we have seen that the von Neumann entropy of the reduced statistical operators of the nonentangled state of Eq. (21) is equal to 1, and that the same value can characterize entangled states with Schmidt number greater than or equal to 3. Accordingly, the von Neumann entropy criterion also does not allow, by itself, an unambiguous identification of nonentangled states.

Concluding, we have shown that a complete and satisfactory criterion for distinguishing entangled from nonentangled boson states should involve consideration of *both* the Schmidt number criterion *and* the von Neumann entropy of the reduced density operator of each single particle, in accordance with the following theorem.

Theorem 3.9. A state vector $|\psi(1,2)\rangle$ describing two identical bosons is nonentangled if and only if either its Schmidt number is equal to 1, or the Schmidt number is equal to 2 and the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1\ or\ 2)})$ is equal to 1. Alternatively, one might say that the state is nonentangled if and only if either its von Neumann entropy is equal to 0, or it is equal to 1 and the Schmidt number is equal to 2.

IV. SUMMARY

To conclude our paper we summarize the previous results and give a general overview of the problem of characterizing the entanglement of quantum systems composed of two identical particles. First of all we recall that this problem can be dealt with in a completely rigorous way by following our original criterion of Ref. [2] expressed in the Definition 2.1. However, here we are interested in analyzing the criteria based on the determination of the Slater and the Schmidt numbers of the (fermionic and bosonic, respectively) Schmidt decomposition of the state vector of the composite system and on the evaluation of the von Neumann entropy of the reduced density operators associated with each single constituent. As usual, we deal separately with the cases of two identical fermions and of two identical bosons.

We start with a system of two identical fermions.

A. Two identical fermions

(1) Slater number of $|\psi(1,2)\rangle=1 \Leftrightarrow S(\rho^{(1\ or\ 2)})=1 \Leftrightarrow$ nonentangled state.

(2) Slater number of $|\psi(1,2)\rangle>1 \Leftrightarrow S(\rho^{(1\ or\ 2)})>1 \Leftrightarrow$ entangled state.

In the first case the state can be obtained by antisymmetrizing a tensor product of two orthogonal single-particle states, while in the second case it cannot.

In the considered case the two criteria are, as indicated, fully equivalent, and they identify precisely the same states as our criterion. It is interesting to notice that, since identical fermions cannot be in the same state, at least the uncertainty corresponding to the impossibility of identifying which particle has which property, when there is no entanglement, is always present. This is why the minimum value of the entropy turns out to be equal to 1. In such a case, when one makes a precise claim concerning the state of a *randomly chosen particle*, there is a probability equal to $1/2$ that the claim is correct. On the other hand, when the Slater number is greater than 1, or, equivalently, the entropy is larger than 1, there is no state such that the statement "this particle, if subjected to the appropriate measurement, will be found in such a state" has a probability equal to or greater than $1/2$. Thus, the nonentangled state is the one in which one has the maximum information about the state of the system.

The case of two identical bosons is slightly more involved. We recall the conclusions we have reached.

B. Two identical bosons

(1) Schmidt number of $|\psi(1,2)\rangle=1 \Leftrightarrow S(\rho^{(1\ or\ 2)})=0 \Rightarrow$ nonentangled state.

(2) Schmidt number of $|\psi(1,2)\rangle=2$ and $S(\rho^{(1\ or\ 2)})\in(0,1) \Rightarrow$ entangled state.

(3) Schmidt number of $|\psi(1,2)\rangle=2$ and $S(\rho^{(1\ or\ 2)})=1 \Rightarrow$ nonentangled state.

(4) Schmidt number of $|\psi(1,2)\rangle>2 \Rightarrow$ entangled state.

In the first case the state is the tensor product of the same single-particle vector; in the second case the state can be obtained by symmetrizing two nonorthogonal vectors and thus no definite property can be attributed to both subsystems. In the third case the state can be obtained by symmetrizing a tensor product of two orthogonal vectors while in the last one it involves more than two linearly independent single-particle states and the von Neumann entropy $S(\rho^{(1\ or\ 2)})$ can take any value within the interval $[0,\log_2(2s+1)]$. The above list exhibits some interesting features. First of all it shows clearly that the Schmidt number cannot be used, unless it has the value 1, to identify nonentangled states, since there are both entangled and nonentangled states with Schmidt number equal to 2. It also shows that the von Neumann entropy criterion does not allow, by itself, a clear-cut identification of the nonentangled states since it can take the value 1 both for a nonentangled state of Schmidt number 2 and for an entangled state of Schmidt number greater than 2.

As discussed in this paper, the von Neumann entropy supplies us, by itself, with important information concerning the state of a randomly chosen constituent: when it lies in the interval $(0,1)$ there is a precise state such that the probability of finding it in an appropriate measurement is greater than $1/2$.

V. CONCLUSIONS

In this paper we have discussed in general the problem of deciding whether a state describing a system of two identical particles is entangled or not. We recalled the general criteria and results derived in Refs. [2,3], which make explicit reference to the possibility of attributing a complete set of objective properties to both constituents. We have shown how this attitude allows one to clarify the somewhat puzzling situation that one meets when resorting to the consideration either of the Slater and the Schmidt number of the (fermionic and bosonic) Schmidt decomposition or of the von Neumann entropy of the reduced statistical operator for detecting entanglement. Our analysis has made clear that some alleged difficulties of the mentioned approaches derive simply from not having appropriately taken into account the peculiar role of the identity within the quantum formalism. In particular, we have shown how the consideration of the von Neumann entropy allows one to determine whether the uncertainty concerning the states arises simply from the identity of the particles or is also a genuine consequence of the entanglement. With reference to this problem we stress once more that, when all our lack of information derives simply from the identity, the existing correlations cannot be used as a quantum mechanical resource to implement any teleportation procedure or to violate Bell's inequality, contrary to what happens for an entangled state.

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